Problem Solving Questions: Inequalities

Eoin Murphy

Monday, 1st February, 2010

1. Show that for all positive x, y, z one has

$$S = \left(\frac{x+y}{x+y+z}\right)^{1/2} + \left(\frac{y+z}{x+y+z}\right)^{1/2} + \left(\frac{z+x}{x+y+z}\right)^{1/2} \leq 6^{1/2}$$

2. Show that for all positive x, y, z one has

$$x+y+z \leq 2\left\{\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right\}$$

3. Suppose $\alpha_i \in \mathbb{R}$ are the n roots to the polynomial $P(x)$.
Prove that if $x \geq \max\{\alpha_i\}$ then

$$P(x) \leq \left\{x - \sum \frac{\alpha_i}{n}\right\}^n$$

4. Prove the following geometric propositions using the synthetic method:

a) Let L be any line cutting the plane in two. Let A,B be two points lying in the same half-plane. Find the point $P \in L$ that minimises the value of

$$S = |AP| + |BP|$$

b) With recourse to part a), prove that among all triangles having the identical area, it is the equilateral which minimises the perimeter.

5. Prove the following geometric propositions using analytic inequalities.

a) Prove that among all triangles having the identical area, it is the equilateral which minimises the perimeter.

b) Prove that among all triangles having the identical perimeter, it is the equilateral which minimises the area.

Heron's formula for the area of a triangle may be useful:

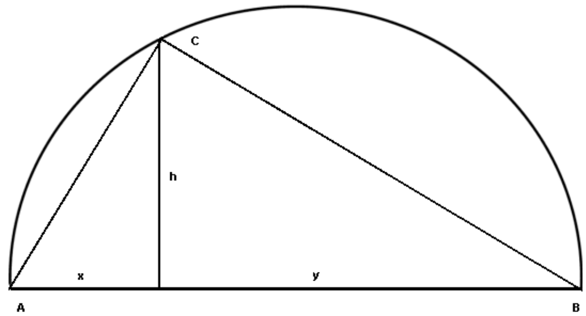
$$\Delta = s(s-a)(s-b)(s-c)$$

where Δ is the area of the triangle, s its semiperimeter.

6. Show that for nonnegative x, y, z that it is invariably the case that

$$1 \leq xyz \Rightarrow 8 \leq (1+x)(1+y)(1+z)$$

7. By consideration of the relationship between h and the lengths x and y, deduce the two dimensional AM-GM inequality.



8. Let $|x| \leq 1$. Show that for any polynomial $P(x)$, with real coefficients a_i we have,

$$(1-x^2)P(x) \leq \sum_{i=1}^{n} a_i^2$$

9. (IMO 2008) If x, y and z are three real numbers, all different from 1, such that $xyz = 1$, then prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

10. (IMO 1978) Let a_i be distinct positive integers. Prove that

$$\frac{a_1}{1^2} + \frac{a_2}{2^2} + ... + \frac{a_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}$$