VISIBLE MEASURES OF MAXIMAL ENTROPY IN DIMENSION ONE

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Abstract
A real one-dimensional analogue of Zdunik’s dichotomy is proven giving dynamical conditions for a multimodal map to have a measure of maximal entropy of dimension one.

1. Introduction

In [11], Zdunik shows the following dichotomy: for every rational map of the Riemann sphere either the measure of maximal entropy is of Hausdorff dimension strictly smaller than the dimension of the Julia set or the map is critically finite with parabolic orbifold.

In the complex case there is a unique ergodic measure of maximal entropy, positive on open subsets of the Julia set. In the real case matters are more complicated with potentially the presence of numerous ergodic measures of maximal entropy.

The real one-dimensional analogue of Zdunik’s dichotomy is as follows. For C-maps (see below) \( f \) with positive entropy, an ergodic measure of maximal entropy \( \mu \) has dimension strictly smaller than 1 or \( f \) is conjugate to a continuous piecewise-linear map on some restrictive interval, this conjugacy is smooth outside the (finite) post-critical set and \( \mu \) is supported on the forward orbit of the restrictive intervals. Interestingly, the conjugacy is \( C^r \) if \( f \) is, for \( r \geq 2 \), so the result is optimal.

A point \( c \) is called critical for smooth map \( f \) if the derivative \( Df(c) = 0 \) (note that inflection points are critical); for continuous piecewise-linear maps turning points are called critical. We denote by \( \text{Crit} \) the collection of critical points. The post-critical set \( PC \) for a map \( f \) is defined by \( PC = \bigcup_{i \geq 1} f^i(\text{Crit}) \).

2000 Mathematics Subject Classification 37A05 (primary).
Definition 1.1. Following [6] we say $f$ is a C-map if $f : I \to I$ is a continuously differentiable map of the closed interval $I$ and its derivative $Df$ has the following properties:

- $Df$ satisfies a Hölder condition of order $\epsilon$, $\epsilon > 0$.
- There are only a finite number of critical points $a_i$, $1 \leq i \leq n$, $Df(a_i) = 0$.
- There exist positive numbers $k_i^- (k_i^+)$ such that
  $$\left| \frac{\log |Df(x)|}{|x - a_i|^{k_i^- (+)}} \right|$$
  is bounded in a left (right) neighbourhood of $a_i$ - this is a non-flatness condition.

C-maps which are also of class $C^r$ shall be called $C^r$ C-maps. We state our results for $C^r$ C-maps. If $f$ is $C^{1+\epsilon}$ (i.e. just a C-map) then the same results hold but the respective conjugacies may have different Hölder constants.

The Hausdorff dimension [9] of a probability measure $\nu$ with support $X$ is defined as

$$\text{HD}(\nu) = \inf \{ \text{HD}(Y) : Y \subset X, \nu(Y) = 1 \}.$$ 

Theorem 1. Let $f$ be a $C^r$, $r \geq 2$, C-map with positive topological entropy. Exactly one of the following statements holds:

- $\text{HD}(\mu) < 1$ for all measures of maximal entropy $\mu$.
- There exists a restrictive interval $J$ of period $k$ such that $f^k_J$ has finite post-critical set $PC$, $f^k$ is $C^r$ conjugate to a continuous piecewise linear map of constant slope on $J \setminus PC$ and $\mu(J) > 0$ for some measure of maximal entropy $\mu$.

The singularities of the conjugacy are of root type. If $f$ has no inflection points (this is typical) then the post-critical set must be contained in the boundary, $PC = \partial J$.

The condition that $f$ has positive topological entropy is necessary. Indeed a C-map which is the identity on $[-1, 0]$ and a Feigenbaum map on $[0, 1]$ has zero topological entropy and an absolutely continuous invariant measure.

For non-renormalisable maps a more dynamical version is possible.
COROLLARY 1. Let $f$ be a $C^r$, $r \geq 2$, $C$-map without non-trivial restrictive intervals. Exactly one of the following statements holds:

- $\text{HD}(\mu) < 1$ for measure of maximal entropy $\mu$.
- $f$ has finite post-critical set $PC$ and $f$ is $C^r$ conjugate to a continuous piecewise linear map of constant slope on $I \setminus PC$.

In the unimodal case we achieve a complete dynamical classification of maps where the Hausdorff dimension of the measure of maximal entropy is one.

DEFINITION 1.2. A $C$-map $f$ is unimodal if it has exactly one critical (turning) point, exactly two fixed points and $f(\partial I) \subset \partial I$. Unimodal $C$-map $f$ is called $C^r$ pre-Chebyshev if

- $f$ is exactly $k$ times renormalisable, for some $0 \leq k < \infty$, and each renormalisation is of type 2 (i.e. of Feigenbaum type);
- Let $f_{ij}^k : J \to J$ be the resultant non-renormalisable map. Then there is a $C^r$ conjugacy $h$ on the interior of $J$ from $f_{ij}^k$ to the full tent map $T_2$.

If $f$ is pre-Chebyshev then the $C^r$ conjugacy $h$ extends to a topological conjugacy on $I$ from $f$ to a symmetric tent map $T_s$ of slope $\pm s$, where the entropy of $f$ is $\log s$. Then if $f$ is $k$ times renormalisable, $s = 2^{2^{-k}}$. Also remark that the full tent map $T_2$ is analytically conjugate to the Chebyshev quadratic polynomial $4x(1-x)$.

THEOREM 2. Suppose $f$ is a $C^r$ unimodal $C$-map, $r \geq 2$, and that $f$ has positive topological entropy and measure of maximal entropy $\mu$. Exactly one of the following statements holds:

- $\text{HD}(\mu) < 1$.
- $f$ is $C^r$ pre-Chebyshev.

We say $f$ has a (non-trivial) restrictive interval $J \subset I$ of period $n$ if there exist interval $J(\neq I)$ disjoint from periodic attractors and $n > 0$ such that $J, f(J), \ldots, f^{n-1}(J)$ are disjoint, $f^n(\partial J) \subset \partial J$ and $f^n(J) \subset J$. Also, $f$ is renormalisable if $f$ has a non-trivial restrictive interval containing at least one critical point.

An interval $U$ is regularly returning if $f^n(\partial U) \cap U = \emptyset$ for all $n > 0$. Then if $f^n(A) = f^n(B) = U$ either $A \cap B = \emptyset$, $A \subset B$ or $A \supset B$. 


An important principle in our considerations is the existence of induced expansion for maps with absolutely continuous invariant probability measures (acips) (Proposition 2.13). The induced maps we construct have nice “$C^r$” properties, necessary to get good conjugacy results. This principle can be applied, for example, to extend the results of [8] to ergodic C-maps with acips with optimal smoothness results for the conjugacies.

In the following section we shall assemble various results to be used in the proofs.

2. Preliminaries

2.1. Topological results

We start with the result of Milnor and Thurston (1977), also Parry (1966), see [1] II.8.

**Fact 2.1.** Assume that $f : I \rightarrow I$ is a continuous, piecewise (strictly) monotone map with positive topological entropy $h_t(f)$ and let $s = \exp(h_t(f))$. Then there exists a continuous, piecewise linear map $T : [0,1] \rightarrow [0,1]$ with slope $\pm s$, and a continuous, monotone increasing map $h : I \rightarrow [0,1]$ which is a semi-conjugacy between $f$ and $T$, i.e.

$$h \circ f = T \circ h.$$ 

Proposition III.4.3 in the same book [1] adds the following properties for $C^1$ multimodal maps without wandering intervals (absence of wandering intervals for C-maps is stated in [10]):

**Fact 2.2.** Let $X$ be the set of points which are eventually mapped into a restrictive interval or into the basin of a periodic attractor.

(i) The semi-conjugacy $h$ can collapse an interval only if it is contained in $X$. Basins of periodic attractors are certainly collapsed.

(ii) The union of $X$ and the set of points which are mapped eventually into an interval $V \neq I$ with $f(V) \subset V$ is dense in $I$. Moreover $f$ is conjugate to $T$ on the complement of $X$.

(iii) If $T$ has a periodic turning point $c$ then $h^{-1}(c)$ is a restrictive interval.
Proposition III.4.4 will also be useful:

**Fact 2.3.** Let $T : I \to I$ be a piecewise linear $l$-modal map with slope $\pm s$, $s > 1$. Then one has the following properties.

(i) $T$ has sensitive dependence on initial conditions.
(ii) $T$ is only finitely often renormalizable.
(iii) $T$ The non-wandering set of $T$ contains a finite number of intervals which are permuted by $T$ and on each of these intervals $T$ is transitive. (The permutation of the intervals may split into disjoint cycles.) On the complement of these intervals, the non-wandering set of $T$ consists of a finite number of periodic points and the remainder is a subshift of finite type.

(iv) The only attractors of $T$ are intervals.

Another well-known fact [4] is:

**Fact 2.4.** For symmetric tent map $T$ with slope $\pm s$ and turning point $c$

(i) If $\sqrt{2} < s \leq 2$ then for every open interval $J$ there exists an $n$ such that $T^n(J) \supset [T^2(c), T(c)]$. In particular $T$ has no restrictive intervals.

(ii) If integer $k$ is such that $\sqrt{2} < s^k \leq 2$ then $T$ has exactly $k$ restrictive intervals $I_k \subset I_{k-1} \subset \ldots \subset I_0 = I$ containing the turning point. $I_i$ has period $2^i$ for $i = 0, 1, \ldots, k$. For each $x \in I$ there exists an integer $n$ such that either $T^n(x) \in I_k$ or such that $T^n(x)$ is one of the $k$ periodic orbits that remain outside of the forward orbit of $I_k$.

**Lemma 2.5.** Let $f$ be a transitive, non-renormalisable continuous map of the interval $I$. Then for any subinterval $U$ there exists $n$ such that $f^n(U) = I$.

**Proof.** Fact 2.1 implies we can assume $f$ is an expanding continuous piecewise linear map. Consider periodic point $p \in U$ of period $k$ and small subinterval $V$ with $p \in \partial V$. Now $p \in f^{2ki}(V) \subset f^{2k(i+1)}(V)$ for all $i \geq 0$. Set $W_i = f^{2k(i+1)}(V) \setminus f^{2k(i)}(V)$. For $i$ sufficiently large one has $W_i = \emptyset$. This follows from the finiteness of the number of critical points, that the $W_i$ are disjoint and that the map is expanding: If $W_i \cap \text{Crit} = \emptyset$ then $|W_{i+1}| > |W_i|$. 
Let $k$-periodic $V_* = \bigcup_{i \geq 1} f^{2ki}(V)$. By transitivity $I \subset \bigcup_{j=0}^{k-1} f^j(V_*)$. Thus some iterate of $U$ contains a fixed point $p'$. One restarts the process to find 2-periodic $V_*'$ and concludes by non-renormalisability that $V_*' = I$.

2.2. Absolutely continuous measures and induced expansion

From Ledrappier [6] we have that for C-maps an ergodic invariant measure $\nu$ of positive entropy satisfies the Rokhlin formula

$$h_\nu(f) = \int \log |Df| \ d\nu$$

if and only if $\nu$ is absolutely continuous with respect to the Lebesgue measure $\lambda$.

Combined with the Dynamic Volume Lemma which for ergodic, invariant, probability measures $\nu$ states

$$\text{HD}(\nu) = \frac{h_\nu}{\int \log |Df|},$$

this implies that

**Lemma 2.6.** $\text{HD}(\nu) = 1$ if and only if $\nu$ is absolutely continuous with respect to Lebesgue.

This holds for $\nu$ an ergodic, invariant, probability measure for C-map $f$.

**Fact 2.7.** [5] A piecewise linear map $S$ with slope strictly greater than 1 has an absolutely continuous invariant measure.

**Definition 2.8.** Given a $C^{1+\epsilon}$ function $f$, we say a map $F : \bigcup_i U_i \to U$ is a Markov map induced by $f$ if $U$ is a subinterval of $I$, $U_i \subset U$ are disjoint intervals and

- $\lambda(U \setminus \bigcup_i U_i) = 0$
- for all $i$, $F(U_i) = U$ and $F|_{U_i} = f^{n_i}$ for some $n_i$
- there exists $\delta > 1$ such that $|Df| \geq \delta$
- there exists a bound on distortion $K > 1$ such that, for all $n > 0$, on each connected component $V$ of the domain of $F^n$,

$$\frac{DF^n(x)}{DF^n(y)} < K$$

for all $x, y \in V$. 

Definition 2.9. If $f$ is $C^r$, for integer $r \geq 2$, then we say $F$ is a $C^r$ Markov map induced by $f$ if in addition to being Markov we have the following condition on higher derivatives (which gives a similar condition on higher derivatives of the nonlinearity):

$$\left| \frac{D^i F(x)}{(DF(x))^i} \right| < L,$$

for some $L > 0$, for all $x$ in the domain of $F$ and for $i = 2, 3, \ldots, r$.

For existence of Markov maps, we shall need to extend a theorem of Ledrappier [6]. Summarily, here $(Y, f)$ is the extension or inverse system for $(I, f)$, $f$ being a $C^r$-map of the interval $I$ with ergodic invariant measure $\mu$. An element of $Y$ is a point $x \in I$ and a sequence $z$ which says which inverse branch of $f$ to take (coded by the partition defined by critical points). $(Y, f)$ is invertible and if $\Pi$ is the projection on $I$ then there exists a unique ergodic invariant measure $\bar{\mu}$ on $Y$ whose image by $\Pi$ is $\mu$.

Fact 2.10. [6] Let $f$ be a $C^r$-map with an invariant ergodic probability measure $\mu$ with positive Lyapunov exponent. Suppose $0 < \chi < \int \log |Df|d\mu$. Then there exist on $Y$ four measurable functions $\alpha, \beta, \gamma, \gamma$ and a constant $K$ such that:

- $\alpha > 0, 1 < \beta < \infty, 0 < \gamma < \gamma < \infty, \bar{\mu}$-almost everywhere;
- let $y = (x, z)$ and $|t| < \alpha(y)$; the point $y_t = (x + t, z)$ lies in $Y$;
- $|\Pi(f^{-n}y) - \Pi(f^{-n}y_t)| \leq \beta(y) \exp(-\chi n)|t|$;
- for any $n$,

$$\frac{1}{K} \leq \frac{Df(\Pi(f^{-n}y))}{Df(\Pi(f^{-n}y_t))} \leq K;$$

$$\gamma(y) \leq \Pi_{n=1}^{\infty} \frac{Df(\Pi(f^{-n}y))}{Df(\Pi(f^{-n}y_t))} \leq \bar{\gamma}(y).$$

Proposition 2.11. Supplementarily to Fact 2.10, there exists measurable $\gamma_1 < \infty \bar{\mu}$-almost everywhere such that for all $n$

$$|\log Df^n(f^{-n}y) - \log Df^n(f^{-n}y_t)| < \gamma_1(y)|t|,$$

($\epsilon$ being the Hölder exponent). If $f$ is also $C^r$, for some integer $r \geq 3$, then there exists measurable functions $\gamma_i, i = 2, 3, \ldots, r - 1$ with $\gamma_i(y) < \infty \bar{\mu}$-almost
everywhere such that for all \( n \)
\[
\left| \frac{D^i f^n(f^{-n}y)}{Df^n(f^{-n}y)} - \frac{D^i f^n(f^{-n}y_t)}{Df^n(f^{-n}y_t)} \right| \leq \gamma_1(y)|t|.
\]

Proof.
\[
\log Df^n(f^{-n}y) = \sum_{j=0}^{n-1} \log Df(f^{-n+j}y) = \sum_{j=1}^{n} \log Df(f^{-j}y)
\]
and
\[
|\log Df(x) - \log Df(x')| = \log \left(1 + \left| \frac{Df(x) - Df(x')}{Df(x')} \right| \right) \leq C \frac{|x - x'|^c}{|Df(x')|},
\]
for some constant \( C \). Thus
\[
|\log Df^n(f^{-n}y) - \log Df^n(f^{-n}y_t)| \leq C \sum_{j=1}^{n} \frac{||\Pi(f^{-j}y) - \Pi(f^{-j}y_t)||^c}{|Df(f^{-j}y)|},
\]
Non-degeneracy of the measure (Definition 1.3 and Proposition 7 in [6]) means that for \( \mathfrak{p} \) almost every \( y \) the sequence \( (f^{-j}y) \) approaches critical points sub-exponentially, i.e. almost everywhere
\[
\liminf_j \frac{1}{j} \log \text{dist}(\text{Crit}, f^{-j}y) = 0.
\]
Non-flatness then implies that \( \frac{1}{|Df(f^{-j}y)|} \) is bounded by a sub-exponential sequence. From Fact 2.10 we also have that \( |\Pi(f^{-j}y) - \Pi(f^{-j}y_t)| \leq \beta(y) \exp(-\chi_j)|t| \); thus almost everywhere we are summing an exponentially decreasing sequence and there exists a function \( \gamma_1, < \infty \) almost everywhere, such that
\[
\sum_{j=1}^{n} \frac{||\Pi(f^{-j}y) - \Pi(f^{-j}y_t)||^c}{|Df(f^{-j}y)|} \leq \sum_{j=1}^{\infty} \frac{||\Pi(f^{-j}y) - \Pi(f^{-j}y_t)||^c}{|Df(f^{-j}y)|} \leq \gamma_1(y)|t|,
\]
completing the first part of the proof.

For \( i \geq 2 \) we can expand \( \frac{D^i f^n}{Df^n} \) as a polynomial
\[
\frac{D^i f^n(x)}{Df^n(x)} = \sum_{j=0}^{n-1} P \left( \frac{D^2 f(f^j x)}{Df(f^j x)}, \frac{D^3 f(f^j x)}{Df(f^j x)}, \ldots, \frac{D^i f(f^j x)}{Df(f^j x)} \right)
\]
where \( P \) is some (calculable) polynomial.

It suffices to show that for each \( i \leq r \) there exists a set of full measure on which
\[
\sum_{j=0}^{\infty} \left| \frac{D^i f(f^{-j}y)}{Df(f^{-j}y)} - \frac{D^i f(f^{-j}y_t)}{Df(f^{-j}y_t)} \right| \leq \theta(y) < \infty, \text{ for some function } \theta.
\]
But
\[
\left| \frac{D^i f(x)}{Df(x)} - \frac{D^i f(x')}{Df(x')} \right| = \left| \frac{D^i f(x)Df(x') - D^i f(x')Df(x)}{Df(x')Df(x)} \right|
\]
\[
= \left| \frac{D^i f(x)(Df(x') - Df(x)) + Df(x)(D^i f(x) - D^i f(x'))}{Df(x')Df(x)} \right| \leq 2C^2 \left| Df(x')Df(x) \right|,
\]
where $C$ is some constant such that $|D^i f(x)| < C$, $|D f(x)| < C$, $|D^i f(x') - D f(x)| < C |x - x'|$ and $|D f(x') - D f(x)| < C |x - x'|$, for $i = 2, 3, \ldots, r - 1$.

Continuing as before we have an exponentially decreasing, thus summable, sequence almost everywhere as required.

**Lemma 2.12.** Suppose $F$ is a Markov map for a $C^r$ function such that for some $K > 0$, for $i = 2, \ldots, r - 1$ and $x, y$ in the same connected component $V$ of the domain of $F$ one has

$$
\left| \frac{DF^i(x)}{DF(x)} \right| < K \left| \frac{x - y}{|V|} \right|.
$$

Then

$$
\left| \frac{D^j F(x)}{(DF(x))^j} \right| < L
$$

for all $x$ in the domain of $F$ and for $j = 2, 3, \ldots, r$ for some constant $L > 1$.

**Proof.** We shall show

$$
\left| \frac{D^j F(x)}{DF(x)} \right| < L \cdot \frac{1}{|V|^{j-1}} \quad (2.3)
$$

Since $\frac{|DF(x)|}{|V|}$ is uniformly bounded (independently of connected component $V$), the above inequality implies the lemma.

As per Lemma 3.3 of [8] one shows that inequality (2.3) is true for $j = 2$.

Afterwards one proceeds inductively writing

$$
\left| \int_x^y \frac{D^j F}{DF} \right| = \left| \int_x^y \frac{D^{j-1} F}{DF} \right| + \left| \int_x^y \frac{D^j F}{DF} \right| \leq K \cdot \frac{|x - y|}{|V|} + |y - x| \cdot \frac{C}{|V|^{j-1}}
$$

for some constant $C$, implying

$$
\left| \frac{D^j F}{DF} \right| \leq \frac{K + C}{|V|^{j-1}} < \frac{K + C}{|V|^{j-1}}.
$$

**Proposition 2.13.** Let $f$ be a $C$-map with an ergodic absolutely continuous invariant probability measure $\mu$ with positive Lyapunov exponent. Then there exists a Markov map induced by $f$. If in addition $f$ is $C^r$ then there exists a $C^r$ Markov map induced by $f$.

**Proof.** We apply Fact 2.10 and Proposition 2.11 to fix $\alpha_0 > 0, \beta_0 < \infty, K < \infty$ such that $A = \{ y \in Y : \alpha(y) > \alpha_0, \beta(y) < \beta_0, \gamma(y) < K, \delta(y) < K \}$ has positive
measure, $\mathcal{P}(A) > 0$. Let $x \in I$ be a density point of $(\Pi|_{A}), \mathcal{P}$, and set $A_x = A \cap \Pi^{-1}x$.

Let $U \ni x$ be a regularly returning interval with $|U| < \frac{\alpha_0}{2}$. By ergodicity almost every point $y \in Y$ accumulates on $A_x$. Thus for almost every $y$ there exist iterates $f^n y = (x', z)$ with $x' \in U$ and $(x, z) \in A_x$ with $n$ arbitrarily large. By Fact 2.10, since $\alpha((x, z)) \geq \alpha_0$ the interval $U$ can be pulled back along the branch defined by $z$ with bounded distortion to an interval containing $\Pi y$, i.e. for $\mathcal{P}$-almost every $y$ there exist $n_y$ and interval $V_y \ni \Pi y$ such that $f^{n_y}$ maps $V_y$ diffeomorphically onto $U$ with bounded distortion. Taking $n_y$ large (and thus $V_y$ small) gives expansion: $Df^{n_y} > \delta > 1$ on $V_y$.

For $y, y'$ their respective intervals $V_y, V_{y'}$ are disjoint or nested since $U$ is regularly returning. A set of full $\mathcal{P}$-measure projects to a set of full $\mu$-measure, so $\mu$-almost every point in $U$ is contained in some $V_y$. Take a disjoint collection of full measure of intervals $V_y$ and define $F$ on $\bigcup V_y \ni x$ by

$$F|_{V_y}(x) = f^{n_y}(x).$$

Now $\frac{Df(x)}{Df(x')} < K^2 \frac{|x-x'|}{|V_y|}$ if $x, x' \in V_y$ and a standard summation argument gives the distortion bound (2.1) so $F$ is a Markov map induced by $f$.

In the $C^r$ case $\frac{D^i f(x)}{D^i f(x')} - \frac{D^i f(x')}{D^i f(x)} < K^2 \frac{|x-x'|}{|V_y|}$ for all $x, x' \in V_y$ for all connected components $V_y$ of the domain of $F$, and for $i = 2, 3, \ldots, r - 1$. The bound on $\frac{D^i F}{(D^i F)^j}$ follows upon applying Lemma 2.12, giving a $C^r$ Markov map.

**Proposition 2.14.** Let $f$ be a C-map possessing an ergodic absolutely continuous invariant measure $\mu$. Then $\mu$ is equivalent to Lebesgue measure $\lambda$ on a finite union of intervals $U$. Moreover there exist $\varepsilon > 0$ and measurable density function $\rho > \varepsilon$ on $U$ such that $d\mu = \rho d\lambda$.

**Proof.** It follows from Proposition 3.6 of [6] (which gives the Rohlin decomposition for $\mathcal{P}$) and the existence of the set $A$ of positive measure in the above proposition that there exists a small interval $V$ on which the density $\rho$ of $\mu$ is greater than some strictly positive constant $\varepsilon_0 > 0$. Note that this result is analogous to Theorem 6 of [7] which treats rational maps.
There exists \( n \) such that \( \bigcup_{i \leq n} f^i(V) = \bigcup_{i \geq 0} f^i(V) = \text{Supp}(\mu) \) by Lemma 2.5. By invariance,
\[
\rho > \frac{\varepsilon_0}{\max(|Df^n|)} > \varepsilon > 0,
\]
as required.

**Fact 2.15.** ([1] V, Theorem 2.2) For any Markov map \( G \) with corresponding Markov partition \( \bigcup U_i \), there exists a unique absolutely continuous (w.r.t. Lebesgue measure \( \lambda \)) \( G \)-invariant probability measure \( \nu \). The density is Hölder continuous and uniformly bounded from above and from below away from zero.

**Lemma 2.16.** Every conjugacy between two \( C^r \), \( r = 2, 3, \ldots \) Markov maps which preserves multipliers is a \( C^r \) diffeomorphism.

Definition 2.8 for Markov maps is more general than that of [8]. Instead of demanding negative Schwarzian we simply have the distortion bounds which suffice - all the proofs of [8] follow through with little modification, including Corollary 2.9 which has the same statement as this lemma. Remark that in this paper we actually use this lemma for \( r > 2 \).

### 3. Proofs

**Proposition 3.1.** Let \( f \) be a \( C^r \), \( r \geq 2 \), C-map and \( \mu \) an ergodic measure of maximal entropy for \( f \). If \( \text{HD}(\mu) = 1 \) then there exists a restrictive interval \( J \) of period \( k \) on which \( f^k|_J \) has finite post-critical set \( PC \) and \( f^k \) is \( C^r \) conjugate to a continuous piecewise linear map of constant slope on \( J \setminus PC \). The measure \( \mu \) has support \( \bigcup_{i=0}^{k-1} f^i(J) \).

**Proof.** By Lemma 2.6, \( \mu \) is absolutely continuous with respect to Lebesgue measure. By Proposition 2.14 the measure is equivalent to Lebesgue and is supported on a cycle of intervals \( X' \), on which \( f \) is transitive.

Let \( X \) be a component of \( X' \) and \( n \) such that \( f^n : X \to X \), and set \( g := f^n \). Then \( \mu \) (when normalised) is the unique (probability) measure of maximal entropy for \( g : X \to X \) and is absolutely continuous. Let \( \log s \) be the entropy of \( f \). Then setting \( t = s^n \) the entropy of \( g \) is \( \log t \).
By Facts 2.1-2.2 and transitivity there is a conjugacy $h$ to a piecewise linear map $T : I \to I$ with slope $\pm t$ and the same entropy $\log t$, 

$$h \circ g = T \circ h.$$ 

Note that $X$ could contain a periodic (restrictive) subinterval. Since $T$ is only finitely often renormalisable the same holds for $g$ and there is a smallest restrictive interval $J$. We suppose (without loss of generality) that $X = J$, i.e. that $g$ is non-renormalisable.

The push-forward of $\mu$, $h_\ast \mu$, is the unique measure of maximal entropy for $T$. But $T$ has an absolutely continuous invariant measure $\overline{\mu}$ by Fact 2.7, thus $\text{HD}(\overline{\mu}) = 1$. For the map $T$ the Lyapunov exponent is $\log t$. By the Dynamic Volume Lemma the measure-theoretical entropy $h_\ast \mu = \log t$ and so $\mu$ maximises entropy. By unicity $h_\ast \mu = \overline{\mu}$. This measure is ergodic and thus equivalent to Lebesgue.

It follows that $h$ and $h^{-1}$ are absolutely continuous: Indeed, $\lambda(A) = 0 \iff \mu(A) = 0 \iff h_\ast \mu(h(A)) \iff \lambda(h(A)) = 0$.

By Lemma 2.13 there exists a Markov map $F$ for $g$, which pushes forward to a Markov map $\overline{F}$ for $T$ by the conjugacy $h$. Now Fact 2.15 gives unique absolutely continuous invariant probability measures $\rho$ and $\overline{\rho}$ for $F, \overline{F}$ with densities Hölder continuous and bounded away from 0 and $\infty$. By unicity $h_\ast \rho = \overline{\rho}$ thus

$$Dh = \frac{d\rho}{d\overline{\rho}}$$

is Hölder continuous and bounded away from 0 and $\infty$ on the domain of $F$. The conjugacy being $C^1$ implies multipliers are preserved so by Lemma 2.16 the conjugacy is upgraded to $C^j$ for all finite $j \leq r$ and thus for $j = r = \infty$ if such is the case. Let $U$ be a connected component of the domain of $F$, and $n > 0$. On $U$, 

$$D(h \circ g^n)(x) = Dh(g^n(x)) \cdot Dg^n(x) = D(T^n \circ h)(x) = t^n \cdot Dh(x),$$ 

so if $Dg^n(x) \neq 0$ then 

$$Dh(g^n(x)) = \frac{t^n}{Dg^n(x)}Dh(x) \quad (3.1)$$

exists and is non-zero. A similar calculation gives existence and continuity of $D^i h(g^n(x))$ where $Dg^n(x) \neq 0$ for all finite $i \leq r$.

We claim that the post-critical set $PC = \bigcup_{i \geq 0} g^i(g(Crit))$ is finite and $h$ is $C^r$ on $X \setminus PC$. Indeed, by Lemma 2.5 there exists $n_0$ such that $g^{n_0}(U) = X$. 

Thus $h$ is $C^r$ (and indeed strictly positive) on $X \setminus \bigcup_{0 \leq i \leq n_0} g^i(g(Crit))$. Because $Dh(g^i(g(Crit))) = \infty$ (upon taking limits) $PC \subset \bigcup_{0 \leq i \leq n_0} g^i(g(Crit))$ as required.

Proof of Theorem 1. One direction is given by the preceding proposition.

According to [2], [3], measures of maximal entropy form a convex set with a finite number of (ergodic) extremal points. A measure of maximal entropy of Hausdorff dimension one therefore has an ergodic component of dimension one.

Thus given the restrictive interval $J$ of period $k$ and $\mu$ such that $\mu(J) > 0$ from the statement, we can assume without loss of generality that $\mu$ is ergodic. Then the support of $\mu$ is $\bigcup_{i=0}^{k-1} f^k(J)$. The measure $\mu|_J$ when normalised is the measure of maximal entropy for $f^k|_J : J \rightarrow J$ and coincides with the $C^r$ pullback of the absolutely continuous measure of maximal entropy for the piecewise linear map. Thus $\mu$ is absolutely continuous and $\text{HD}(\mu) = 1$.

We remark that from equation (3.1) and the non-flatness of critical points it follows immediately that the singularities are of root type: For each $s \in PC$ there exist numbers $k_s^{(+)}$ such that

$$\left| \log |Dh(x)| x - s|^{k_s^{(+)}} \right|$$

is bounded for $x$ in left(right) neighbourhoods of $s$.

We now show that if $f$ (and thus $g$) has no inflection points, then $PC = \partial X$. This is equivalent to showing that $g(Crit \cup \partial X) \subset \partial X$.

Lemma 3.2. For each non-renormalisable transitive $C^1$ map without inflection points $f' : I \rightarrow I$, for each point $y$ in the interior of $I$, there exists an infinite sequence $y = y_0, y_1, \ldots$ satisfying $f'(y_{i+1}) = y_i$, $Df'(y_i) \neq 0$ and $y_i \neq y_j$ for all $i \neq j$.

Proof. Firstly note that there exists $x$ in the interior of $I$ with $f'(x) = y$ and $Df'(x) \neq 0$, for otherwise there is a contradiction with transitivity and non-renormalisability (all critical points are turning points). We can in this way construct a sequence which satisfies the conditions of the lemma unless $y$ is part of a periodic orbit without non-critical (or boundary) preimages (other than itself). The
partition induced by this periodic orbit defines restrictive intervals, contradicting non-renormalisability.

**Lemma 3.3.** Suppose $g$ has no inflection points, then $PC = \partial X$.

**Proof.** By transitivity, $g(Crit) \cup g^2(Crit) \supset \partial X$. It remains to show $PC \subset \partial X$. Suppose there exists $y \in PC \setminus \partial X$. Then $Dh(y) = Dh(y_i) = \infty$ for all $i$, where $y_i$ is given by Lemma 3.2. But from above, the conjugacy is $C^r$ on all of $X$ except the finite post-critical set. The contradiction proves the lemma.

In this case all branches are therefore full and a simple calculation gives that if the slope is $\pm t$, then $T$ has $t - 1$ turning points.

**Proof of Corollary 1.** It follows immediately from Theorem 1.

**Proof of Theorem 2.** A unimodal map has no inflection points. Suppose $\text{HD}(\mu) = 1$. Then $\mu$ is absolutely continuous so the semi-conjugacy $h$ to the symmetric tent map $T$ given by Fact 2.1 collapses no restrictive intervals and $h$ is a conjugacy on $I$. By unicity of measures of maximal entropy for unimodal maps, $\mu$ is the pullback of the absolutely continuous invariant measure for $T$. Arguing as above there is a restrictive interval containing the critical point $c$ on which $h$ is a $C^r$ conjugacy and on which the renormalisation of $T$ is the full tent map $T_2$. Applying Fact 2.4 gives the remainder of the theorem.

**References**


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