

The determinant of an  $n \times n$  square matrix is a sum of signed terms, each term of which is a product of  $n$  matrix elements, one taken from each row of the matrix in such a way that every column is also represented. The sign relates to the permutation structure of the term. For example, the determinant of a  $3 \times 3$  matrix is the sum of six terms

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{array}{l} a_1 b_2 c_3 - a_1 c_2 b_3 \\ + b_1 c_2 a_3 - b_1 a_2 c_3 \\ + c_1 a_2 b_3 - c_1 b_2 a_3 \end{array}$$

and, if the elements of each term are placed in column order 1 2 3, then the sign is *plus* if the elements, for example,  $bca$  form an even permutation of  $abc$  (involving an even number of interchanges, two say in  $bca \rightarrow bac \rightarrow abc$ ) and the sign is *minus* if the elements, for example,  $acb$  form an odd permutation of  $abc$  (with an odd number of interchanges such as, here, the single interchange,  $b$  and  $c$ ). Some properties are

- The determinant of a matrix with a Row of Zeros is 0. Each term includes an element taken from the row and consequently every term is zero.
- The determinant of a matrix with a Column of Zeros also has the value zero. Examples of determinants with a row of zeros and a column of zeros are

$$\det \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix} = 0, \quad \det \begin{bmatrix} 2 & 0 & 4 \\ 5 & 0 & 7 \\ 8 & 0 & 1 \end{bmatrix} = 0.$$

- The determinant of a Triangular Matrix is the product of its diagonal elements since all other terms are zero. Consider the determinant of the upper triangular matrix

$$\det \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} = 2 \times 5 \times 7 = 70.$$

- The determinant of a Product of Matrices is the product of their determinants.

$$\det(AB) = \det(A) \det(B).$$

- The same is not true for addition in general

$$\det(A + B) \neq \det(A) + \det(B).$$

- If  $B$  is the matrix that results when a single row or single column of  $A$  is multiplied by a scalar  $k$ , then  $\det(B) = k \det(A)$
- If  $B$  is the matrix that results when two rows or two columns of  $A$  are interchanged, then  $\det(B) = -\det(A)$
- If  $B$  is the matrix that results when a multiple of one row of matrix  $A$  is added to another row of  $A$ , then  $\det(B) = \det(A)$ . A similar result holds when a multiple of one column of  $A$  is added to another column of  $A$ .

## LAPLACE EXPANSION OF A DETERMINANT IN TERMS OF MINORS

Pierre-Simon Laplace (Normandy 1749–1827 Paris) had a remarkably modern view of the impact of comets on the Earth

“... the small probability of collision of the Earth and a comet can become very great in adding over a long sequence of centuries. It is easy to picture the effects of this impact on the Earth. The axis and the motion of rotation have changed, the seas abandoning their old position . . . , a large part of men and animals drowned in this universal deluge, or destroyed by the violent tremor imparted to the terrestrial globe.”

Laplace introduced a method of evaluating the determinant of a matrix by expansion using, for example, the first row and signed minor sub-determinants

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{vmatrix} = 2 \begin{vmatrix} 6 & 7 \\ 9 & 1 \end{vmatrix} - 3 \begin{vmatrix} 5 & 7 \\ 8 & 1 \end{vmatrix} + 4 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}.$$

The determinant may alternatively be expanded using the second row, or any other row or column. The sign associated with the minor obtained by deleting row  $i$  and column  $j$  of a matrix is

$$(-1)^{i+j}$$

and the signed minor is known as the  $ij$ th cofactor of the matrix.

See Minors and Cofactors in section 4.1 of *Contemporary Linear Algebra* by H. Anton and R. Busby, or in section 2.1 of *Elementary Linear Algebra* by H. Anton and C. Rorres.