

LECTURE NOTES
MA3415: INTRODUCTION TO LIE ALGEBRAS

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1. LIE ALGEBRAS AND LIE GROUPS

1.1. Definition and examples. Let A be an associative algebra over a field $\mathbb{k}$. Define the commutator

$$[-, -] : A \times A \rightarrow A, \quad [x, y] = xy - yx.$$

This map is bilinear and satisfies

- (1) $[x, x] = 0$ for all $x \in A$ (anti-commutativity).
- (2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in A$ (Jacobi identity).

1.1. Verify the Jacobi identity.

Sometimes we have an operation on a vector space as above that is not induced by an associative structure. This gives rise to the notion of a Lie algebra.

Definition 1.1. A **Lie algebra** over a field $\mathbb{k}$ is a $\mathbb{k}$ -vector space L with a bilinear map

$$[-, -] : L \times L \rightarrow L$$

called a Lie bracket, such that

- (1) $[x, x] = 0$ for all $x \in L$ (anti-commutativity).
- (2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$ (Jacobi identity).

Remark 1.2. Anti-commutativity implies $[x, y] = -[y, x]$ as

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Conversely, if $[x, y] = -[y, x]$ for all $x, y \in L$, then $[x, x] = -[x, x]$ and therefore $2[x, x] = 0$. This implies $[x, x] = 0$ if $\text{char } \mathbb{k} \neq 2$. $\diamond$

Definition 1.3. Let L be a Lie algebra. A **Lie subalgebra** of L is a subspace $M \subset L$ such that $[x, y] \in M$ for all $x, y \in M$. Note that M equipped with an induced bracket operation is itself a Lie algebra.

Example 1.4. Let A be an associative $\mathbb{k}$ -algebra. Then A , equipped with a commutator operation

$$[x, y] = xy - yx, \quad x, y \in A,$$

is a Lie algebra, denoted by $L(A)$ (note that A and $L(A)$ coincide as vector spaces, but have different binary operations).

In particular, if V is a vector space, then the set of endomorphisms $\text{End}(V)$ is an associative algebra. The corresponding Lie algebra is called the **general linear Lie algebra** and is denoted by $\mathfrak{gl}(V)$. The algebra $\text{End}(\mathbb{k}^n)$ is isomorphic to the algebra $M_n(\mathbb{k})$ of $n \times n$ matrices over $\mathbb{k}$. The corresponding Lie algebra is called again a general linear Lie algebra and is denoted by $\mathfrak{gl}_n(\mathbb{k})$ or $\mathfrak{gl}(n, \mathbb{k})$. $\diamond$

Example 1.5. Define a subspace

$$\mathfrak{sl}_n(\mathbb{k}) = \mathfrak{sl}(n, \mathbb{k}) = \{ A \in \mathfrak{gl}_n(\mathbb{k}) \mid \text{tr } A = 0 \} \subset \mathfrak{gl}_n(\mathbb{k}).$$

For any $A, B \in \mathfrak{gl}_n(\mathbb{k})$, we have

$$\text{tr}[A, B] = \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0$$

and therefore $[A, B] \in \mathfrak{sl}_n(\mathbb{k})$. This implies that $\mathfrak{sl}_n(\mathbb{k}) \subset \mathfrak{gl}_n(\mathbb{k})$ is a Lie subalgebra. It is called the **special linear Lie algebra**.

Consider the algebra

$$\mathfrak{sl}_2(\mathbb{k}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{k}) \mid a + d = 0 \right\}.$$

It is a 3-dimensional algebra with a basis

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The multiplication rules are

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

Note that the matrix product of $x, y \in M_2(\mathbb{k})$ is

$$xy = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin \mathfrak{sl}_2(\mathbb{k}).$$

This means that $\mathfrak{sl}_2(\mathbb{k}) \subset M_2(\mathbb{k})$ is not closed under the matrix multiplication, although it is closed under the commutator. This is an example of a Lie algebra that is not induced by an associative structure. $\diamond$

Example 1.6. Define the algebra of upper-triangular matrices

$$\mathfrak{b}_n(\mathbb{k}) = \{ (a_{ij}) \in \mathfrak{gl}_n(\mathbb{k}) \mid a_{ij} = 0 \text{ for } i > j \},$$

the algebra of strictly upper-triangular matrices

$$\mathfrak{n}_n(\mathbb{k}) = \{ (a_{ij}) \in \mathfrak{gl}_n(\mathbb{k}) \mid a_{ij} = 0 \text{ for } i \geq j \},$$

the algebra of diagonal matrices

$$\mathfrak{t}_n(\mathbb{k}) = \{ (a_{ij}) \in \mathfrak{gl}_n(\mathbb{k}) \mid a_{ij} = 0 \text{ for } i \neq j \}.$$

All of them are Lie subalgebras of $\mathfrak{gl}_n(\mathbb{k})$. $\diamond$

Example 1.7. Let V be a $\mathbb{k}$ -vector space of dimension n and $\sigma : V \times V \rightarrow \mathbb{k}$ be a bilinear form. Define

$$\mathfrak{o}(V, \sigma) = \{ A \in \mathfrak{gl}(V) \mid \sigma(Av, w) = -\sigma(v, Aw) \forall v, w \in V \}.$$

It is a Lie subalgebra of $\mathfrak{gl}(V)$. Indeed, if $A, B \in \mathfrak{o}(V, \sigma)$, then

$$\begin{aligned} \sigma([A, B]v, w) &= \sigma(ABv, w) - \sigma(BAv, w) = -\sigma(Bv, Aw) + \sigma(Av, Bw) \\ &= \sigma(v, BAw) - \sigma(v, ABw) = -\sigma(v, [A, B]w) \end{aligned}$$

and therefore $[A, B] \in \mathfrak{o}(V, \sigma)$. If $V = \mathbb{k}^n$ and σ is given by the matrix S , then

$$\mathfrak{o}(V, \sigma) = \{ A \in \mathfrak{gl}_n(\mathbb{k}) \mid (Av)^t Sw = -v^t SAw \} = \{ A \in \mathfrak{gl}_n(\mathbb{k}) \mid A^t S = -SA \}.$$

Let us consider some examples.

- (1) If σ is symmetric non-degenerate and $\mathbb{k} = \mathbb{C}$ (or any algebraically closed field with $\text{char } \mathbb{k} \neq 2$), then one can choose a basis of V such that σ is given by the identity matrix. The algebra

$$\mathfrak{so}(n, \mathbb{k}) := \mathfrak{o}(V, \sigma) = \{ A \in \mathfrak{gl}_n(\mathbb{k}) \mid A^t = -A \}$$

consists of skew-symmetric matrices and is called the **(special) orthogonal Lie algebra**.

- (2) If σ is symmetric non-degenerate and $\mathbb{k} = \mathbb{R}$ then one can choose a basis of V such that σ is given by the diagonal matrix $S = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ for some $p + q = n$. The corresponding algebra $\mathfrak{o}(V, \sigma)$ is denoted by $\mathfrak{so}(p, q)$.

- (3) Let σ be skew-symmetric non-degenerate. One can show that $n = 2l$ is even and there exists a basis of V such that σ is given by the matrix $S = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$. The algebra

$$\mathfrak{sp}(2l, \mathbb{k}) = \mathfrak{o}(V, \sigma) = \{ A \in \mathfrak{gl}_n(\mathbb{k}) \mid A^t S = -SA \}$$

is called the **symplectic Lie algebra**. $\diamond$

1.2. Derivations.

Definition 1.8. The algebras in this definition are not necessarily associative or Lie algebras.

- (1) A $\mathbb{k}$ -algebra (or an algebra over a field $\mathbb{k}$) is a $\mathbb{k}$ -vector space A equipped with a bilinear map

$$A \times A \rightarrow A, \quad (a, b) \mapsto a \cdot b.$$

- (2) A $\mathbb{k}$ -linear map $f : A \rightarrow B$ between two $\mathbb{k}$ -algebras is called a $\mathbb{k}$ -algebra homomorphism if $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in A$. It is called a $\mathbb{k}$ -algebra isomorphism if f is bijective.
- (3) Let A be a $\mathbb{k}$ -algebra. A linear map $\delta : A \rightarrow A$ is called a **derivation** if it satisfies the **Leibniz rule**

$$\delta(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y) \quad \forall a, b \in A.$$

The set of all derivations of A is denoted by $\text{Der } A$.

Remark 1.9. Note that

- (1) An algebra homomorphism between Lie algebras is also called a Lie (algebra) homomorphism.
- (2) In the case of unital associative algebras a subalgebra $B \subset A$ is additionally required to contain 1_A .

◇

Lemma 1.10. Let A be a $\mathbb{k}$ -algebra. For any $\delta, \delta' \in \text{Der } A$, the commutator

$$[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$$

is again a derivation. In particular, $\text{Der } A$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Proof. Let $\delta, \delta' \in \text{Der } A$. It is left as an exercise to show that, for any $x, y \in A$, we have

$$[\delta, \delta'](xy) = [\delta, \delta'](x) \cdot y + x \cdot [\delta, \delta'](y).$$

This implies that $[\delta, \delta'] \in \text{Der } A$. One can also show that if $\delta, \delta' \in \text{Der } A$ and $c \in \mathbb{k}$ then

$$\delta + \delta' \in \text{Der } A, \quad c\delta \in \text{Der } A.$$

This implies that $\text{Der } A \subset \mathfrak{gl}(A)$ is a subspace closed under the bracket operation. □

Lemma 1.11. Let L be a Lie algebra. Then, for any $x \in L$, the map

$$\text{ad } x = \text{ad}_x : L \rightarrow L, \quad y \mapsto [x, y]$$

is a derivation, called an inner derivation. The map

$$\text{ad} : L \rightarrow \text{Der } L \subset \mathfrak{gl}(L), \quad x \mapsto \text{ad } x$$

is a homomorphism of Lie algebras, called the adjoint representation of L .

Proof. The Leibniz rule for ad_x can be written as

$$\text{ad}_x([y, z]) = [\text{ad}_x(y), z] + [y, \text{ad}_x(z)] \quad \forall y, z \in L$$

or

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

Using anti-commutativity we can rewrite this in the form

$$[x, [y, z]] = -[z, [x, y]] - [y, [z, x]]$$

which is equivalent to the Jacobi identity. This implies that $\text{ad}_x \in \text{Der } L$.

For any $\delta \in \text{Der } L$, we have

$$[\delta, \text{ad}_y](z) = \delta(\text{ad}_y(z)) - \text{ad}_y(\delta(z)) = \delta([y, z]) - [y, \delta(z)] = [\delta(y), z] = \text{ad}_{\delta(y)}(z)$$

and therefore

$$[\delta, \text{ad}_y] = \text{ad}_{\delta(y)}.$$

In particular, for $\delta = \text{ad}_x$, we obtain

$$[\text{ad}_x, \text{ad}_y] = \text{ad}_{[x, y]}$$

and this means that $\text{ad} : L \rightarrow \text{Der } L$ is a Lie algebra homomorphism. □

1.3. Vector fields. This section assumes some familiarity with manifolds. Given a smooth ($= C^\infty$) manifold M of dimension d and a point $p \in M$, one can define the **tangent space** $T_p M$ of M at the point p . This is the set of equivalence classes of (smooth) curves $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ with $\gamma(0) = p$, where

$$\gamma \sim \nu \quad \Leftrightarrow \quad \lim_{t \rightarrow 0} \frac{\gamma(t) - \nu(t)}{t} = 0.$$

It is a d -dimensional vector space with a basis

$$\left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_d} \right|_p,$$

where $(x_1, \dots, x_d)$ is a local coordinate system around p . A curve γ with local coordinates $(\gamma_1, \dots, \gamma_d)$ corresponds to the vector

$$\dot{\gamma}(0) := \sum_{i=1}^d \left. \frac{d\gamma_i}{dt} \right|_{t=0} \left. \frac{\partial}{\partial x_i} \right|_p.$$

Any tangent vector $[\gamma] \in T_p M$ induces a derivation

$$D_\gamma : C^\infty(M) \rightarrow \mathbb{R}, \quad f \mapsto (f \circ \gamma)'(0) = \sum_{i=1}^d \left. \frac{d\gamma_i}{dt} \right|_{t=0} \left. \frac{\partial f}{\partial x_i} \right|_p$$

that satisfies

$$D_\gamma(fg) = D_\gamma(f) \cdot g(p) + f(p) \cdot D_\gamma(g).$$

There is a bijection between $T_p M$ and derivations. We will identify the objects $[\gamma]$, $\dot{\gamma}(0)$ and D_γ .

The union $TM = \bigcup_{p \in M} T_p M$ has a structure of a smooth manifold of dimension $2d$. It is called the **tangent bundle** of M . There is smooth map $\pi : TM \rightarrow M$, sending $T_p M$ to p and called the natural projection. A smooth section

$$X : M \rightarrow TM, \quad p \mapsto X_p \in T_p M$$

of π is called a **vector field** on M . It induces a **Lie derivative**

$$L_X : C^\infty(M) \rightarrow C^\infty(M), \quad (L_X f)(p) = X_p(f), \quad f \in C^\infty(M), \quad p \in M,$$

satisfying the Leibniz rule. Therefore $L_X \in \text{Der } C^\infty(M)$ and one can show that this induces a bijection between the set of all vector fields and the Lie algebra of derivations $\text{Der } C^\infty(M)$. In particular, for any vector fields X, Y the commutator $[L_X, L_Y]$ is again a Lie derivative $L_{[X, Y]}$ for some vector field $[X, Y]$. This bracket equips the space of vector fields with a structure of a Lie algebra. This was historically one of the first examples of a Lie algebra.

Remark 1.12. In local coordinates we can write

$$X = \sum_{i=1}^d X_i \partial_i, \quad Y = \sum_{i=1}^d Y_i \partial_i,$$

where $\partial_i = \frac{\partial}{\partial x_i}$ and $X_i, Y_i \in C^\infty(U)$ for some open neighborhood U of p . Then

$$\begin{aligned} [L_X, L_Y]f &= \sum_{i,j} (X_i \partial_i (Y_j \partial_j (f)) - Y_j \partial_j (X_i \partial_i (f))) = \sum_{i,j} (X_i \partial_i (Y_j) \partial_j (f) - Y_j \partial_j (X_i) \partial_i (f)) \\ &= \sum_{i=1}^d \left(\sum_{j=1}^d (X_j \partial_j (Y_i) - Y_j \partial_j (X_i)) \right) \partial_i (f), \\ [X, Y] &= \sum_{i=1}^d \left(\sum_{j=1}^d (X_j \partial_j (Y_i) - Y_j \partial_j (X_i)) \right) \partial_i. \end{aligned}$$

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1.4. Lie groups. This section will be rather informal and should serve just as a motivation.

Definition 1.13. A **Lie group** G is a smooth manifold endowed with a group structure such that the map

$$G \times G \rightarrow G, \quad (x, y) \mapsto xy^{-1}$$

is smooth.

Theorem 1.14. Let G be a Lie group, $e \in G$ be the unit and $L(G) = T_e G$ be the tangent space. For any $x \in L(G)$ there exists a unique smooth group homomorphism $\gamma_x : \mathbb{R} \rightarrow G$ (called a 1-parameter subgroup) such that $\dot{\gamma}_x(0) = x$. This correspondence induces a bijection between $L(G)$ and 1-parameter subgroups of G .

Example 1.15. Let $G = \mathrm{GL}_n(\mathbb{R})$, called the **general linear group**. Then $L(G) = T_e G = \mathfrak{gl}_n(\mathbb{R})$. For any $A \in \mathfrak{gl}_n(\mathbb{R})$ the map

$$\gamma_A : \mathbb{R} \rightarrow \mathrm{GL}_n(\mathbb{R}), \quad t \mapsto \exp(tA) = \sum_{k \geq 0} \frac{t^k A^k}{k!}$$

is a group homomorphism and satisfies

$$\dot{\gamma}_A(0) = \left. \frac{d}{dt} \exp(tA) \right|_{t=0} = A.$$

Note that $\gamma_A(1) = \exp(A)$ and

$$\left. \frac{d}{dt} \exp(tA) B \exp(-tA) \right|_{t=0} = \left. \frac{d}{dt} (1 + tA) B (1 - tA) \right|_{t=0} = AB - BA = [A, B].$$

◇

Generally, motivated by this example, for any Lie group G , we define the map

$$\exp : L(G) \rightarrow G, \quad \exp(x) = \gamma_x(1)$$

and define

$$[x, y] = \left. \frac{d}{dt} \exp(tx) y \exp(-tx) \right|_{t=0} \quad \forall x, y \in L(G).$$

This bracket equips $L(G)$ with a Lie algebra structure. The map $\exp : L(G) \rightarrow G$ is a diffeomorphism in a neighborhood of $0 \in L(G)$.

Given a smooth map $f : M \rightarrow N$ between manifolds and $p \in M$, there is a linear map $df : T_p M \rightarrow T_{f(p)} N$ between tangent spaces. In particular, given a smooth group homomorphism (we call it a Lie group homomorphism) $\varphi : G \rightarrow H$, there is a linear map

$$d\varphi : L(G) = T_e G \rightarrow T_e H = L(H).$$

Theorem 1.16. Given a Lie group homomorphism $\varphi : G \rightarrow H$, the map $d\varphi : L(G) \rightarrow L(H)$ is a Lie algebra homomorphism and the diagram

$$\begin{array}{ccc} L(G) & \xrightarrow{d\varphi} & L(H) \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\varphi} & H \end{array}$$

commutes.

Theorem 1.17. Let G be a Lie group. There is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $L \subset L(G)$. It is given by sending H to its Lie algebra

$$L = \{x \in L(G) \mid \exp(tx) \in H \ \forall t \in \mathbb{R}\}.$$

Example 1.18. Consider the group

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det A = 1\},$$

called the **special linear group**. Its Lie algebra consists of matrices $A \in \mathfrak{gl}_n(\mathbb{R})$ such that

$$\exp(tA) \in \mathrm{SL}_n(\mathbb{R}), \quad \forall t \in \mathbb{R}.$$

If A has eigenvalues (λ_i) , then e^{tA} has eigenvalues $(e^{t\lambda_i})$ and

$$\det e^{tA} = \prod_i e^{t\lambda_i} = e^{t \sum \lambda_i} = e^{t \operatorname{tr} A}.$$

Therefore if $e^{tA} \in \mathrm{SL}_n(\mathbb{R})$, then $e^{t \operatorname{tr} A} = \det e^{tA} = 1$ and $\operatorname{tr} A = 0$. We conclude that the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$ is

$$\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid \operatorname{tr} A = 0\},$$

the special linear Lie algebra defined in Example 1.5. $\diamond$

Example 1.19. Let V be an $\mathbb{R}$ -vector space of dimension n and σ be a bilinear form on V . Define a Lie group

$$\mathrm{O}(V, \sigma) = \{A \in \mathrm{GL}(V) \mid \sigma(Av, Aw) = \sigma(v, w) \forall v, w \in V\} \subset \mathrm{GL}(V).$$

If σ is given by the matrix S then

$$(1) \quad \mathrm{O}(V, \sigma) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A^t S A = S\}.$$

The corresponding Lie algebra $L \subset \mathfrak{gl}(V)$ consists of $A \in \mathfrak{gl}(V)$ such that $\exp(tA) \in \mathrm{O}(V, \sigma)$, that is,

$$\sigma(e^{tA}v, e^{tA}w) = \sigma(v, w) \quad \forall t \in \mathbb{R}.$$

Taking the derivative at $t = 0$ we obtain

$$\frac{d}{dt} \sigma((1 + tA)v, (1 + tA)w)|_{t=0} = \sigma(Av, w) + \sigma(v, Aw) = 0.$$

Therefore the Lie algebra of $\mathrm{O}(V, \sigma)$ is

$$L = \{A \in \mathfrak{gl}(V) \mid \sigma(Av, w) = -\sigma(v, Aw) \forall v, w \in V\} = \mathfrak{o}(V, \sigma)$$

defined in Example 1.7. In matrix form

$$(2) \quad L = \mathfrak{o}(V, \sigma) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A^t S + S A = 0\}.$$

Let us consider some examples.

- (1) If σ is given by the identity matrix, we obtain the **orthogonal group**

$$\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^t A = 1\}$$

with the Lie algebra

$$\mathfrak{so}(n) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A^t + A = 0\}$$

defined in Example 1.7. The group $\mathrm{O}(n)$ has a connected component

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det A = 1\}.$$

with the same Lie algebra $\mathfrak{so}(n)$.

- (2) If σ is given by the matrix $S = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ with $n = p + q$, the corresponding Lie group (1) is denoted by $\mathrm{O}(p, q)$. Its Lie algebra is $\mathfrak{so}(p, q)$ defined in Example 1.7.
- (3) If σ is given by the skew-symmetric matrix $S = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ with $n = 2l$, we obtain the symplectic group

$$\mathrm{Sp}(n, \mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^t S A = S\}$$

with the Lie algebra

$$\mathfrak{sp}(n, \mathbb{R}) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \mid A^t S + S A = 0\}.$$

defined in Example 1.7. $\diamond$

Example 1.20. The (real) Lie group

$$U(n) = \{A \in GL_n(\mathbb{C}) \mid AA^* = 1\}, \quad A^* = \overline{A}^t,$$

is called the **unitary group**. Its Lie algebra

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid A + A^* = 0\}$$

is a Lie algebra over $\mathbb{R}$ (but not over $\mathbb{C}$), called the **unitary Lie algebra**. For example, if $n = 1$, then

$$U(1) = \{z \in \mathbb{C}^* \mid |z| = 1\}, \quad \mathfrak{u}(1) = \{z \in \mathbb{C} \mid z + \bar{z} = 0\} = i\mathbb{R} \simeq \mathbb{R}.$$

The exponential map $\exp : \mathfrak{u}(1) \rightarrow U(1)$ is given by

$$\mathbb{R} \rightarrow U(1), \quad \varphi \mapsto e^{i\varphi}.$$

◇

Example 1.21. The (real) Lie group

$$SU(n) = \{A \in GL_n(\mathbb{C}) \mid AA^* = 1, \det A = 1\}$$

is called the **special unitary group**. Its Lie algebra

$$\mathfrak{su}(n) = \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid A + A^* = 0, \operatorname{tr} A = 0\}$$

is a Lie algebra over $\mathbb{R}$ (but not over $\mathbb{C}$), called the **special unitary Lie algebra**. ◇

Remark 1.22. Let G be a Lie group and $L = L(G)$ be its Lie algebra. For any $g \in G$, the internal automorphism

$$\operatorname{Int}_g : G \rightarrow G, \quad h \mapsto ghg^{-1}$$

induces a Lie algebra automorphism

$$\operatorname{Ad} g = \operatorname{Ad}_g : L \rightarrow L.$$

For example, for $G = GL_n(\mathbb{R})$, we have $\operatorname{Ad}_g(A) = gAg^{-1}$ for all $g \in GL_n(\mathbb{R})$ and $A \in \mathfrak{gl}_n(\mathbb{R})$.

The group homomorphism

$$\operatorname{Ad} : G \rightarrow GL(L), \quad g \mapsto \operatorname{Ad}_g$$

is called the adjoint representation of G . Its differential $d\operatorname{Ad} : L \rightarrow \mathfrak{gl}(L)$ coincides with the adjoint representation $\operatorname{ad} : L \rightarrow \mathfrak{gl}(L)$ of L introduced in Lemma 1.11. Using Theorem 1.16, we obtain a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\operatorname{ad}} & \mathfrak{gl}(L) \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\operatorname{Ad}} & GL(L) \end{array}$$

◇

Not every Lie algebra homomorphism $L(G) \rightarrow L(H)$ can be lifted to a Lie group homomorphism $G \rightarrow H$. However, the following is true

Theorem 1.23. *Let G be a connected and simply connected Lie group. Then the map $\varphi \mapsto d\varphi$ between Lie group homomorphisms $G \rightarrow H$ and Lie algebra homomorphisms $L(G) \rightarrow L(H)$ is a bijection.*

Theorem 1.24. *There is a bijection between isomorphism classes of connected simply connected Lie groups and (finite-dimensional) Lie algebras.*

2. NILPOTENT AND SOLVABLE ALGEBRAS

In this section we will assume that $\text{char } \mathbb{k} = 0$ and that all our Lie algebras and vector spaces are finite-dimensional.

2.1. Ideals. Given a Lie algebra L and subspaces $A, B \subset L$, define $[A, B] \subset L$ to be a subspace consisting of all linear combinations of commutators $[x, y]$ for $x \in A, y \in B$.

Definition 2.1. Let L be a Lie algebra.

- (1) A subspace $I \subset L$ is called an **ideal** of L if for any $x \in L$ and $y \in I$, we have $[x, y] \in I$. Equivalently, $[L, I] \subset I$.
- (2) The **center** of L is $Z(L) = \{x \in L \mid [x, y] = 0 \ \forall y \in L\} = \{x \in L \mid [x, L] = 0\}$.
- (3) The **derived algebra** of L is $[L, L]$.
- (4) L is called **abelian** (or commutative) if $[x, y] = 0$ for all $x, y \in L$. Equivalently $[L, L] = 0$.

Remark 2.2. Note that

- (1) Every ideal $I \subset L$ is also a Lie subalgebra.
- (2) $[L, I] \subset I$ implies also $[I, L] \subset I$.
- (3) The center $Z(L)$ is an ideal of L .
- (4) The derived algebra $[L, L]$ is an ideal of L .

◇

Example 2.3. If $f : L \rightarrow L'$ is a Lie algebra homomorphism, then $\ker f$ is an ideal of f . Indeed, given $x \in L$ and $y \in \ker f$, we have

$$f([x, y]) = [f(x), f(y)] = [f(x), 0] = 0$$

and therefore $[x, y] \in \ker f$.

◇

Example 2.4. If L is abelian then any subspace $I \subset L$ is an ideal of L .

◇

Example 2.5. Consider the algebra $L = \mathfrak{sl}_2(\mathbb{k})$. We know that it has a basis x, y, h with multiplication

$$[h, x] = 2x, \quad [h, y] = 2y, \quad [x, y] = h.$$

This implies that $L' = [L, L] = L$. Let us show that the only ideals of L are zero and L . Let $I \subset L$ be a non-zero ideal and $ax + by + ch \in I$ be a non-zero element. If $a \neq 0$, we apply $\text{ad } y$ twice and obtain $a[y, [y, x]] = 2ay \in I$. If $a = 0$ and $b \neq 0$, we apply $\text{ad } h$ and obtain $-2by \in I$. If $a = b = 0$ and $c \neq 0$, we apply $\text{ad } y$ and obtain $2y \in I$. Finally, $y \in I$ generates the whole L (as an ideal).

◇

Example 2.6. Let $L = \mathfrak{gl}_n(\mathbb{k})$. Every element of the form λI_n (for $\lambda \in \mathbb{k}$) is central as $[\lambda I_n, x] = \lambda[I_n, x] = 0$. One can show that $Z(L) = \mathbb{k}I_n$.

◇

Given an ideal $I \subset L$, the quotient space L/I can be equipped with a Lie algebra structure. The Lie bracket on L/I is defined by

$$[x + I, y + I] = [x, y] + I, \quad x, y \in L.$$

This Lie algebra is called a **quotient algebra**. The map

$$\pi : L \rightarrow L/I, \quad x \mapsto x + I$$

is a Lie algebra homomorphism.

Remark 2.7. The above multiplication is well-defined. If $x + I = x' + I$ and $y + I = y' + I$ then $a = x' - x \in I$, $b = y' - y \in I$ and

$$[x', y'] + I = [x + a, y + b] + I = [x, y] + ([x, b] + [a, y] + [a, b]) + I = [x, y] + I.$$

◇

Example 2.8. Let $L = \mathfrak{b}_n(\mathbb{k}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{k} \right\}$ be the Lie algebra of upper-triangular matrices and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{k} \right\}$. Then I is an ideal of L and L/I is isomorphic to the abelian algebra $\mathbb{k}^2$ (of diagonal matrices).

◇

2.2. Representations.

Definition 2.9. Let L be a Lie algebra.

- (1) An L -**representation** (or representation of L) is a Lie algebra homomorphism $\rho : L \rightarrow \mathfrak{gl}(V)$, where V is a vector space called the **space of the representation**.
- (2) An L -**module** (or module over L) is a vector space V with a bilinear map

$$L \times V \rightarrow V, \quad (x, v) \mapsto xv$$

such that

$$[x, y]v = xyv - yxv, \quad \forall x, y \in L, v \in V.$$

Remark 2.10. There is a 1 – 1 correspondence between L -representations and L -modules. Given a representation $\rho : L \rightarrow \mathfrak{gl}(V)$, we equip V with an L -module structure by

$$(x, v) \mapsto xv := \rho(x)v, \quad x \in L, v \in V.$$

This map satisfies the axiom of an L -module

$$[x, y]v = \rho([x, y])v = [\rho(x), \rho(y)]v = (\rho(x)\rho(y) - \rho(y)\rho(x))v = xyv - yxv.$$

We will usually identify L -representations and L -modules and we will use notation $\rho(x)v$ and xv interchangeably. We will say that L acts on V . $\diamond$

Example 2.11.

- (1) Any vector space V (in particular $\mathbb{k}$) can be equipped with a **trivial action** of L by setting $xv = 0$ for all $x \in L$ and $v \in V$.
- (2) If V is a vector space then $\mathfrak{gl}(V)$ acts on V in an obvious way. This representation is called the **standard representation** of $\mathfrak{gl}(V)$.
- (3) We have seen that the map

$$\text{ad} : L \rightarrow \mathfrak{gl}(L), \quad x \mapsto \text{ad } x$$

is a Lie algebra homomorphism. We call it the adjoint representation of L . Note that

$$\ker(\text{ad}) = \{x \in L \mid \text{ad } x = 0\} = \{x \in L \mid [x, L] = 0\} = Z(L)$$

is the center of L . $\diamond$

Example 2.12. Given $\mathbb{k}$ -vector spaces V, W , we define their tensor product $V \otimes W = V \otimes_{\mathbb{k}} W$ as follows. It is a vector space generated by elements of the form $v \otimes w$ for $v \in V, w \in W$ subject to the relations

- (1) $(v + v') \otimes w = v \otimes w + v' \otimes w$ for $v, v' \in V$ and $w \in W$.
- (2) $v \otimes (w + w') = v \otimes w + v \otimes w'$ for $v \in V$ and $w, w' \in W$.
- (3) $a(v \otimes w) = (av) \otimes w = v \otimes (aw)$ for $a \in \mathbb{k}, v \in V$ and $w \in W$.

If V has a basis $(e_1, \dots, e_m)$ and W has a basis $(f_1, \dots, f_n)$ then $V \otimes W$ has a basis

$$(e_i \otimes f_j \mid 1 \leq i \leq m, 1 \leq j \leq n).$$

If V, W are L -representations, then $V \otimes W$ can be equipped with an L -representation structure by the rule

$$x(v \otimes w) = (xv) \otimes w + v \otimes (xw), \quad x \in L, v \in V, w \in W. \quad \diamond$$

Definition 2.13. Let L be a Lie algebra.

- (1) A linear map $f : V \rightarrow W$ between L -representations is called a **homomorphism** of representations if

$$f(xv) = x(f(v)) \quad \forall x \in L, v \in V.$$

It is called an isomorphism of representations if it is an isomorphism of vector spaces.

- (2) Given an L -representation V , a subspace $W \subset V$ is called a **subrepresentation** of V if $xv \in W$ for all $x \in L, v \in W$.

- (3) Given an L -representation V , an element $v \in V$ is called **L -invariant** if $xv = 0$ for all $x \in L$. The vector space of all L -invariant elements in V is denoted by V^L .

Remark 2.14. To motivate the last definition, consider a matrix $A \in \mathfrak{gl}_n(\mathbb{R})$ and a vector $v \in \mathbb{R}^n$ that is invariant under the action of $\exp(tA)$ for all $t \in \mathbb{R}$. Then $\exp(tA)v = v$ and taking the derivative at $t = 0$, we obtain

$$\frac{d}{dt} \exp(tA)v|_{t=0} = A \exp(tA)v|_{t=0} = Av = 0.$$

Therefore on the Lie algebra level we require $Av = 0$. $\diamond$

Example 2.15. If V is an L -representation and $W \subset V$ is a subrepresentation, then the quotient space V/W can be equipped with a structure of an L -representation

$$x(v + W) = xv + W, \quad x \in L, v \in V.$$

$\diamond$

Example 2.16. If V, W are L -representations, then the vector space $\text{Hom}(V, W) = \text{Hom}_{\mathbb{k}}(V, W)$ of all linear maps $V \rightarrow W$ can be equipped with an L -representation structure by the rule

$$(xf)(v) = x(f(v)) - f(xv), \quad x \in L, f \in \text{Hom}(V, W), v \in V.$$

One can verify that this action indeed satisfies the axiom of an L -representation. On the other hand the axiom would not be satisfied without the minus sign above.

In particular, the dual space $V^* = \text{Hom}(V, \mathbb{k})$ (where $\mathbb{k}$ is considered as a trivial representation) is equipped with an L -representation structure

$$(xf)(v) = -f(xv), \quad x \in L, f \in V^*, v \in V.$$

Note that an element $f \in \text{Hom}(V, W)$ is L -invariant (that is, $xf = 0$ for all $x \in L$) if and only if

$$f(xv) = x(f(v)) \quad \forall x \in L, v \in V.$$

This means that f is a homomorphism of representations. $\diamond$

2.3. Nilpotent Lie algebras.

Definition 2.17. Let L be a finite-dimensional Lie algebra.

- (1) Define the **central series** of ideals in L by

$$L^0 = L, \quad L^n = [L, L^{n-1}], \quad n \geq 1.$$

- (2) L is called **nilpotent** if $L^n = 0$ for some $n \geq 1$.

Remark 2.18. For $n \geq 1$, the following conditions are equivalent

- (1) $L^n = 0$.
- (2) $[x_1, [x_2, \dots, [x_n, x_{n+1}]]] = 0 \quad \forall x_1, \dots, x_{n+1} \in L$.
- (3) $\text{ad}_{x_1} \circ \dots \circ \text{ad}_{x_n} = 0 \quad \forall x_1, \dots, x_n \in L$.

In particular, if L is nilpotent then ad_x is nilpotent for all $x \in L$. ◇

Example 2.19.

- (1) Any abelian Lie algebra is nilpotent.
- (2) Let

$$L = \{ (a_{ij}) \in \mathfrak{gl}_n(\mathbb{k}) \mid a_{ij} = 0 \text{ for } i \geq j \} \subset \mathfrak{gl}_n(\mathbb{k})$$

be the Lie algebra of strictly upper-triangular matrices. Let us show that L is nilpotent. $L^0 = L$ is spanned by matrices E_{ij} with $j - i \geq 1$ (where E_{ij} has 1 on the intersection of i -th row and j -th column and zero otherwise). $L^1 = [L, L]$ is generated by

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il},$$

where $i < j, k < l$ and we assumed that $j \leq l$. Note that $l - i \geq 2$. Continuing this process we obtain that L^k is spanned by E_{ij} with $j - i \geq k + 1$. This implies that $L^{n-1} = 0$ and L is nilpotent. ◇

Theorem 2.20 (Engel). *Let L be a Lie algebra and (V, ρ) be a non-zero finite-dimensional representation of L such that $\rho(x) \in \text{End}(V)$ is nilpotent for all $x \in L$. Then*

- (1) *There exists $0 \neq v \in V$ such that $\rho(x)v = 0$ for each $x \in L$ (an L -invariant element).*
- (2) *Then there exists a basis of V such that all $\rho(x)$ are strictly upper-triangular with respect to this basis.*
- (3) *There exists a filtration by subrepresentations*

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = V$$

such that $LV_k \subset V_{k-1}$ (equivalently, L acts trivially on V_k/V_{k-1}) for $1 \leq k \leq n$.

Proof. We postpone the proof of (1) and show first that it implies (2) and (3).

$1 \Rightarrow 2$. We choose $v_1 = v$ from (1). Then $Lv_1 = 0$ and therefore $\mathbb{k}v_1 \subset V$ is a subrepresentation. Applying induction by dimension of V to the representation $V' = V/\mathbb{k}v_1$, we obtain a basis $v'_2, \dots, v'_n$ of V' such that $\rho_{V'}(x)$ are strictly upper-triangular with respect to this basis. We can lift these basis vectors to vectors $v_2, \dots, v_n$ in V . Then $v_1, v_2, \dots, v_n$ form a basis of V such that all $\rho(x)$ are strictly-upper triangular with respect to this basis.

$2 \Rightarrow 3$. We define $V_k = \langle v_1, \dots, v_k \rangle$ for $1 \leq k \leq n$. As the matrix of $\rho(x)$ is strictly upper-triangular, it maps v_k to a linear combination of $v_1, \dots, v_{k-1}$. This implies that $xv_k \in V_{k-1}$ and generally $xV_k \subset V_{k-1}$. □

Remark 2.21. Given a nilpotent operator $A \in \text{End } V$, one can find a basis of V making A a strictly upper-triangular matrix (consider the Jordan normal form of A). However, it is not true that given nilpotent operators $A, B \in \text{End } V$, one can find a basis of V making A and B simultaneously strictly upper-triangular. We can not apply the previous theorem as it may happen that the Lie algebra generated by A, B contains non-nilpotent elements. For example consider the standard representation $V = \mathbb{k}^2$ of $\mathfrak{sl}_2(\mathbb{k})$. The elements $x, y \in \mathfrak{sl}_2(\mathbb{k})$ induce nilpotent operators on V . But one can not make them simultaneously strictly upper-triangular. ◇

Theorem 2.22 (Engel). *A Lie algebra L is nilpotent if and only if $\text{ad } x \in \mathfrak{gl}(L)$ is nilpotent for all $x \in L$.*

Proof. If $L^n = 0$ then $(\text{ad } x)^n = 0$ for all $x \in L$.

Conversely, assume that $\text{ad } x$ is nilpotent for every $x \in L$. Applying the above result to the adjoint representation $\text{ad} : L \rightarrow \mathfrak{gl}(L)$, we can find a filtration by subspaces

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = L$$

such that $(\text{ad } x)V_k \subset V_{k-1}$ for all $x \in L$ and $k \geq 1$. Then $[L, V_k] \subset V_{k-1}$ and therefore $[L, [L, \dots]] = 0$ (n products). $\square$

Proof of Theorem 2.20 (1). We will prove the theorem by induction on $\dim L$. Taking $\rho(L) \subset \text{End}(V)$ instead of L , we can assume that $L \subset \text{End}(V)$.

We will need later the fact that if $L \subset \text{End}(V)$ consists of nilpotent operators then L acts nilpotently on itself (via the adjoint representation). To see this, note that

$$\text{ad}_x(y) = xy - yx = L_x(y) - R_x(y), \quad L_x(y) = xy, \quad R_x(y) = yx.$$

As x acts nilpotently on V , there exists $n \geq 1$ such that $x^n = 0$. Therefore $L_x^n = R_x^n = 0$. As operators L_x, R_x commute with each other, we conclude

$$(3) \quad \text{ad}_x^{2n} = (L_x - R_x)^{2n} = 0.$$

If $\dim L = 1$ and $0 \neq x \in L$ then x is nilpotent and kills some $0 \neq v \in V$. Therefore $Lv = 0$. Now assume that $\dim L \geq 2$ and let $0 \neq I \subset L$ be a maximal proper subalgebra (it is non-trivial as L contains 1-dimensional subalgebras).

(1) Let us show that I is an ideal. Let

$$N = \{x \in L \mid [x, I] \subset I\}$$

be the normalizer of I (it is a Lie subalgebra of L). We have $I \subset N \subset L$ and we claim that $N = L$. The Lie algebra I acts on L/I nilpotently (by 3) and we conclude by induction (as $\dim I < \dim L$) that there exists a nonzero element

$$x + I \in L/I, \quad x \notin I$$

such that $[I, x + I] \subset I$. This implies that $[x, I] \subset I$ and $x \in N$. Therefore $I \subsetneq N$ and $N = L$ (by maximality of I). This implies that $[L, I] \subset I$ and $I \subset L$ is an ideal.

(2) The vector space

$$W = V^I = \{v \in V \mid Iv = 0\}$$

is nonzero by induction (as $\dim I < \dim L$) and is stable under L :

$$mxv = xmv - [x, m]v = 0, \quad m \in I, x \in L, v \in W$$

and therefore $I(xv) = 0$ and $xv \in W$. This means that W is a representation of L such that $IW = 0$.

(3) By the previous statements the Lie algebra L/I acts on W . This implies that L/I acts on W and by induction (as $\dim L/I < \dim L$) there exists $0 \neq v \in W$ such that $(L/I)v = 0$. Then also $Lv = 0$. $\square$

In the course of the proof of Engel's theorem we proved

Lemma 2.23. *If $x \in \mathfrak{gl}(V)$ is a nilpotent operator on V , then ad_x is a nilpotent operator on $\mathfrak{gl}(V)$.*

2.4. Solvable Lie algebras.

Definition 2.24. Let L be a finite-dimensional Lie algebra.

- (1) Define the **derived series** of ideals in L by

$$L^{(0)} = L, \quad L^{(n)} = [L^{(n-1)}, L^{(n-1)}], \quad n \geq 1.$$

- (2) L is called **solvable** if $L^{(n)} = 0$ for some $n \geq 1$.

Remark 2.25. If $I, J \subset L$ are ideals then, using the Jacobi identity, we obtain

$$[x, [a, b]] = [[x, a], b] + [a, [x, b]] \in [I, J] + [I, J] = [I, J] \quad \forall x \in L, a \in I, b \in J.$$

This implies that $[I, J] \subset L$ is also an ideal. In particular, the above spaces $L^{(n)}$ are ideals. $\diamond$

Example 2.26. Any abelian or nilpotent Lie algebra is solvable. To see this we note that $L^{(0)} = L = L^0$ and generally $L^{(i)} \subset L^i$ by induction

$$L^{(i+1)} = [L^{(i)}, L^{(i)}] \subset [L, L^i] = L^{i+1}.$$

Therefore if L is nilpotent then $L^{(n)} \subset L^n = 0$ for some n . Thus L is solvable. $\diamond$

Example 2.27. The algebra of upper-triangular matrices

$$L = \{(a_{ij}) \in \mathfrak{gl}_n(\mathbf{k}) \mid a_{ij} = 0 \text{ for } i \leq j\}$$

is solvable. Indeed, $L' = [L, L]$ consists of strictly upper-triangular matrices and we have seen that this algebra is nilpotent and therefore solvable. Therefore $L^{(n)} = (L')^{(n-1)} = 0$ and L is also solvable. $\diamond$

Lemma 2.28. Let L be a Lie algebra.

- (1) If $I \subset L$ is an ideal and L is solvable then $I, L/I$ are solvable.
- (2) If $I \subset L$ is an ideal and $I, L/I$ are solvable then L is solvable.
- (3) If $I, J \subset L$ are solvable ideals then $I + J$ is solvable.

Proof. (1) Let $L^{(n)} = 0$. Then $I^{(n)} \subset L^{(n)} = 0$ and I is solvable. Let $\pi : L \rightarrow L/I$ be the projection. Then we prove by induction that $\pi(L^{(k)}) = (L/I)^{(k)}$

$$\pi(L^{(k+1)}) = \pi([L^{(k)}, L^{(k)}]) = [\pi(L^{(k)}), \pi(L^{(k)})] = [(L/I)^{(k)}, (L/I)^{(k)}] = (L/I)^{(k+1)}.$$

In particular $(L/I)^{(n)} = \pi(L^{(n)}) = 0$ and L/I is solvable.

- (2) Assume that $I^{(m)} = 0$ and $(L/I)^{(n)} = 0$. Then $\pi(L^{(n)}) = (L/I)^{(n)} = 0$ in L/I and therefore $(L)^{(n)} \subset I$. This implies that $(L)^{(m+n)} \subset I^{(m)} = 0$ and L is solvable.
- (3) There is a natural isomorphism $(I + J)/I \simeq J/(I \cap J)$. If J is solvable then also the quotient $J/(I \cap J)$ is solvable by (1). Therefore $(I + J)/I$ is solvable and if I is solvable then $I + J$ is solvable by (2). $\square$

Remark 2.29. The quotient algebra $L/[L, L]$ is abelian. This implies that if L is solvable then there is a filtration by ideals

$$L = L^{(0)} \supset L^{(1)} \supset \dots \supset L^{(n)} = 0$$

such that each factor $L^{(k)}/L^{(k+1)}$ is abelian. Conversely, if there exists a filtration by ideals

$$L = I_0 \supset I_1 \supset \dots \supset I_n = 0$$

such that each factor I_k/I_{k+1} is abelian, then L is solvable. Indeed, assume that we proved that I_k is solvable. As I_{k-1}/I_k is abelian, it is solvable. By the previous result we conclude that I_{k-1} is solvable. Continuing this process we obtain that $I_0 = L$ is solvable. $\diamond$

Theorem 2.30 (Lie). *Let L be a solvable Lie algebra over an algebraically closed field $\mathbb{k}$ of characteristic 0. Let (V, ρ) be a non-zero representation of L . Then*

- (1) *There exists a common eigenvector $0 \neq v \in V$ of L , that is, an eigenvector of $\rho(x)$ for all $x \in L$. Equivalently, $\mathbb{k}v \subset V$ is a subrepresentation.*
- (2) *There exists a basis of V such that $\rho(x)$ is upper-triangular in this basis for all $x \in L$.*
- (3) *There exists a chain of subrepresentations*

$$0 = V_0 \subset V_1 \subset \dots \subset V_n = V$$

such that $\dim V_k/V_{k-1} = 1$ for $1 \leq k \leq n$.

Proof. $1 \Rightarrow 2$. Let $v_1 = v \in V$ from (1). Then $\mathbb{k}v_1 \subset V$ is a subrepresentation. Applying induction to the quotient representation $V' = V/\mathbb{k}v_1$ we find a basis $(v'_2, \dots, v'_n)$ of V' such that $\rho_{V'}(x)$ are upper-triangular in this basis. We lift this basis to some vectors $v_2, \dots, v_n$ in V . Then $v_1, \dots, v_n$ form a basis of V . And $\rho(x)$ are upper-triangular in this basis.

$1 \Rightarrow 3$. Choose a basis as above and define $V_k = \langle v_1, \dots, v_k \rangle$. These vector spaces satisfy the required conditions.

Now we will prove (1). As L is solvable, $[L, L] \neq L$. Indeed, if $L^{(1)} = [L, L] = L$, then $L^{(2)} = [L^{(1)}, L^{(1)}] = [L, L] = L$ and generally $L^{(k)} = L$, contradicting to solvability of L . The quotient $L/[L, L]$ is a nonzero abelian Lie algebra. Therefore we can choose a codimension 1 subspace $[L, L] \subset I \subset L$ which is automatically an ideal. By induction there exists $0 \neq v \in V$ and $\lambda : I \rightarrow \mathbb{k}$ such that $yv = \lambda(y)v$ for all $y \in I$ (λ is automatically linear). The space

$$W = \{v \in V \mid yv = \lambda(y)v \ \forall y \in I\}$$

is nonzero and is an L -subrepresentation by the next lemma. Given $x \in L \setminus I$, the map $x : W \rightarrow W$ has an eigenvector $0 \neq v \in W$ (as $\mathbb{k}$ is algebraically closed). Then v is a common eigenvector of $\mathbb{k}x + I = L$. $\square$

Lemma 2.31. *Let L be a Lie algebra, $I \subset L$ be an ideal, V be an L -representation. For any linear map $\lambda : I \rightarrow \mathbb{k}$, the space*

$$W = \{v \in V \mid xv = \lambda(x)v \ \forall x \in I\}$$

is an L -subrepresentation of V .

Proof. We can assume that $W \neq 0$. Let us show first that $[L, I]W = 0$. For any $x \in L$, we have to show that $\lambda([x, I]) = 0$. Let $0 \neq v \in W$ and let $v_k = x^{k-1}v$ for $k \geq 1$. Let us show by induction that, for any $y \in I$,

$$(4) \quad yv_k = \lambda(y)v_k + \sum_{i=1}^{k-1} a_i v_i, \quad a_i = a_i(y) \in \mathbb{k}.$$

For $k = 1$, we have $yv = \lambda(y)v$. Further, we have

$$yv_{k+1} = yx^k v = (xy - [x, y])x^{k-1}v = x \left(\lambda(y)v_k + \sum_{i=1}^{k-1} a_i v_i \right) - \sum_{i=1}^k a'_i v_i = \lambda(y)v_{k+1} + \sum_{i=1}^k b_i v_i.$$

Let $n \geq 1$ be maximal such that $v_1, \dots, v_n$ are linearly independent and let $U \subset V$ be the vector space generated by these vectors. Then $xv_k = v_{k+1} \in U$ for $1 \leq k \leq n$. Therefore $xU \subset U$. Equation (4) implies that every $y \in I$ induces an endomorphism of U given by an upper-triangular matrix with entries $\lambda(y)$ on the diagonal (with respect to the basis $v_1, \dots, v_n$). This implies

$$\text{Tr}_U(y) = n\lambda(y) \quad \forall y \in I$$

and in particular

$$0 = \text{Tr}_U([x, y]) = n\lambda([x, y]).$$

Therefore $\lambda([x, y]) = 0$. This proves $[L, I]W = 0$.

Finally, for all $x \in L$ and $v \in W$

$$yxv = xyv - [x, y]v = xyv = \lambda(y)xv, \quad \forall y \in I,$$

where $[x, y]v = 0$ was proved above. This implies that $xv \in W$. Hence W is an L -subrepresentation. $\square$

Corollary 2.32. *If L is a solvable Lie algebra over a field of characteristic zero then $[L, L]$ is nilpotent.*

Proof. Consider the adjoint representation $\text{ad} : L \rightarrow \mathfrak{gl}(L)$. By Lie theorem we can find a basis of L (over an algebraic closure of $\mathbb{k}$) such that all $\text{ad } x$ are upper-triangular. Then each $\text{ad}[x, y] = [\text{ad } x, \text{ad } y]$ is strictly upper-triangular. This implies that for all $x \in [L, L]$ the matrix of $\text{ad } x$ is strictly upper-triangular and therefore nilpotent. By Engel theorem $[L, L]$ is a nilpotent Lie algebra. $\square$

3. SEMISIMPLE LIE ALGEBRAS

In this chapter we will assume that over field $\mathbb{k}$ is algebraically closed of characteristic zero.

3.1. Jordan-Chevalley decomposition. The Jordan normal form of a linear operator $x \in \mathfrak{gl}(V)$ is a sum of Jordan blocks

$$J_{n,\lambda} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

Each Jordan block $J_{n,\lambda}$ can be written as a sum of a diagonal matrix λI_n and a nilpotent matrix which commute. This implies that x also can be written as a sum of a diagonal and a nilpotent matrix which commute.

Definition 3.1. Let V be a vector space over a field $\mathbb{k}$. An endomorphism $x \in \mathfrak{gl}(V)$ is called semisimple if it is diagonalizable.

Remark 3.2. Given $x \in \mathfrak{gl}(V)$, we can consider a one-dimensional Lie algebra $L = \mathbb{k}x \subset \mathfrak{gl}(V)$ and consider V as a representation of L . Then x is semisimple if and only if V is a direct sum of 1-dimensional (therefore simple) modules over L . This is the reason for the name semisimple. $\diamond$

Lemma 3.3. Let V be a vector space and $x \in \text{End } V$ be an endomorphism.

- (1) x is semisimple if and only if the minimal polynomial of x does not have multiple roots.
- (2) If $W \subset V$ is an x -invariant subspace then the restriction $x|_W$ is a semisimple endomorphism of W .

Proof. 1. The minimal polynomial of the Jordan block $J_{n,\lambda}$ is $(t - \lambda)^n$. If x has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ and its Jordan normal form has corresponding maximal Jordan blocks of sizes $n_1, \dots, n_k$, then the minimal polynomial of x is $\prod_{i=1}^k (t - \lambda_i)^{n_i}$. In particular, if x is semisimple, then the minimal polynomial of x is $\prod_{i=1}^k (t - \lambda_i)$ and has no multiple roots. Conversely, if x is not semisimple, then $n_i \geq 2$ for some $1 \leq i \leq k$ and the minimal polynomial of x has root λ_i of multiplicity ≥ 2 .

2. If $p(t)$ is the minimal polynomial of x and $q(t)$ is the minimal polynomial of $x|_W$, then $p(x|_W) = 0$ and therefore $q \mid p$. As x is semisimple, p does not have multiple roots. This implies that q also does not have multiple roots and $x|_W$ is semisimple. For an alternative approach see Remark 3.24, where semisimple modules are considered. $\square$

Theorem 3.4 (Jordan-Chevalley decomposition). Let V be a vector space over a field $\mathbb{k}$ and $x \in \text{End } V$. Then

- (1) There exists a unique decomposition $x = s + n$, where s is semisimple, n is nilpotent and $sn = ns$.
- (2) There exist polynomials $p, q \in \mathbb{k}[t]$ without constant term, such that $s = p(x)$ and $n = q(x)$.

Proof. Let $(\lambda_1, \dots, \lambda_k)$ be distinct eigenvalues of x with multiplicities $(m_1, \dots, m_k)$. The characteristic polynomial of x is $\det(t \text{Id} - x) = \prod (t - \lambda_i)^{m_i} \in \mathbb{k}[t]$. The vector spaces

$$V_i = \ker(x - \lambda_i)^{m_i}$$

are x -stable and $V = \bigoplus_{i=1}^k V_i$.

Let us prove uniqueness of the decomposition. As s commutes with n , it also commutes with $x = s + n$ and with $(x - \lambda_i)^{m_i}$. Therefore $sV_i \subset V_i$. As s, x commute and $x - s$ is nilpotent, the eigenvalues of x and s on V_i coincide (we can decompose V_i into the eigenspaces of s , put x to Jordan normal form on each summand and then compare the eigenvalues). But x has the only eigenvalue λ_i on V_i . As s is semisimple, it acts on V_i by multiplication with λ_i . This proves uniqueness of s .

Taking the above description of s as a definition and putting $n = x - s$, we obtain a required decomposition.

Applying Chinese remainder theorem we can find a polynomial $p \in \mathbb{k}[t]$ such that

$$p(t) \equiv \lambda_i \pmod{(t - \lambda_i)^{m_i}}, \quad p(t) \equiv 0 \pmod{t} \quad 1 \leq i \leq k.$$

The operator $(x - \lambda_i)^{m_i}$ is zero on V_i . Therefore $p(x)$ acts on V_i by multiplication with λ_i and $p(x) = s$. Setting $q(t) = t - p(t)$, we obtain $q(x) = x - p(x) = n$. $\square$

Lemma 3.5. *Let $x \in \mathfrak{gl}(V)$ have the Jordan decomposition $x = s + n$. If $W \subset V$ is x -invariant (that is, $xW \subset W$), then W is also s - and n -invariant.*

Proof. The statement follows from the presentation $s = p(x)$, $n = q(x)$, where p, q are polynomials without constant term. $\square$

Lemma 3.6. *Let $x \in L = \mathfrak{gl}(V)$ have the Jordan decomposition $x = s + n$. Then $\text{ad}_x \in \mathfrak{gl}(L)$ has the Jordan decomposition $\text{ad}_x = \text{ad}_s + \text{ad}_n$.*

Proof. We just have to show that ad_s is semisimple and ad_n is nilpotent. We have seen before, that if n is nilpotent then ad_n is nilpotent. Let us choose a basis $(v_1, \dots, v_n)$ such that s is diagonal with respect to it and has eigenvalues $\lambda_1, \dots, \lambda_n$. Define elementary matrices E_{ij} by the rule $E_{ij}v_k = \delta_{jk}v_i$. Then $s = \sum_i \lambda_i E_{ii}$ and

$$\text{ad}_s(E_{ij}) = \sum_k \lambda_k [E_{kk}, E_{ij}] = (\lambda_i - \lambda_j)E_{ij}.$$

This implies that $\text{ad}_s \in \mathfrak{gl}(L)$ is diagonal in the basis consisting of elementary matrices. $\square$

3.2. Cartan's criterion.

Lemma 3.7. *Let $A \subset B \subset \mathfrak{gl}(V)$ be subspaces and let*

$$M = \{x \in \mathfrak{gl}(V) \mid [x, B] \subset A\}.$$

If $x \in M$ satisfies $\text{tr}(xy) = 0$ for all $y \in M$, then x is nilpotent.

Proof. Let $x = s + n$ be the Jordan decomposition. Then $\text{ad}_x = \text{ad}_s + \text{ad}_n$ is the Jordan decomposition of ad_x . As $\text{ad}_x(B) = [x, B] \subset A$, we have also $\text{ad}_s(B) \subset A$ (ad_s is a polynomial of ad_x with the zero constant term). Let $(v_1, \dots, v_n)$ be a basis for the Jordan normal form of x . Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. Define $E \subset \mathbb{k}$ to be a $\mathbb{Q}$ -vector space generated by the elements $\lambda_1, \dots, \lambda_n \in \mathbb{k}$. We want to show that x is nilpotent, that is, $\lambda_1 = \dots = \lambda_n = 0$ or equivalently, $E = 0$. Let $f : E \rightarrow \mathbb{Q}$ be a linear map. Define $y \in \text{End } V$ such that $yv_i = f(\lambda_i)v_i$. Let $E_{ij} \in \text{End } V$ be defined by $E_{ij}(v_k) = \delta_{jk}v_i$ (elementary matrix). Then

$$\text{ad}_y(E_{ij}) = \left[\sum_k f(\lambda_k)E_{kk}, E_{ij} \right] = f(\lambda_i - \lambda_j)E_{ij}.$$

Similarly $\text{ad}_s(E_{ij}) = (\lambda_i - \lambda_j)E_{ij}$. This means that ad_y and ad_s are diagonalizable in the same basis of $\text{End } V$. Let $\lambda_{ij} = \lambda_i - \lambda_j$. Applying Chinese remainder theorem, we can find a polynomial $p \in \mathbb{k}[t]$ such that

$$p(t) \equiv f(\lambda_{ij}) \pmod{t - \lambda_{ij}} \quad \forall i, j, \quad p(t) \equiv 0 \pmod{t}.$$

This polynomial does not have a constant term and satisfies $p(\lambda_{ij}) = f(\lambda_{ij})$. Therefore $p(\text{ad}_s) = \text{ad}_y$. We obtain from $\text{ad}_s(B) \subset A \subset B$ that $\text{ad}_y(B) \subset A$. This implies that $y \in M$. By assumption

$$0 = \text{tr}(xy) = \sum_i f(\lambda_i)\lambda_i.$$

Applying f to this equation, we obtain $\sum_i f(\lambda_i)^2 = 0$. As $f(\lambda_i) \in \mathbb{Q}$, we obtain $f(\lambda_i) = 0$ for all i . This implies that every linear map $f : E \rightarrow \mathbb{Q}$ is zero. Therefore $E = 0$ as required. $\square$

Theorem 3.8 (Cartan). *Let $L \subset \mathfrak{gl}(V)$ be a Lie algebra. Then L is solvable if and only if*

$$\text{tr}(xy) = 0 \quad \forall x \in [L, L], y \in L.$$

Proof. $\Rightarrow$. If L is solvable then there exists a basis of V such that all elements of L are upper-triangular in this basis. Then any $x \in [L, L]$ is strictly upper triangular. Therefore, for any $y \in L$, the matrix xy is also strictly upper-triangular and $\text{tr}(xy) = 0$.

$\Leftarrow$. We will show that every $x \in [L, L]$ is nilpotent. This implies that ad_x is nilpotent and then by Engel's theorem the algebra $[L, L]$ is nilpotent. Therefore L is solvable.

Let

$$A = [L, L], \quad B = L \subset \mathfrak{gl}(V), \quad M = \{y \in \mathfrak{gl}(V) \mid [y, L] \subset [L, L]\}.$$

If $x = \sum_i [x_i, x'_i] \in [L, L] \subset M$ and $y \in M$, then

$$\text{tr}(xy) = \sum \text{tr}([x_i, x'_i]y) = \sum \text{tr}(x_i[x'_i, y]).$$

Here $x'_i \in L$ and $y \in M$ imply $[x'_i, y] \in [L, L]$ (by the definition of M). Therefore $\text{tr}(x_i[x'_i, y]) = 0$ by assumption. We conclude that $\text{tr}(xy) = 0$ for all $y \in M$. By the previous lemma this implies that x is nilpotent and this is what we wanted to show. $\square$

3.2.1. Killing form.

Definition 3.9. Let $\sigma : L \times L \rightarrow \mathbb{k}$ be a bilinear form. Then

- (1) σ is called symmetric if $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in L$.
- (2) σ is called L -invariant if $\sigma([x, y], z) = \sigma(x, [y, z])$ for all $x, y, z \in L$.
- (3) σ is called non-degenerate if $\sigma(x, L) = 0$ for $x \in L$ implies $x = 0$.

Given a representation $\rho : L \rightarrow \mathfrak{gl}(V)$, we define a symmetric bilinear form

$$\sigma_V : L \times L \rightarrow \mathbb{k}, \quad \sigma_V(x, y) = \text{tr}(\rho(x)\rho(y)) \quad \forall x, y \in L.$$

In particular, consider the adjoint representation $\text{ad} : L \rightarrow \mathfrak{gl}(L)$ and the corresponding symmetric L -invariant bilinear form

$$\kappa(x, y) = \kappa_L(x, y) = \text{tr}(\text{ad}_x \text{ad}_y) \quad x, y \in L.$$

Corollary 3.10 (Cartan). *A Lie algebra L is solvable if and only if $\kappa([L, L], L) = 0$.*

Proof. Assume that $\kappa([L, L], L) = 0$. Consider the adjoint map $\text{ad} : L \rightarrow \mathfrak{gl}(L)$ and its image $\bar{L} = \text{ad}(L) \subset \mathfrak{gl}(L)$ which is a Lie algebra. For any $x, y, z \in L$, we have

$$\text{tr}([\text{ad}_x, \text{ad}_y] \text{ad}_z) = \text{tr}(\text{ad}_{[x, y]}, \text{ad}_z) = \kappa([x, y], z) = 0.$$

This implies that the conditions of Theorem 3.8 are satisfied for the algebra $\bar{L} \subset \mathfrak{gl}(L)$. Therefore $\bar{L}$ is solvable. The kernel of $\text{ad} : L \rightarrow \bar{L}$ is equal

$$\{x \in L \mid \text{ad}_x(y) = 0 \ \forall y \in L\} = \{x \in L \mid [x, y] = 0 \ \forall y \in L\} = Z(L)$$

which is an abelian ideal and therefore solvable. As $L/Z(L) = \bar{L}$ is also solvable, we obtain that L is solvable.

Conversely, if L is solvable, then $\bar{L}$ is solvable. Then applying Theorem 3.8 we obtain that $\kappa([x, y], z) = 0$ for all $x, y, z \in L$. $\square$

Later we will need the following result about Killing forms.

Lemma 3.11. *Let $I \subset L$ be an ideal of a Lie algebra. Then the Killing form κ_I of I coincides with the restriction of the Killing form κ_L to I .*

Proof. If $A \in \text{End}(L)$ satisfies $A(L) \subset I$, then $\text{tr}(A) = \text{tr}(A|_I)$ (write A in a basis of L that contains a basis of I). In particular, for any $x, y \in I$ the operators ad_x, ad_y map L to I . Applying the first observation to $A = \text{ad}_x \text{ad}_y$, we obtain

$$\kappa_L(x, y) = \text{tr}(\text{ad}_x \text{ad}_y) = \text{tr}(A) = \text{tr}(A|_I) = \kappa_I(x, y).$$

$\square$

3.3. Semisimple Lie algebras.

Definition 3.12. Let L be a Lie algebra.

- (1) L is called simple if it is not abelian and it does not contain ideals except 0 and L .
- (2) L is called semisimple if it does not contain abelian ideals except zero.

Remark 3.13. If L is simple, then it is semisimple. Indeed, the only nonzero ideal of L is L itself and by assumption it is not abelian. $\diamond$

Example 3.14. We have seen that $\mathfrak{sl}_2$ does not contain ideals except zero and $\mathfrak{sl}_2$. This implies that $\mathfrak{sl}_2$ is a simple Lie algebra and therefore also semisimple. $\diamond$

We proved earlier that if $I, J \subset L$ are solvable ideals then $I+J$ is also a solvable ideal. Therefore the sum

$$\text{rad } L = \sum_{\substack{I \subset L \\ \text{solvable}}} I$$

is the maximal solvable ideal of L , called the radical of L .

Lemma 3.15. Let L be a Lie algebra. The following conditions are equivalent.

- (1) L is semisimple.
- (2) L does not contain solvable ideals except zero.
- (3) $\text{rad } L = 0$.

Proof. $2 \Leftrightarrow 3$ is trivial.

$2 \Rightarrow 1$. If L does not contain solvable ideals (except zero) then in particular it does not contain abelian ideals (except zero).

$1 \Rightarrow 2$. Let $0 \neq I \subset L$ be a solvable ideal. Then $I^{(n)} = 0$ for some $n \geq 1$. Let n be the minimal number with this property. Then $I^{(n-1)} \neq 0$ and $[I^{(n-1)}, I^{(n-1)}] = I^{(n)} = 0$. This means that $I^{(n-1)}$ is a non-trivial abelian ideal in L . A contradiction. $\square$

Theorem 3.16. Let L be a Lie algebra. The following conditions are equivalent.

- (1) L is semisimple.
- (2) The Killing form κ_L is non-degenerate.
- (3) L is a direct sum $I_1 \oplus \cdots \oplus I_n$ of simple ideals (simple as Lie algebras).
- (4) L is a direct sum $L_1 \oplus \cdots \oplus L_n$ of simple Lie algebras.

Proof. $1 \Rightarrow 2$. Let L be a semisimple Lie algebra. Define

$$I = \{x \in L \mid \kappa(x, L) = 0\} = L^\perp.$$

This is an ideal in L by Exercise 3.3. By Lemma 3.11 $\kappa_I = \kappa|_I = 0$. Therefore by the Cartan criterion I is solvable. As L is semisimple, we obtain that $I = 0$. This means that κ is non-degenerate.

$2 \Rightarrow 1$. If L is not semisimple, then there exists an abelian ideal $0 \neq I \subset L$. For any $x, z \in L$ and $y \in I$, we have

$$(\text{ad}_x \text{ad}_y)^2 z = [x, [y, [x, [y, z]]]] \in [x, [I, I]] = 0$$

as $y \in I$ and $[x, [y, z]] \in I$. This implies that $(\text{ad}_x \text{ad}_y)^2 = 0$ and $\text{ad}_x \text{ad}_y$ is a nilpotent operator. Therefore its trace is zero and

$$\kappa(x, y) = \text{tr}(\text{ad}_x \text{ad}_y) = 0 \forall x \in L, y \in I.$$

This implies that κ is degenerate, a contradiction.

$1 \Rightarrow 3$. If L is simple then we are done. Assume that $I \subset L$ is a non-trivial ideal. Then

$$I^\perp = \{x \in L \mid \kappa(x, I) = 0\}$$

is an ideal by Exercise 3.3. The Killing form is zero on the ideal $I \cap I^\perp$. Therefore this is a solvable ideal. By semisimplicity of L , we obtain $I \cap I^\perp = 0$. This implies that $L = I \oplus I^\perp$ (compare dimensions). Moreover $[I, I^\perp] \subset I \cap I^\perp = 0$. Ideals of $I, I^\perp$ are automatically ideals of

L . This implies that there are no solvable ideals in $I, I^\perp$ and these algebras are semisimple. Then by induction we can decompose $I, I^\perp$ into a direct sum of simple ideals.

3 $\Rightarrow$ 4. Trivial.

4 $\Rightarrow$ 1. Let $L = L_1 \oplus \cdots \oplus L_n$ be a direct sum of simple Lie algebras. Let $I \subset L$ be an abelian ideal. Then the image of I under the projection $\pi_k : L \rightarrow L_k$ is an abelian ideal, but the only abelian ideal of L_k is zero. This implies that $\pi_k(I) = 0$ for all $1 \leq k \leq n$. Therefore $I = 0$. This implies that L is semisimple. $\square$

Remark 3.17. If L is a semisimple Lie algebra with a decomposition $L = I_1 \oplus \cdots \oplus I_n$ into a sum of simple ideals as above, then any simple ideal $I \subset L$ is equal to one of I_k (not just isomorphic). Indeed, $[I, L] \subset I$ is non-zero as I is non-abelian. Therefore $I = [I, L] = [I, I_1] \oplus \cdots \oplus [I, I_n]$. Only one of this summands can be nonzero and then we have $I = [I, I_k] = I_k$. $\diamond$

Theorem 3.18. *Let L be a semisimple Lie algebra. Then the adjoint map $\text{ad} : L \rightarrow \text{Der } L$ is an isomorphism.*

Proof. We know that $\ker \text{ad} = Z(L)$. This is an abelian ideal. Therefore $\ker \text{ad} = 0$ and ad is a monomorphism. Let $D = \text{Der } L$ and $M = \text{ad}(L) \subset D$. For any derivation $\delta \in D$ and for any $x \in L$, we have

$$[\delta, \text{ad}_x] = \text{ad}_{\delta x} \in M.$$

Therefore $M \subset D$ is an ideal. The subspace

$$I = M^\perp = \{x \in D \mid k_D(x, M) = 0\}$$

is also an ideal. As $M \simeq L$ is semisimple, we obtain that $\kappa_D|_M = \kappa_M$ is non-degenerate and therefore $I \cap M = 0$. This implies $D = I \oplus M$. For any $\delta \in I$ and $x \in L$, we have

$$\text{ad}_{\delta x} = [\delta, \text{ad}_x] \in [I, M] \subset I \cap M = 0.$$

But ad is a monomorphism, hence $\delta x = 0$ for all $x \in L$. Therefore $\delta = 0$ and $I = 0$. We conclude that $\text{ad}(L) = M = D$. $\square$

3.4. Complete reducibility.

Definition 3.19. Let L be a Lie algebra and V be an L -module.

- (1) V is called simple (or irreducible) if it does not contain submodules except 0 and V .
- (2) V is called semisimple (or completely reducible) if it is a direct sum of simple modules.

Remark 3.20. Any ideal of a Lie algebra L can be considered as a subrepresentation of an adjoint representation on L . This implies that (non-abelian) L is a simple algebra if and only if the adjoint representation is simple. We have seen that every semisimple algebra is a direct sum of simple ideals. This implies that the adjoint representation is a sum of simple representations and is therefore semisimple. Conversely, if L is a Lie algebra such that its adjoint representation is semisimple (without summand with trivial module structure), then L is a direct sum of simple ideals. This implies that L is a semisimple Lie algebra. $\diamond$

Lemma 3.21. Let L be a Lie algebra and V be an L -module. The following conditions are equivalent

- (1) V is semisimple.
- (2) V is a sum of simple L -modules.
- (3) Any submodule $W \subset V$ has a complement (a submodule $W' \subset V$ such that $V = W \oplus W'$).

Proof. $1 \Rightarrow 2$ is trivial.

$2 \Rightarrow 1$. Let $V = \sum_{i=1}^n S_i$ with simple S_i and with minimal possible n . By induction on n , $V' = S_1 \oplus \cdots \oplus S_{n-1}$ is a direct sum. If $S_n \cap V' \neq 0$, then $S_n \cap V' = S_n$ and $S_n \subset V'$. This implies that $V = V' = S_1 \oplus \cdots \oplus S_{n-1}$. If $S_n \cap V' = 0$, then $V = V' \oplus S_n = S_1 \oplus \cdots \oplus S_n$.

$1 \Rightarrow 3$. Let $W \subset V = S_1 \oplus \cdots \oplus S_n$. Consider a projection $\pi : V \rightarrow S_1 \oplus \cdots \oplus S_{n-1} = V'$. By induction $V' = \pi(W) \oplus W'$ for some submodule $W' \subset V'$. If $W \cap S_n = 0$, then $W \rightarrow \pi(W)$ is an isomorphism and $V = W \oplus (W' \oplus S_n)$. If $W \cap S_n \neq 0$, then $W \cap S_n = S_n$ and $S_n \subset W$. This implies $V = W \oplus W'$.

$3 \Rightarrow 1$. Choose a simple submodule $S_1 \subset V$. By assumption, it has a complement $V_1 \subset V$. Choose a simple submodule $S_2 \subset V_1$. By assumption, the module $S_1 \oplus S_2$ has a complement $V_2 \subset V$. Continuing this procedure, we obtain $V = S_1 \oplus \cdots \oplus S_n$. $\square$

Example 3.22. Consider a one dimensional Lie algebra $L = \mathbb{k}x$ and a 2-dimensional L -module $V = \mathbb{k}v_1 \oplus \mathbb{k}v_2$ with an action of x given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then the subspace $W = \mathbb{k}v_1 \subset V$ is an L -submodule. But it does not have a complement. This means that V is not-semisimple.

On the other hand, if $L = \mathbb{k}x \subset \mathfrak{gl}(V)$ and x is a semisimple endomorphism, then V is a direct sum of 1-dimensional L -submodules. This implies that V is a semisimple L -module. $\diamond$

Example 3.23. The standard representation $\mathbb{k}^n$ of $L = \mathfrak{gl}_n(\mathbb{k})$ is simple. Indeed, for any non-zero $v, w \in \mathfrak{gl}_n(\mathbb{k})$ there exists an endomorphism $x \in \mathfrak{gl}_n(\mathbb{k})$ such that $xv = w$. This implies that any submodule $W \subset V$ that contains v should be equal to the whole space V . $\diamond$

Remark 3.24. Any submodule and any quotient module of a semisimple module is again semisimple. If $\pi : V \rightarrow W$ is an epimorphism and $V = \sum_i S_i$ is a sum of simple modules, then $W = \sum_i \pi(S_i)$ is a sum of simple (and zero) modules and therefore W is semisimple. If $W \subset V$ is a submodule of a semisimple module, then there exists a complement $V = W \oplus W'$. Therefore there exists a surjective homomorphism $V \rightarrow W$ and W is semisimple. $\diamond$

Lemma 3.25 (Schur lemma). Let L be a Lie algebra and V be a simple L -module. Then $\text{End}_L(V) = \mathbb{k} \text{Id}_V$.

Proof. Let $f : V \rightarrow V$ be a homomorphism of modules. Let $\lambda \in \mathbb{k}$ be an eigenvalue of f and let

$$V_\lambda = \{v \in V \mid f(v) = \lambda v\} \subset V$$

be the corresponding eigenspace. It is a submodule of V : if $x \in L$ and $v \in V_\lambda$ then

$$f(xv) = xf(v) = x\lambda v = \lambda xv$$

and therefore $xv \in V_\lambda$. As V is simple, we obtain $V_\lambda = V$. This implies $f = \lambda \text{Id}_V$. Therefore $\text{End}_L(V) = \mathbb{k} \text{Id}_V$. $\square$

3.4.1. *Casimir operator.* Let $\sigma : L \times L \rightarrow \mathbb{k}$ be a symmetric non-degenerate L -invariant form. Let $(x_1, \dots, x_n)$ be a basis of L and $(y_1, \dots, y_n)$ be the dual basis of L with respect to σ

$$\sigma(x_i, y_j) = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

For any L -representation V , define a linear operator

$$C_\sigma = C_\sigma^V : V \rightarrow V, \quad C_\sigma(v) = \sum_{i=1}^n x_i y_i v \quad \forall v \in V,$$

called the Casimir operator.

Lemma 3.26.

- (1) *The Casimir operator C_σ is independent of the choice of a basis in L .*
- (2) *$C_\sigma : V \rightarrow V$ is a homomorphism of L -representations.*

Proof. 1. Let (x'_i) be a different basis of L and let (y'_i) be the dual basis of L with respect to σ . Let

$$x'_k = \sum_i a_{ki} x_i, \quad y'_l = \sum_j b_{jl} y_j.$$

Then

$$\delta_{kl} = \sigma(x'_k, y'_l) = \sum_{i,j} a_{ki} b_{jl} \sigma(x_i, y_j) = \sum_{i,j} a_{ki} b_{jl} \delta_{ij} = \sum_i a_{ki} b_{il} = (AB)_{kl}$$

and $AB = I_n$. Therefore

$$\sum_k x'_k y'_k = \sum_{i,j} \left(\sum_k a_{ki} b_{jk} \right) x_i y_j = \sum_{i,j} (BA)_{ji} x_i y_j = \sum_i x_i y_i.$$

2. For any $x \in L$, let

$$[x_k, x] = \sum_i a_{ki} x_i, \quad [x, y_l] = \sum_j b_{jl} y_j.$$

Then

$$a_{kl} = \sigma([x_k, x], y_l) = \sigma(x_k, [x, y_l]) = b_{kl}.$$

Therefore

$$\begin{aligned} C_\sigma(xv) - xC_\sigma(v) &= \sum_k (x_k y_k x - x x_k y_k) v = \sum_k (x_k [y_k, x] + [x_k, x] y_k) \\ &= \left(- \sum_{k,i} b_{ik} x_k y_i + \sum_{k,i} a_{ki} x_i y_k \right) v = 0. \end{aligned}$$

□

Lemma 3.27. *Let L be a semisimple Lie algebra and $\rho : L \rightarrow \mathfrak{gl}(V)$ be a faithful representation (injective map). Then*

- (1) *The bilinear form*

$$\sigma : L \times L \rightarrow \mathbb{k}, \quad \sigma(x, y) = \text{tr}(\rho(x)\rho(y))$$

is non-degenerate.

- (2) *The Casimir operator on V satisfies $\text{tr } C_\sigma^V = \dim L$.*

Proof. 1. We can assume that $L \subset \mathfrak{gl}(V)$. The radical $I = \{x \in L \mid \sigma(x, L) = 0\}$ is an ideal in L . We have $I \subset \mathfrak{gl}(V)$ and

$$\text{tr}(xy) = \sigma(x, y) = 0 \quad \forall x, y \in I.$$

By the Cartan criterion this implies that I is solvable. As L is semisimple, we obtain $I = 0$ and therefore σ is non-degenerate.

2. The dual bases (x_i) and (y_i) of L satisfy $\text{tr}(x_i y_j) = \sigma(x_i, y_j) = \delta_{ij}$. Therefore the Casimir operator $C_\sigma^V = \sum_i x_i y_i$ satisfies

$$\text{tr}(C_\sigma^V) = \sum_i \text{tr}(x_i y_i) = \sum_i \delta_{ii} = \dim L.$$

□

3.4.2. Weyl theorem.

Theorem 3.28 (Weyl). *Let L be a semisimple Lie algebra. Then any representation of L is semisimple.*

We will need some preparations before proving this theorem. Given an L -representation V , we can construct its subrepresentations

$$V^L = \{v \in V \mid Lv = 0\}, \quad LV = \langle xv \mid x \in L, v \in V \rangle,$$

where the angular brackets denote the subspace generated by given vectors.

Lemma 3.29. *Let L be a semisimple Lie algebra. Then*

- (1) *For any L -representation V , we have $V = V^L \oplus LV$.*
- (2) *For any surjective homomorphism $V \rightarrow W$ of L -representations, the induced map $V^L \rightarrow W^L$ is surjective.*

Proof. If $V^L = V$, then $LV = 0$ and the statement is obvious. Assume that $V^L \neq V$. This implies that the corresponding representation map $\rho : L \rightarrow \mathfrak{gl}(V)$ is nonzero. The image $M = \rho(L) \subset \mathfrak{gl}(V)$ is again a semisimple algebra (L is a sum of simple ideals and therefore M is also a sum of simple ideals). By Lemma 3.27, the corresponding bilinear form σ on M is non-degenerate. Let $C = C_\sigma^V$ be the corresponding Casimir operator on V . Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of C having multiplicities $m_1, \dots, m_k$. Generalized eigenspaces

$$V_i = \ker(C - \lambda_i)^{m_i} \subset V$$

are subrepresentations of V (as $Cxv = xCv$ for all $x \in L, v \in V$) and $V = \bigoplus_{i=1}^k V_i$. If $k > 1$, then by induction $V_i = V_i^L \oplus LV_i$ and the statement of the theorem follows. If $k = 1$, then $\text{tr} C = \lambda_1 \dim V = \dim M$ by Lemma 3.27. Therefore $\lambda_1 \neq 0$ and C is invertible. Injectivity of C implies that $V^L = 0$ (C acts as zero on V^L). Surjectivity of C implies $LV = V$. Therefore $V = V^L \oplus LV$.

For the second statement we note that a homomorphism $V \rightarrow W$ maps V^L to W^L and maps LV to LW . Because of the proved decomposition, we obtain that the map $V^L \rightarrow W^L$ is surjective. □

Proof of the Weyl theorem. Let $W \subset V$ be a subrepresentation. It induces a surjective linear map

$$\varphi : \text{Hom}(V, W) \rightarrow \text{Hom}(W, W), \quad f \mapsto f|_W.$$

These Hom-spaces have a natural structure of L -representations and φ is a homomorphism between them. By the previous result it induces a surjective map

$$\text{Hom}(V, W)^L \rightarrow \text{Hom}(W, W)^L.$$

The space $\text{Hom}(V, W)^L$ consists of linear maps $f : V \rightarrow W$ that are L -invariant, that is, $f(xv) = xf(v)$ for all $x \in L$ and $v \in V$. This means that $f : V \rightarrow W$ is a homomorphism of L -representations. By the surjectivity of the above map, we can find $f \in \text{Hom}(V, W)^L$ such that $f|_W = \text{Id}_W$. Then $V = W \oplus \ker f$, which means that W has a complement. This implies that V is semisimple. □

3.5. Reductive algebras.

Definition 3.30. A Lie algebra is called reductive if $Z(L) = \text{rad } L$.

Example 3.31. We always have $Z(L) \subset \text{rad } L$. If L is semisimple, then $\text{rad } L = 0$ and therefore $Z(L) = \text{rad } L = 0$, hence L is reductive. If L is abelian, then $Z(L) = L$ and therefore $Z(L) = \text{rad } L = L$, hence L is reductive. $\diamond$

Theorem 3.32. Let L be a Lie algebra. The following conditions are equivalent

- (1) L is reductive.
- (2) Adjoint representation of L is a direct sum of simple representations.
- (3) L is a direct sum of $Z(L)$ and a semisimple Lie algebra.

Proof. $1 \Rightarrow 2$. Consider the adjoint representation $\text{ad} : L \rightarrow \mathfrak{gl}(L)$. Its kernel is $\ker \text{ad} = Z(L)$. We can consider L as a representation over $M = L/Z(L)$. We have $M = L/Z(L) = L/\text{rad } L$, hence M is semisimple. By the Weyl Theorem L can be decomposed as a direct sum of simple M -modules (which are also simple L -modules).

$2 \Rightarrow 3$. Let $L = I_1 \oplus \dots \oplus I_n$ be a decomposition into a sum of simple modules. Then each I_k (considered as an algebra) does not contain non-trivial ideals. Hence I_k is either simple or abelian. If I_k is abelian, then $[L, I_k] = [I_k, I_k] = 0$ and $I_k \subset Z(L)$. This implies that we can decompose $L = I \oplus J$, where $I \subset Z(L)$ and J is a semisimple Lie algebra. The algebra $L/I \simeq J$ is semisimple, hence I is the maximal solvable ideal. This implies that $I = Z(L) = \text{rad } L$ and we obtain the required decomposition $L = Z(L) \oplus J$.

$3 \Rightarrow 1$. Our assumption implies that $L/Z(L)$ is a semisimple Lie algebra. Therefore $Z(L)$ is the maximal solvable ideal and $Z(L) = \text{rad } L$. $\square$

Theorem 3.33. If L has a faithful simple representation $\rho : L \rightarrow \mathfrak{gl}(V)$ (faithful means that ρ is injective), then L is reductive and $\dim Z(L) \leq 1$. If moreover $\rho(L) \subset \mathfrak{sl}(V)$, then L is semisimple.

Proof. Let $I = \text{rad } L$ be the maximal solvable ideal. By Lie's Theorem there exists $0 \neq v_0 \in V$ such that $Iv \subset \mathbb{k}v$. Let $\lambda : I \rightarrow \mathbb{k}$ be defined by $xv_0 = \lambda(x)v_0$, $x \in I$ (it is linear). We proved earlier (see the proof of Lie theorem) that the subspace

$$V_\lambda = \{v \in V \mid xv = \lambda(x)v \ \forall x \in I\}$$

is an L -submodule of V . As V is simple, we obtain $V_\lambda = V$. This implies that every $x \in I$ acts on V as $\lambda(x) \text{Id}_V$. Therefore $I \subset \mathbb{k} \text{Id}_V$ and $I \subset Z(L)$ and $\dim I \leq 1$. This implies $I = \text{rad } L = Z(L)$ and $\dim Z(L) \leq 1$.

If $\rho(L) \subset \mathfrak{sl}(V)$, then $\rho(I) \subset \mathfrak{sl}(V)$, but $\lambda \text{Id}_V \in \mathfrak{sl}(V)$ only if $\lambda = 0$. This implies that $I = \text{rad } L = 0$ and L is semisimple. $\square$

Example 3.34. Consider the standard representation V of the Lie algebra $\mathfrak{gl}(V)$. Then V is a simple representation and by the previous theorem $L = \mathfrak{gl}(V)$ is reductive and $\dim Z(L) \leq 1$. It is clear that $\mathbb{k} \text{Id} \subset Z(L)$. Comparing dimensions, we obtain $Z(L) = \mathbb{k} \text{Id}$. In a similar we can show that $\mathfrak{sl}(V)$ is semisimple (if $\dim V \geq 2$). $\diamond$

Exercises.

3.1. Show that the form σ_V is symmetric and L -invariant.

3.2. Let V be the standard representation of $L = \mathfrak{gl}(V)$. Show that σ_V is non-degenerate.

3.3. Show that if $\sigma : L \times L \rightarrow \mathbb{k}$ is a bilinear L -invariant form and $I \subset L$ is an ideal, then

$$I^\perp = \{x \in L \mid \sigma(x, I) = 0\}$$

is also an ideal of L . Moreover, $\dim I^\perp + \dim I \geq \dim L$.

3.4. Let $\sigma : L \times L \rightarrow \mathbb{k}$ be a symmetric non-degenerate L -invariant form.

(1) Show that the induced map

$$\sigma' : L \rightarrow L^*, \quad \sigma'(x)(y) = \sigma(x, y) \quad \forall x, y \in L$$

is an isomorphism of L -modules. In particular, it induces an isomorphism $L \otimes L \rightarrow L^* \otimes L$.

(2) Show that the natural map

$$\varphi : L^* \otimes L \rightarrow \text{Hom}(L, L), \quad \varphi(f \otimes x)(y) = f(y)x \quad \forall f \in L^*, x, y \in L$$

is an isomorphism of L -modules.

(3) Consider the composition

$$L \otimes L \rightarrow L^* \otimes L \rightarrow \text{Hom}(L, L)$$

of the above isomorphisms and let $\Omega \in L \otimes L$ be the preimage of Id_L . Show that Ω can be written in the form $\Omega = \sum_{i=1}^n x_i \otimes y_i$, where $(x_1, \dots, x_n)$ is a basis of L and $(y_1, \dots, y_n)$ is a dual basis of L with respect to σ .

4. CLASSIFICATION OF SEMISIMPLE LIE ALGEBRAS

4.1. Representations of $\mathfrak{sl}_2$. We consider the algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{k})$ of an algebraically closed field $\mathbb{k}$ of characteristic zero. We know that $\mathfrak{sl}_2$ is simple, hence semisimple. By the Weyl's Theorem every representation of $\mathfrak{sl}_2$ is a direct sum of simple representations. Therefore we need to classify just simple representations of $\mathfrak{sl}_2$.

Theorem 4.1. *Let x, y, h be the standard generators of $\mathfrak{sl}_2$ with*

$$(5) \quad [x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

For any $n \geq 0$, there exists exactly one simple representation $V(n)$ of $\mathfrak{sl}_2$ of dimension $n+1$ (up to isomorphism). Representation $V = V(n)$ is a direct sum of 1-dimensional h -eigenspaces

$$V = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n,$$

where V_k is the subspace of h -eigenvectors with eigenvalue k , and x, y induce isomorphisms

$$x : V_k \rightarrow V_{k+2}, \quad y : V_{k+2} \rightarrow V_k$$

whenever $V_k, V_{k+2} \neq 0$.

Proof. Let V be a simple representation. For any $\lambda \in \mathbb{k}$ define the eigenspace

$$V_\lambda = \{v \in V \mid hv = \lambda v\}.$$

Then

$$xV_\lambda \subset V_{\lambda+2}, \quad yV_\lambda \subset V_{\lambda-2}.$$

Indeed, if $v \in V_\lambda$, then

$$h xv = ([h, x] + xh)v = (2x + xh)v = 2xv + \lambda xv = (\lambda + 2)xv.$$

Therefore $xv \in V_{\lambda+2}$. Similarly $yv \in V_{\lambda-2}$. There exists $\lambda \in \mathbb{k}$ such that $V_\lambda \neq 0$ and $V_{\lambda+2} = 0$. Let $0 \neq v \in V_\lambda$. Then $xv \in V_{\lambda+2} = 0$ and we obtain

$$hv = \lambda v, \quad xv = 0.$$

We will prove by induction that

$$hy^i v = (\lambda - 2i)y^i v, \quad xy^i v = i(\lambda - i + 1)y^{i-1}v.$$

The statement for $i = 0$ was just proved. As $v \in V_\lambda$, we obtain $yv \in V_{\lambda-2}$ and generally $y^i v \in V_{\lambda-2i}$. Therefore $hy^i v = (\lambda - 2i)y^i v$. Assume that the second formula is proved for $i \geq 0$. Then

$$\begin{aligned} xy^{i+1}v &= xy(y^i v) = ([x, y] + yx)y^i v = (h + yx)y^i v = (\lambda - 2i)y^i v + i(\lambda - i + 1)y^i v \\ &= (\lambda(i+1) - i^2 - i)y^i v = (i+1)(\lambda - i)y^i v. \end{aligned}$$

This proves the induction step. Let $n \geq 0$ be the maximal integer such that $y^n v \neq 0$. Then the vector space

$$\langle v, yv, \dots, y^n v \rangle \subset V$$

is a submodule of V (it is closed under the action of x, y, h). As V is simple, the above subspace should be the whole space V . The above vectors are linearly independent as $y^i v \in V_{\lambda-2i}$ are contained in different eigenspaces. This implies that $\dim V = n+1$. As $y^{n+1}v = 0$, we obtain

$$0 = xy^{n+1}v = (n+1)(\lambda - n)y^n v,$$

hence $\lambda - n$ and $\lambda = n$. The above formulas uniquely determine representation V .

To show existence, we use the above formulas as a motivation for our construction. Let V be a vector space with a basis $(v_0, \dots, v_n)$ (v_i corresponds to $y^i v$ in the above discussion). Define the linear operators x, y, h on V by the rule

$$(6) \quad xv_i = i(n - i + 1)v_{i-1}, \quad yv_i = v_{i+1}, \quad hv_i = (n - 2i)v_i, \quad 0 \leq i \leq n,$$

where $v_{-1} = v_{n+1} = 0$. In order to show that this rule defines a representation of $\mathfrak{sl}_2$, we need to verify the commutator relations (5). Let us check just the first one

$$[x, y]v_i = (xy - yx)v_i = xv_{i+1} - yxv_i = (i+1)(n-i)v_i - i(n-i+1)v_i = (n-2i)v_i = hv_i.$$

This implies that V is indeed a representation of $\mathfrak{sl}_2$. We have to check V is a simple representation. Let $W \subset V$ be a non-zero subrepresentation and $0 \neq v \in W$. Assume that $v = \sum_i a_i v_i$ and $0 \leq i \leq n$ is minimal with the property $a_i \neq 0$. Then $y^{n-i}v = a_i v_n \in W$, hence $v_n \in W$. Applying the operator x , we obtain that $v_{n-1}, \dots, v_1, v_0$ are also contained in W . Therefore $W = V$ and V is a simple module. $\square$

Remark 4.2. Note that

- (1) The 1-dimensional representation $V(0)$ is the trivial representation $\mathbb{k}$.
- (2) The 2-dimensional representation $V(1)$ is the standard representation $\mathbb{k}^2$ of $\mathfrak{sl}_2(\mathbb{k})$.
- (3) The 3-dimensional representation $V(2)$ is the adjoint representation of $\mathfrak{sl}_2$.

$\diamond$

Corollary 4.3. *Let V be a finite-dimensional representation of $\mathfrak{sl}_2$. Then*

- (1) $V = \bigoplus_{k \in \mathbb{Z}} V_k$, where $V_k = \{v \in V \mid hv = kv\}$ is the h -eigenspace (also called a weight space) with eigenvalue k (also called a weight).
- (2) x maps V_k to V_{k+2} and y maps V_k to V_{k-2} .
- (3) $x^k : V_{-k} \rightarrow V_k$ and $y^k : V_k \rightarrow V_{-k}$ are isomorphisms, for any $k \geq 1$.

Proof. 1. Every finite-dimensional representation V of $\mathfrak{sl}_2$ is a direct sum of simple representations $V(n)$ classified in the previous theorem. By this theorem, if $V = V(n)$, then $V = \bigoplus_{k \in \mathbb{Z}} V_k$, the direct sum of h -eigenspaces. This implies the first statement of the corollary for simple modules, hence also for direct sums of simple modules.

2. We have seen that it is true for simple modules. Therefore it is also true for arbitrary modules.
3. It is enough to prove the statement for a simple module $V = V(n)$. We know that

$$V = V_{-n} \oplus V_{-n+2} \oplus \dots \oplus V_{n-2} \oplus V_n.$$

If k and n have different parities, then $V_k = V_{-k} = 0$ and the statement is automatically true. If $k > n$, then again $V_k = V_{-k} = 0$ and the statement is true. Assume that k and n have the same parity and $k \leq n$. By the previous theorem, there is a sequence of isomorphisms

$$V_{-n} \xrightarrow{x} V_{-n+2} \xrightarrow{x} \dots \xrightarrow{x} V_{n-2} \xrightarrow{x} V_n.$$

In particular, $x^k : V_{-k} \rightarrow V_k$ is an isomorphism. The proof for y^k is the same. $\square$

Remark 4.4. Let V be a representation of $\mathfrak{sl}_2$. Then we can decompose V as a direct sum of simple modules $V = \bigoplus_{n \geq 0} V(n)^{m_n}$. We can determine multiplicities m_n as follows. Let $V = \bigoplus_{k \in \mathbb{Z}} V_k$ be the weight space decomposition. A weight $k \geq 0$ appears in $V(n)$ (with multiplicity one) if and only if n and k have the same parity and $n \geq k$. This implies

$$\dim V_k = m_k + m_{k+2} + \dots$$

Therefore

$$m_k = \dim V_k - \dim V_{k+2}.$$

$\diamond$

4.1.1. *Action of $\mathfrak{sl}_2$ on the ring of polynomials.* We will describe now an alternative construction of simple representations of $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{k})$. Consider the standard representation

$$\mathbb{k}^2 = \mathbb{k}u \oplus \mathbb{k}v$$

of $\mathfrak{sl}_2$. Explicitly, x, y, h act on $\mathbb{k}^2$ as

$$(7) \quad xu = 0, \quad xv = u, \quad yu = v, \quad yv = 0, \quad hu = u, \quad hv = -v.$$

This action of $\mathfrak{sl}_2$ on $\mathbb{k}^2$ can be extended to the action of $\mathfrak{sl}_2$ by derivations on the ring of polynomials $\mathbb{k}[u, v]$

$$\rho : \mathfrak{sl}_2 \rightarrow \text{Der } \mathbb{k}[u, v].$$

Note that any derivation $\delta : \mathbb{k}[u, v] \rightarrow \mathbb{k}[u, v]$ satisfies the Leibniz rule

$$\delta(fg) = \delta(f) \cdot g + f \cdot \delta(g), \quad \forall f, g \in \mathbb{k}[u, v].$$

This requirement implies that δ is determined already by its values $\delta(u)$ and $\delta(v)$. More precisely, we can write δ as a vector field

$$\delta = p\partial_u + q\partial_v, \quad \partial_u = \frac{\partial}{\partial u}, \quad \partial_v = \frac{\partial}{\partial v}, \quad p, q \in \mathbb{k}[u, v]$$

that acts as

$$\delta(f) = p \frac{\partial f}{\partial u} + q \frac{\partial f}{\partial v}, \quad \forall f \in \mathbb{k}[u, v].$$

In particular,

$$\delta(u) = p, \quad \delta(v) = q$$

and they determine δ uniquely. Applying these formulas to (7), we obtain the following representation $\rho : \mathfrak{sl}_2 \rightarrow \text{Der } \mathbb{k}[u, v]$

$$\rho(x) = u\partial_v, \quad \rho(y) = v\partial_u, \quad \rho(h) = u\partial_u - v\partial_v.$$

To verify that ρ is indeed a representation, that is, a Lie algebra homomorphism, we need just to verify that $\rho([a, b])$ and $[\rho(a), \rho(b)]$ (for $a, b \in \mathfrak{sl}_2$) act identically on u and v . For example

$$\begin{aligned} \rho([x, y])u &= \rho(h)u = u, & [\rho(x), \rho(y)]u &= \rho(x)v - \rho(y)u = u. \\ \rho([x, y])v &= \rho(h)v = -v, & [\rho(x), \rho(y)]v &= \rho(x)u - \rho(y)v = -v. \end{aligned}$$

Remark 4.5. Alternatively, we can exponentiate the action of $\text{SL}_2(\mathbb{R})$ on $\mathbb{R}[u, v]$ to the action of $\mathfrak{sl}_2(\mathbb{R})$. The Lie group $\text{SL}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ in the obvious way and induces an action on $C^\infty(\mathbb{R}^2)$ (as well as on $\mathbb{R}[u, v]$)

$$(gf)(p) = f(g^T p), \quad g \in \text{SL}_2(\mathbb{R}), \quad f \in C^\infty(\mathbb{R}^2), \quad p \in \mathbb{R}^2.$$

This action induces the action of $\mathfrak{sl}_2(\mathbb{R})$ on $C^\infty(\mathbb{R}^2)$ determined by the rule

$$\begin{aligned} (Af)(p) &= \frac{d}{dt} f(\exp(tA)^T p)|_{t=0} = (\partial_u f, \partial_v f) \circ A^T \begin{pmatrix} u \\ v \end{pmatrix} \\ &= (u, v) A \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} f, \quad A \in \mathfrak{sl}_2(\mathbb{R}), \quad f \in C^\infty(\mathbb{R}^2), \quad p = (u, v) \in \mathbb{R}^2, \end{aligned}$$

where $\partial_u = \frac{\partial}{\partial u}$, $\partial_v = \frac{\partial}{\partial v}$ are considered as differential operators on $C^\infty(\mathbb{R}^2)$. The rule

$$A \mapsto (u, v) A \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix}$$

defines a representation of $\mathfrak{sl}_2(\mathbb{R})$ on $C^\infty(\mathbb{R}^2)$ or $\mathbb{R}[u, v]$. In particular, x, y, h act as operators

$$\begin{aligned} x &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto (u, v) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} = u\partial_v, \\ y &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto (u, v) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} = v\partial_u, \\ h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto (u, v) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} = u\partial_u - v\partial_v. \end{aligned}$$

◇

Representation $\mathbb{k}[u, v]$ is infinite dimensional and can be decomposed as a direct sum of finite-dimensional spaces

$$\mathbb{k}[u, v]^n = \langle u^0 v^n, u^1 v^{n-1}, \dots, u^n v^0 \rangle$$

which are spaces of homogeneous polynomials of total degree n . It is clear that operators x, y, h don't change the total degree, hence $\mathbb{k}[u, v]^n$ are subrepresentations. Let us show that there is an isomorphism between representations $\mathbb{k}[u, v]^n$ and $V(n)$ constructed in the previous theorem. We consider an isomorphism of vector spaces

$$\varphi : V(n) \rightarrow \mathbb{k}[u, v]^n, \quad v_i \mapsto \frac{1}{(n-i)!} u^{n-i} v^i.$$

Note that

$$h\varphi(v_i) = \frac{1}{(n-i)!} (u\partial_u - v\partial_v)(u^{n-i} v^i) = \frac{n-i-i}{(n-i)!} u^{n-i} v^i = (n-2i)\varphi(v_i).$$

$$x\varphi(v_i) = \frac{1}{(n-i)!} u \partial_v (u^{n-i} v^i) = \frac{i(n-i+1)}{(n-i+1)} u^{n-i+1} v^{i-1} = i(n-i+1) \varphi(v_{i-1}).$$

$$y\varphi(v_i) = \frac{1}{(n-i)!} v \partial_u (u^{n-i} v^i) = \frac{1}{(n-i-1)!} u^{n-i-1} v^{i+1} = \varphi(v_{i+1}).$$

This implies that φ is an isomorphism of $\mathfrak{sl}_2$ -representations.

4.2. Jordan decomposition in semisimple Lie algebras.

Remark 4.6. When we classified representations of $\mathfrak{sl}_2$, we noticed an interesting phenomenon. The action of h on every simple $\mathfrak{sl}_2$ -representation (hence all representations) was diagonal. It turns out that this is a general rule for semisimple Lie algebras. Namely, if $L \subset \mathfrak{gl}(V)$ is a semisimple algebra and $h \in L$ is a semisimple element (our example is $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2$) then, for any L -representation, the action of h on it is semisimple. $\diamond$

Lemma 4.7. *Let $L \subset \mathfrak{gl}(V)$ be a semisimple subalgebra. If $x \in L$ and $x = s + n$ is the Jordan decomposition in $\mathfrak{gl}(V)$, then $s, n \in L$.*

Proof. Let

$$N = \{y \in \mathfrak{gl}(V) \mid [y, L] \subset L\}$$

be the normalizer of L . Then N is a Lie algebra and $L \subset N$ is an ideal. For any L -submodule $W \subset V$, define the Lie algebra

$$L_W = \{y \in \mathfrak{gl}(V) \mid yW \subset W, \operatorname{tr}(y|_W) = 0\}.$$

As $L = [L, L]$ (for semisimple L), we obtain $\operatorname{tr}(y|_W) = 0 \ \forall y \in L$. Therefore $L \subset L_W$. Let M be the intersection of N and all L_W (for all L -submodules $W \subset V$). Then M is a Lie algebra and $L \subset M$ is an ideal.

We claim that $s, n \in M$. There is the Jordan decomposition $\operatorname{ad} x = \operatorname{ad} s + \operatorname{ad} n$ in $\mathfrak{gl}(L)$. As $\operatorname{ad} x(L) = [x, L] \subset L$, we obtain that $\operatorname{ad} s(L) \subset L$ and $\operatorname{ad} n(L) \subset L$ and therefore $s, n \in N$. For every L -submodule $W \subset V$, we have $xW \subset W$ and $\operatorname{tr}(x|_W) = 0$ as $x \in L = [L, L]$. Therefore the same is true for s and n , hence $s, n \in L_W$. This implies $s, n \in M$.

We claim that $M = L$ (hence $s, n \in L$). We can write $M = L \oplus L'$ for some L -submodule $L' \subset M$. Then $[L, L'] \subset L \cap L' = 0$, therefore every $y \in L'$ acts on a simple L -submodule $W \subset V$ as an L -endomorphism, hence as a scalar (by Schur's lemma). As $\operatorname{tr}(y|_W) = 0$, we obtain $y|_W = 0$. But V is a direct sum of simple L -submodules, hence $y = 0$. This implies $L' = 0$ and $M = L$. $\square$

Definition 4.8. Let L be a semisimple Lie algebra. Then

- (1) $x \in L$ is called semisimple if $\operatorname{ad} x \in \mathfrak{gl}(L)$ is semisimple.
- (2) $x \in L$ is called nilpotent if $\operatorname{ad} x \in \mathfrak{gl}(L)$ is nilpotent.

Theorem 4.9. *Let L be a semisimple Lie algebra.*

- (1) *For any $x \in L$, there exists a unique decomposition $x = s + n$ in L (called the abstract Jordan decomposition) such that $[s, n] = 0$, $\operatorname{ad} s$ is semisimple and $\operatorname{ad} n$ is nilpotent.*
- (2) *If $f : L \rightarrow M$ is a homomorphism of semisimple Lie algebras and $x = s + n$ is an abstract Jordan decomposition in L , then $f(x) = f(s) + f(n)$ is an abstract Jordan decomposition in M .*
- (3) *If $\rho : L \rightarrow \mathfrak{gl}(V)$ is a representation and $x = s + n$ is an abstract Jordan decomposition in L , then $\rho(x) = \rho(s) + \rho(n)$ is a usual Jordan decomposition in $\mathfrak{gl}(V)$.*

Proof. 1. The adjoint map $\operatorname{ad} : L \rightarrow \mathfrak{gl}(L)$ is injective as its kernel is equal to $Z(L) = 0$ (as L is semisimple). Applying the previous lemma to $L \simeq \operatorname{ad}(L) \subset \mathfrak{gl}(L)$, we obtain a decomposition $x = s + n$ in L such that $\operatorname{ad} x = \operatorname{ad} s + \operatorname{ad} n$ is the Jordan decomposition.

2. Consider M as an L -module via f and let $V = L \oplus M$ be the direct sum of L -modules. The representation map $\rho : L \rightarrow \mathfrak{gl}(V)$ is injective as already $\operatorname{ad} : L \rightarrow \mathfrak{gl}(L)$ is injective. Applying the previous lemma to $L \simeq \rho(L) \subset \mathfrak{gl}(V)$, we obtain a decomposition $x = s + n$ in L such that $[s, n] = 0$ and

- (1) $\rho(s) \in \mathfrak{gl}(V)$ is semisimple, hence $\operatorname{ad} s, \operatorname{ad} f(s)$ are semisimple.
- (2) $\rho(n) \in \mathfrak{gl}(V)$ is nilpotent, hence $\operatorname{ad} n, \operatorname{ad} f(n)$ are nilpotent.

This implies that $\operatorname{ad} x = \operatorname{ad} s + \operatorname{ad} n$ is the Jordan decomposition in $\mathfrak{gl}(L)$ (hence $x = s + n$ is the abstract Jordan decomposition in L) and $\operatorname{ad} f(x) = \operatorname{ad} f(s) + \operatorname{ad} f(n)$ is the Jordan decomposition in $\mathfrak{gl}(M)$ (hence $f(x) = f(s) + f(n)$ is the abstract Jordan decomposition in M).

3. As L is semisimple, we have $L = [L, L]$. This implies that $\rho(L) \subset \mathfrak{sl}(V)$. We proved that the Lie algebra homomorphism $\rho : L \rightarrow \mathfrak{sl}(V)$ preserves abstract Jordan decompositions. Therefore

we can assume that $L = \mathfrak{sl}(V) \subset \mathfrak{gl}(V) = M$. Let $x \in L$ and $x = s + n$ be the usual Jordan decomposition in $M = \mathfrak{gl}(V)$. We will show that it is also the abstract Jordan decomposition of x in L . By the previous lemma $s, n \in L$. By Lemma 3.6, we have a Jordan decomposition $\text{ad } x = \text{ad } s + \text{ad } n$ in $\mathfrak{gl}(M)$. It is also a Jordan decomposition in $\mathfrak{gl}(L)$, hence $x = s + n$ is an abstract Jordan decomposition in L as required. $\square$

Remark 4.10. Let L be a semisimple Lie algebra and $\rho : L \rightarrow \mathfrak{gl}(V)$ be a representation. The previous theorem implies that if $x \in L$ is semisimple (nilpotent), then $\rho(x)$ is semisimple (nilpotent). $\diamond$

4.2.1. *Weight spaces.* Let $x \in \mathfrak{gl}(V)$ be a semisimple endomorphism. For any $\lambda \in \mathbb{k}$, define the eigenspace

$$V_\lambda = \{v \in V \mid xv = \lambda v\}.$$

Then $V = \bigoplus_\lambda V_\lambda$. We will say that x acts on V_λ with weight λ . On the other hand, if $x, y \in \mathfrak{gl}(V)$ are semisimple and commute, then y preserves each summand V_λ :

$$x(yv) = yxv = \lambda(yv) \quad \forall v \in V_\lambda.$$

This implies that we can consider y as a semisimple endomorphism of V_λ and decompose V_λ according to the eigenvalues of y . In this way V decomposes into a direct sum $V = \bigoplus_{\lambda, \mu} V_{\lambda, \mu}$ of simultaneous eigenspaces

$$V_{\lambda, \mu} = \{v \in V \mid xv = \lambda v, yv = \mu v\}.$$

Generally, we obtain the following result.

Lemma 4.11. *Let $H \subset \mathfrak{gl}(V)$ be an abelian subalgebra consisting of semisimple elements. Then V has a simultaneous eigenspace (or weight space) decomposition*

$$V = \bigoplus_{\lambda \in H^*} V_\lambda,$$

where the weight spaces V_λ are defined by

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \ \forall h \in H\}, \quad \lambda \in H^*.$$

Proof. Let $H = \mathbb{k}x \oplus J$, where $0 \neq x \in H$. By induction we can decompose V into a direct sum of simultaneous eigenspaces of J . Then x acts on each of these eigenspaces as a semisimple endomorphism and we can decompose them further into a direct sum of x -eigenspaces. These smaller summands will be simultaneous eigenspaces of H . $\square$

Remark 4.12. Let L be a semisimple Lie algebra, and $H \subset L$ be an abelian subalgebra consisting of semisimple elements. For any representation $\rho : L \rightarrow \mathfrak{gl}(V)$, the image $\rho(H)$ consists of semisimple endomorphisms by Theorem 4.9. Then we also obtain a weight space decomposition

$$V = \bigoplus_{\lambda \in H^*} V_\lambda, \quad V_\lambda = \{v \in V \mid \rho(h)v = \lambda(h)v \ \forall h \in H\}.$$

The spaces V_λ are called weight spaces of V and the elements $\lambda \in H^*$ with $V_\lambda \neq 0$ are called weights of V . $\diamond$

4.3. Root space decomposition.

Definition 4.13. Let L be a semisimple Lie algebra. A subalgebra $H \subset L$ is called a Cartan subalgebra if

- (1) H is abelian and consists of semisimple elements.
- (2) H is maximal among all subalgebras satisfying (a).

Example 4.14. Let $L = \mathfrak{sl}_n$ and let $H \subset L$ consist of all diagonal matrices. Then H is abelian and all of its elements are semisimple. We will see later that H is a Cartan subalgebra. $\diamond$

Definition 4.15. Let L be a semisimple Lie algebra and $H \subset L$ be a Cartan subalgebra. We have a decomposition corresponding to $\text{ad} : H \rightarrow \mathfrak{gl}(L)$

$$L = \bigoplus_{\alpha \in H^*} L_\alpha, \quad L_\alpha = \{x \in L \mid [h, x] = \alpha(h) \cdot x \ \forall h \in H\}.$$

The finite set

$$R = \{\alpha \in H^* \setminus \{0\} \mid L_\alpha \neq 0\} \subset H^*$$

is called the root system of L relative to H . The elements of R are called roots, the spaces L_α are called root spaces. Decomposition

$$L = L_0 \oplus \bigoplus_{\alpha \in R} L_\alpha$$

is called the root space (or Cartan) decomposition.

Example 4.16. Let $L = \mathfrak{sl}_n$ and $H \subset L$ consist of the diagonal matrices. Let $\varepsilon_i \in H^*$, for $1 \leq i \leq n$, be defined by

$$\varepsilon_i(h) = a_i \quad h = \sum_{i=1}^n a_i E_{ii} \in H.$$

These vectors in H^* are not linearly independent as $\dim H = n - 1$. Given an element $h = \sum_i a_i E_{ii} \in H$, we have

$$\text{ad}_h(E_{ij}) = [h, E_{ij}] = (a_i - a_j)E_{ij} = (\varepsilon_i - \varepsilon_j)(h) \cdot E_{ij}.$$

This means that there are root spaces $L_{\alpha_{ij}} = \mathbb{k}E_{ij}$ for the roots

$$\alpha_{ij} = \varepsilon_i - \varepsilon_j \in H^* \quad \forall i \neq j.$$

The case $i = j$ corresponds to the zero weight $0 \in H^*$. Therefore L_0 consists of all diagonal matrices in L , hence $L_0 = H$. We obtain the following root space decomposition

$$L = H \oplus \bigoplus_{i \neq j} \mathbb{k}E_{ij}.$$

Note that $L_0 = H$ and all root spaces $L_{\alpha_{ij}} = \mathbb{k}E_{ij}$ are one-dimensional. We will see later that this is true in general. $\diamond$

Lemma 4.17. *We have*

- (1) $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.
- (2) If κ is the Killing form on L , then $\kappa(L_\alpha, L_\beta) = 0$ whenever $\alpha + \beta \neq 0$.
- (3) The Killing form κ induces a non-degenerate pairing $L_\alpha \times L_{-\alpha} \rightarrow \mathbb{k}$. In particular, the restriction of κ to L_0 is non-degenerate.

Proof. 1. If $x \in L_\alpha$, $y \in L_\beta$ and $h \in H$, then

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h) \cdot [x, y].$$

This implies that $[x, y] \in L_{\alpha+\beta}$.

2. If, additionally, $z \in L_\gamma$, then

$$\text{ad}_x \text{ad}_y(z) = [x, [y, z]] \in L_{\alpha+\beta+\gamma}.$$

This implies that $\text{ad}_x \text{ad}_y$ maps L_γ to $L_{\alpha+\beta+\gamma}$. Therefore, if $\alpha + \beta \neq 0$, then $\text{ad}_x \text{ad}_y$ is nilpotent. In particular, it has trace zero and

$$\kappa(x, y) = \text{tr}(\text{ad}_x \text{ad}_y) = 0.$$

3. If $x \in L_\alpha$ and $\kappa(x, L_{-\alpha}) = 0$, then $\kappa(x, L) = 0$ by (2). As κ is non-degenerate on L , we obtain $x = 0$. Similarly, if $x \in L_{-\alpha}$ and $\kappa(L_\alpha, x) = 0$, then $x = 0$. This implies that the restriction of κ to $L_\alpha \times L_{-\alpha}$ is non-degenerate. $\square$

Remark 4.18. For any $\alpha \in R$, the root space L_α consists of nilpotent elements. Indeed, for every $x \in L_\alpha$, we proved that ad_x maps L_β to $L_{\alpha+\beta}$. As L is finite dimensional, we can find some $n \geq 1$ such that $L_{n\alpha} = 0$ and $L_{n\alpha+\beta} = 0$ for all $\beta \in R$, hence $\text{ad}_x^n = 0$. $\diamond$

Lemma 4.19. Let $C_L(H) = \{x \in L \mid [x, H] = 0\}$ be the centralizer of H in L . Then

$$L_0 = C_L(H) = H.$$

Proof. The first equation follows from the definitions, namely,

$$L_0 = \{x \in L \mid \text{ad}_h(x) = 0 \ \forall h \in H\} = \{x \in L \mid [H, x] = 0\}.$$

For the second equation, we note first that $H \subset L_0$ as H is abelian. Let $x \in L_0$ and $x = s + n$ be its Jordan decomposition in L . Then $\text{ad } x = \text{ad } s + \text{ad } n$ is the Jordan decomposition in $\mathfrak{gl}(L)$. As

$$\text{ad } x(H) = [x, H] = 0,$$

we conclude that also $[s, H] = \text{ad } s(H) = 0$. This implies that $H + \mathbb{k}s$ is abelian and consists of semisimple elements. By the maximality of H , we obtain $s \in H$. Note also that $s, n \in L_0$. To show that $L_0 = H$, we need to show that the only nilpotent element of L_0 is zero as this would imply $n = 0$ and therefore $x = s \in H$.

We claim that L_0 is nilpotent. Given $x \in L_0$ with a Jordan decomposition $x = s + n$, we proved that $s \in H$, hence $\text{ad } s|_{L_0} = 0$. This implies that $\text{ad } x|_{L_0} = \text{ad } n|_{L_0}$ is nilpotent on L_0 . By Engel theorem L_0 is nilpotent. Therefore L_0 is solvable and by Lie theorem we can find a basis of L such that $\text{ad } x$ is upper-triangular for all $x \in L_0$. If $x \in L_0$ is nilpotent, then $\text{ad } x$ is nilpotent and is strictly upper-triangular. Therefore for any $y \in L_0$, $\text{ad}_x \text{ad}_y$ is strictly upper-triangular and has zero trace. Therefore

$$\kappa(x, y) = \text{tr}(\text{ad}_x \text{ad}_y) = 0 \quad \forall y \in L_0.$$

By the previous lemma, κ is non-degenerate on L_0 , hence $x = 0$. This implies that the only nilpotent element of L_0 is trivial, as required. $\square$

From now on we will denote the Killing form on L by $(-, -)$. Its restriction to $H = L_0$ is non-degenerate, hence induces an isomorphism

$$H \simeq H^*, \quad h \mapsto (h, -).$$

We will denote the induced bilinear form on H^* also by $(-, -)$. For any $\alpha \in H^*$, let h'_α denote the corresponding element in H . For any $\alpha \in H^*$ and $h \in H$, define

$$\langle \alpha, h \rangle = \langle h, \alpha \rangle := \alpha(h).$$

We have $h'_\alpha \mapsto \alpha = (h'_\alpha, -)$ under the above isomorphism, hence

$$\langle \alpha, h \rangle = \alpha(h) = (h'_\alpha, h).$$

In particular,

$$(\alpha, \beta) = (h'_\alpha, h'_\beta) = \langle \alpha, h'_\beta \rangle \quad \forall \alpha, \beta \in H^*.$$

Lemma 4.20. Let $\alpha \in R$. Then

(1) If $x \in L_\alpha$ and $y \in L_{-\alpha}$, then

$$[x, y] = (x, y)h'_\alpha.$$

(2) $(\alpha, \alpha) \neq 0$.

- (3) If $x_\alpha \in L_\alpha$ and $y_\alpha \in L_{-\alpha}$ are such that $(x_\alpha, y_\alpha) = \frac{2}{(\alpha, \alpha)}$ (they are unique up to a scalar) and $h_\alpha = \frac{2h'_\alpha}{(\alpha, \alpha)}$, then $\langle \alpha, h_\alpha \rangle = 2$ and the elements $x_\alpha, y_\alpha, h_\alpha$ satisfy the relations of $\mathfrak{sl}_2$. We will denote this algebra by $(\mathfrak{sl}_2)_\alpha$.

Proof. 1. For any $h \in H$, we have

$$([x, y], h) = (x, [y, h]) = -(x, [h, y]) = \langle \alpha, h \rangle (x, y) = (h'_\alpha, h)(x, y) = ((x, y)h'_\alpha, h).$$

Using the fact that $[x, y] \in L_0 = H$, $(x, y)h'_\alpha \in H$ and the Killing form is non-degenerate on H , we conclude that $[x, y] = (x, y)h'_\alpha$.

2. Assume that $\langle \alpha, \alpha \rangle = \langle \alpha, h'_\alpha \rangle = 0$. Let $0 \neq x \in L_\alpha$. As the Killing form is non-degenerate on $L_\alpha \times L_{-\alpha}$, there exists $y \in L_{-\alpha}$ such that $(x, y) \neq 0$. Then

$$h = [x, y] = (x, y)h'_\alpha \neq 0.$$

Consider the vector space M generated by x, y, h . We have

$$[h, x] = \langle \alpha, h \rangle x = \langle \alpha, (x, y)h'_\alpha \rangle x = 0$$

as $\langle \alpha, h'_\alpha \rangle = 0$. Similarly $[h, y] = 0$. This implies that M is a solvable Lie algebra. By Lie theorem, there exists a basis of L such that M acts on L by upper-triangular matrices. This implies that $\text{ad}_h = [\text{ad}_x, \text{ad}_y]$ acts by a strictly upper-triangular matrix, hence is nilpotent. But $h \in H$ is also semisimple, hence $h = 0$. A contradiction.

3. We have

$$\langle \alpha, h_\alpha \rangle = \frac{2}{(\alpha, \alpha)} \langle \alpha, h'_\alpha \rangle = \frac{2}{(\alpha, \alpha)} (\alpha, \alpha) = 2.$$

Moreover,

$$\begin{aligned} [x_\alpha, y_\alpha] &= (x_\alpha, y_\alpha)h'_\alpha = \frac{2h'_\alpha}{(\alpha, \alpha)} = h_\alpha, \\ [h_\alpha, x_\alpha] &= \langle \alpha, h_\alpha \rangle x_\alpha = 2x_\alpha. \end{aligned}$$

Similarly $[h_\alpha, y_\alpha] = -2y_\alpha$. □

Theorem 4.21. *We have*

- (1) R spans H^* .
- (2) The vector space

$$V = \mathbb{k}h_\alpha \oplus_{k \in \mathbb{Z} \setminus \{0\}} L_{k\alpha}$$

is an irreducible representation of $\mathfrak{sl}_2 = \langle x_\alpha, y_\alpha, h_\alpha \rangle$.

- (3) For any root $\alpha \in R$, $\dim L_\alpha = 1$.
- (4) If $\alpha, \beta \in R$, then $\langle \beta, h_\alpha \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.
- (5) For any root $\alpha \in R$, the only multiples of α which are also roots are $\pm\alpha$.
- (6) For any $\alpha \in R$, define a reflection

$$s_\alpha : H^* \rightarrow H^*, \quad s_\alpha(\lambda) = \lambda - \langle \lambda, h_\alpha \rangle \alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \alpha,$$

which leaves the hyperplane $\alpha^\perp = \{\lambda \in H^* \mid (\alpha, \lambda) = 0\}$ invariant and maps α to $-\alpha$. Then $s_\alpha(R) \subset R$.

- (7) For any roots α and $\beta \neq \pm\alpha$, the vector space

$$V = \bigoplus_{k \in \mathbb{Z}} L_{\beta+k\alpha}$$

is an irreducible representation of $\mathfrak{sl}_2 = \langle x_\alpha, y_\alpha, h_\alpha \rangle$.

- (8) If $\alpha, \beta, \alpha + \beta \in R$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

Proof. 1. Assume that R does not span H^* . Then there exists $0 \neq h \in H$ such that $\langle R, h \rangle = 0$. Then, for any $\alpha \in R$ and $z \in L_\alpha$,

$$[h, z] = \langle \alpha, h \rangle z = 0,$$

hence $[h, L_\alpha] = 0$. This implies that $[h, L] = 0$ and $h \in Z(L)$. As L is semisimple, its center is zero, hence $h = 0$. A contradiction.

2. The vector space V is a representation of $\mathfrak{sl}_2$. Indeed, h_α preserves each $L_{k\alpha}$ and $[h_\alpha, h_\alpha] = 0$. The element $x_\alpha \in L_\alpha$ maps $L_{k\alpha}$ to $L_{(k+1)\alpha}$ for $k \neq 0, -1$ and maps h_α to L_α . On the other hand, for any $z \in L_{-\alpha}$, we have $[x_\alpha, z] = (x_\alpha, z)h'_\alpha \in \mathbb{k}h_\alpha$. Therefore $[x_\alpha, L_{-\alpha}] \subset \mathbb{k}h_\alpha$. This implies $[x_\alpha, V] \subset V$. Similarly $[y_\alpha, V] \subset V$. For any $z \in L_{k\alpha}$, we have

$$[h_\alpha, z] = \langle k\alpha, h_\alpha \rangle z = 2kz,$$

hence z has weight $2k$. On the other hand, the elements of $\mathbb{k}h_\alpha$ have weight zero. Representation V of $\mathfrak{sl}_2$ is irreducible as $V_0 = \mathbb{k}h_\alpha$ has dimension one (here we use classification of irreducible representations of $\mathfrak{sl}_2$).

3. From the previous statement we obtain that the weight space $V_2 = L_\alpha$ has dimension one.

4. Let us consider L as a representation of $\mathfrak{sl}_2 = \langle x_\alpha, y_\alpha, h_\alpha \rangle$. We know that all weights of this representation are integers. In particular, for $z \in L_\beta$,

$$[h_\alpha, z] = \langle \beta, h_\alpha \rangle z,$$

hence z has weight $\langle \beta, h_\alpha \rangle$ which is an integer.

5. Assume that $\alpha, \beta \in R$ and $\beta = c\alpha$ for some $c \in \mathbb{k} \setminus \{\pm 1\}$. Then by (4)

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = 2c \in \mathbb{Z}, \quad \frac{2(\alpha, \beta)}{(\beta, \beta)} = \frac{2}{c} \in \mathbb{Z}.$$

Therefore $c = \pm \frac{1}{2}, \pm 2$. If $c = \pm \frac{1}{2}$, we can exchange α and β and assume $c = \pm 2$. If $c = -2$, we can substitute β by $-\beta$ and assume $\beta = 2\alpha$. Consider the irreducible representation

$$V = \mathbb{k}h_\alpha \oplus_{k \in \mathbb{Z} \setminus \{0\}} L_{k\alpha}$$

from (2). By the property of irreducible $\mathfrak{sl}_2$ representations, the map $x_\alpha : V_2 \rightarrow V_4$ is an isomorphism if $V_4 \neq 0$. But $V_2 = L_\alpha$ is one-dimensional, generated by x_α , and $[x_\alpha, x_\alpha] = 0$. Therefore $x_\alpha : V_2 \rightarrow V_4$ is zero and $V_4 = 0$. We obtain from $L_\beta = L_{2\alpha} = V_4 = 0$, that β is not a root.

6. Let $\beta \in R$ and $n = \langle \beta, h_\alpha \rangle$. We have to show that $s_\alpha(\beta) = \beta - n\alpha$ is a root. As before, we consider $V = L$ as a representation of $\mathfrak{sl}_2$. We have seen that elements of L_β have weight $\langle \beta, h_\alpha \rangle = n$. Assume that $n \geq 0$. Then $y_\alpha^n : V_n \rightarrow V_{-n}$ is an isomorphism. In particular, for any $0 \neq z \in L_\beta \subset V_n$, the element $y_\alpha^n(z)$ is non-zero. But

$$y_\alpha^n(z) = \text{ad}_{y_\alpha}^n(z) \in L_{\beta-n\alpha},$$

hence $\beta - n\alpha \in R$ as required. The proof for $n < 0$ is similar (consider $m = -n$ and an isomorphism $x_\alpha^m : V_{-m} \rightarrow V_m$).

7. The vector space V is a representation of $\mathfrak{sl}_2$. Indeed, h_α preserves each $L_{\beta+k\alpha}$, $x_\alpha \in L_\alpha$ maps $L_{\beta+k\alpha}$ to $L_{\beta+(k+1)\alpha}$ and $y_\alpha \in L_{-\alpha}$ maps $L_{\beta+k\alpha}$ to $L_{\beta+(k-1)\alpha}$. For any $z \in L_{\beta+k\alpha}$, we have

$$[h_\alpha, z] = \langle \beta + k\alpha, h_\alpha \rangle z = (\langle \beta, h_\alpha \rangle + 2k)z,$$

hence elements of $L_{\beta+k\alpha}$ have weight $\langle \beta, h_\alpha \rangle + 2k$. All of these weight spaces are one-dimensional, hence V is an irreducible representation.

8. We know that $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ and the latter space is one-dimensional. We obtain from (7) that the map $x_\alpha : L_\beta \rightarrow L_{\alpha+\beta}$ between nonzero weight spaces is an isomorphism. Therefore $[x_\alpha, L_\beta] \neq 0$ and $[L_\alpha, L_\beta] = L_{\alpha+\beta}$. $\square$

Lemma 4.22. *We have*

- (1) $(h, h') = \sum_{\alpha \in R} \langle \alpha, h \rangle \langle \alpha, h' \rangle$, for any $h, h' \in H$.
- (2) $(\beta, \gamma) = \sum_{\alpha \in R} (\alpha, \beta)(\alpha, \gamma)$, for any $\beta, \gamma \in H^*$.
- (3) $(\beta, \gamma) \in \mathbb{Q}$ for any $\beta, \gamma \in R$.
- (4) Let $E \subset H^*$ be the vector space over $\mathbb{Q}$ generated by R . Then the form $(-, -)$ is positive definite on E , that is, $(\gamma, \gamma) > 0$ for any $0 \neq \gamma \in E$.

Proof. 1. Consider the action of $\text{ad}_h \text{ad}_{h'}$ on every root space L_α and take the trace.

2. We have

$$(\beta, \gamma) = (h'_\beta, h'_\gamma) = \sum_{\alpha \in R} \langle \alpha, h'_\beta \rangle \langle \alpha, h'_\gamma \rangle = \sum_{\alpha \in R} (\alpha, \beta)(\alpha, \gamma).$$

3. We have $(\beta, \beta) = \sum_{\alpha \in R} (\alpha, \beta)^2$, hence $\frac{1}{(\beta, \beta)} = \sum_{\alpha \in R} \frac{(\alpha, \beta)^2}{(\beta, \beta)^2} \in \mathbb{Q}$ as $\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$. We obtain from $\frac{2(\beta, \gamma)}{(\beta, \beta)} \in \mathbb{Z}$ that $(\beta, \gamma) \in \mathbb{Q}$.

4. If $0 \neq \gamma \in E \subset H^*$ then

$$(\gamma, \gamma) = \sum_{\alpha \in R} (\alpha, \gamma)^2.$$

We can write $\gamma = \sum_{\beta \in R} c_\beta \beta$ for some $c_\beta \in \mathbb{Q}$. Then $(\alpha, \gamma) = \sum_{\beta \in R} c_\beta (\alpha, \beta) \in \mathbb{Q}$ and $(\alpha, \gamma)^2 \geq 0$, for all $\alpha \in R$. There exists $\alpha \in R$ such that $(\alpha, \gamma) \neq 0$ as R generates H^* . This implies that $\sum_{\alpha \in R} (\alpha, \gamma)^2 > 0$. $\square$

4.4. Abstract root systems. Assume that $\mathbb{k} = \mathbb{C}$ and let $E \subset H^*$ be a real vector space generated by R . Let us formalize the properties of $R \subset E$.

Definition 4.23. Let E be a real vector space with a positive definite inner product (we call E a Euclidean vector space). An abstract root system is a finite subset $R \subset E \setminus \{0\}$ such that

- (1) R generates E .
- (2) For any $\alpha, \beta \in R$, the number $\frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ is an integer.
- (3) For any $\alpha \in R$, define a reflection

$$s_\alpha : E \rightarrow E, \quad \lambda \mapsto \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha.$$

Then $s_\alpha(R) \subset R$.

- (4) For any root $\alpha \in R$, the only multiples of α which are also roots are $\pm\alpha$.

Remark 4.24. Note that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\lambda) = \lambda$ if λ is orthogonal to α (i.e. $(\alpha, \lambda) = 0$). Therefore s_α should be interpreted as a reflection with respect to the hyperplane

$$H_\alpha = \alpha^\perp = \{\lambda \in E \mid (\alpha, \lambda) = 0\}.$$

Note that $s_\alpha^2 = \text{Id}$. ◇

For any root $\alpha \in E$, define the corresponding coroot $\alpha^\vee \in E^*$ (it was denoted by h_α previously) by

$$\langle \lambda, \alpha^\vee \rangle = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}.$$

Then

$$\langle \alpha, \alpha^\vee \rangle = 2, \quad s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Definition 4.25. The Weyl group W of the root system $R \subset E$ is the subgroup of $\text{GL}(E)$ generated by reflections s_α for $\alpha \in R$.

Lemma 4.26. *The Weyl group is a finite subgroup of $O(E)$ which preserves $R \subset E$.*

Proof. Every s_α is orthogonal as

$$\begin{aligned} (s_\alpha \lambda, s_\alpha \mu) &= \left(\lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha, \mu - \frac{2(\alpha, \mu)}{(\alpha, \alpha)}\alpha \right) \\ &= (\lambda, \mu) - \frac{4(\alpha, \lambda)(\alpha, \mu)}{(\alpha, \alpha)} + \frac{4(\alpha, \lambda)(\alpha, \mu)}{(\alpha, \alpha)^2}(\alpha, \alpha) = (\lambda, \mu). \end{aligned}$$

This implies that $W \subset O(E)$. We know that $s_\alpha(R) = R$. This implies that W preserves R . If some $w \in W$ acts trivially on R , then w acts trivially on E as R generates E . Therefore W is a subgroup of the group $\text{Aut } R$ of all permutations of R . As R is a finite set, $\text{Aut } R$ and W are also finite. □

Example 4.27. Let $R \subset E$ be the root system of $\mathfrak{sl}_n$. We identify E with $\{x \in \mathbb{R}^n \mid \sum x_i = 0\}$ and let

$$R = \{\alpha_{ij} = e_i - e_j \mid i \neq j\}.$$

For any root $\alpha = \alpha_{ij}$, the corresponding reflection s_α acts as

$$s_\alpha(x_1, \dots, x_i, \dots, x_j, \dots, x_n) = (x_1, \dots, x_j, \dots, x_i, \dots, x_n),$$

that is, permutes coordinates i and j . We can identify s_α with a transposition (ij) considered as an element of the symmetric group S_n . Then the Weyl group W , generated by all such transpositions, coincides with the symmetric group S_n . ◇

Lemma 4.28. *For any $w \in W$ and $\alpha \in R$, we have $s_{w\alpha} = ws_\alpha w^{-1}$.*

Proof. If $\lambda \in H_\alpha$, then $(w\alpha, w\lambda) = (\alpha, \lambda) = 0$ hence $w\lambda \in H_{w\alpha}$. This implies $wH_\alpha = H_{w\alpha}$. The element $ws_\alpha w^{-1}$ acts as identity on $wH_\alpha = H_{w\alpha}$ and maps $w\alpha$ to $ws_\alpha(\alpha) = -w\alpha$. The same is true about $s_{w\alpha}$. Therefore they are identical. □

4.5. Pairs of roots.

Theorem 4.29. *Let $\alpha, \beta \in R$ be two roots such that $\beta \neq \pm\alpha$ and $\|\beta\| \leq \|\alpha\|$. Let θ be the angle between α and β . Then we have only the following possibilities*

	$\langle \beta, \alpha^\vee \rangle$	$\langle \alpha, \beta^\vee \rangle$	θ
1	0	0	$\pi/2$
2	1	1	$\pi/3$
3	-1	-1	$2\pi/3$
4	1	2	$\pi/4$
5	-1	-2	$3\pi/4$
6	1	3	$\pi/6$
7	-1	-3	$5\pi/6$

Proof. We have $(\alpha, \beta) = \|\alpha\| \|\beta\| \cos \theta$. Let $m = \langle \beta, \alpha^\vee \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ and $n = \langle \alpha, \beta^\vee \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$ and . Then $m, n \in \mathbb{Z}$ and

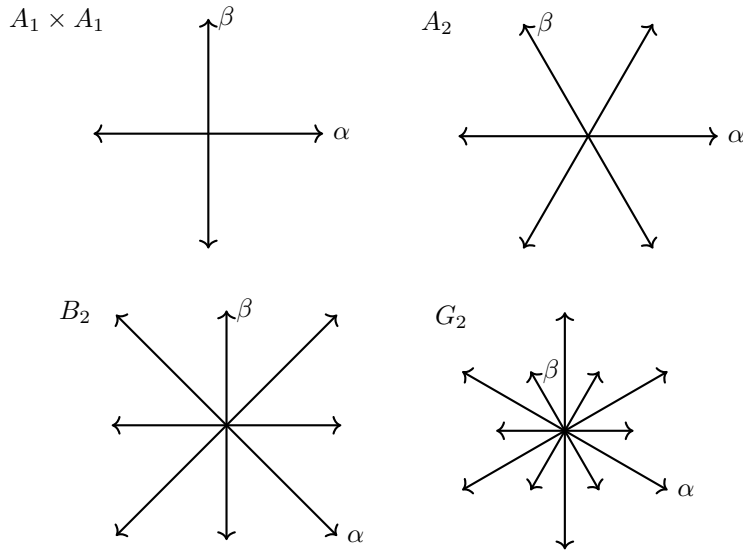
$$mn = 4 \frac{(\alpha, \beta)^2}{\|\alpha\|^2 \cdot \|\beta\|^2} = 4 \cos^2 \theta.$$

As $\beta \neq \pm\alpha$, we have $\cos \theta \neq \pm 1$. Therefore $mn \in \{0, 1, 2, 3\}$. We have

$$\frac{m}{n} = \frac{(\beta, \beta)}{(\alpha, \alpha)} \leq 1,$$

hence $|m| \leq |n|$. This implies that we can only have $m \in \{0, \pm 1\}$. If $m = 0$ then $n = 0$. Otherwise $m = \pm 1$ and $n = \pm 1, \pm 2, \pm 3$. The angle θ is determined from $4 \cos^2 \theta = mn$. $\square$

The root system that appears in (1) is called $A_1 \times A_1$, the root system in (2) and (3) is called A_2 , the root system in (4) and (5) is called B_2 and the root system in (6) and (7) is called G_2 .



Corollary 4.30. *Let $\alpha, \beta \in R$. Then*

- (1) *If $(\alpha, \beta) > 0$, then $\alpha - \beta \in R$.*
- (2) *If $(\alpha, \beta) < 0$, then $\alpha + \beta \in R$.*

Proof. It is enough to prove (1). Let $m = \langle \alpha, \beta^\vee \rangle > 0$, $n = \langle \beta, \alpha^\vee \rangle > 0$. Then $m = 1$ or $n = 1$. If $m = 1$, then

$$s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha - \beta \in R.$$

If $n = 1$, then

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha = \beta - \alpha \in R,$$

hence $\alpha - \beta \in R$. $\square$

4.6. Bases of root systems.

Definition 4.31. A subset $\Pi \subset R$ is called a base of R if

- (1) Π is a basis of E .
- (2) If $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$ is a root, then all n_α are simultaneously in $\mathbb{Z}_{\geq 0}$ or simultaneously in $\mathbb{Z}_{\leq 0}$.

The set

$$R_+ = \left\{ \beta = \sum_{\alpha \in \Pi} n_\alpha \alpha \in R \mid n_\alpha \geq 0 \ \forall \alpha \in \Pi \right\}$$

is called the positive root system (or the set of positive roots) and the set $R_- = -R_+$ is called the negative root system (or the set of negative roots). We have a decomposition $R = R_+ \sqcup R_-$, called a **polarization** of the roots system R corresponding to Π .

One can construct bases of R as follows. Let $H_\alpha = \alpha^\perp = \{ \lambda \in E \mid (\alpha, \lambda) = 0 \}$, for any root α , and let

$$\gamma \in E \setminus \bigcup_{\alpha \in R} H_\alpha,$$

called a regular vector. Define

$$R_+(\gamma) = \{ \alpha \in R \mid (\alpha, \gamma) > 0 \}, \quad R_-(\gamma) = \{ \alpha \in R \mid (\alpha, \gamma) < 0 \}.$$

Then $R_-(\gamma) = -R_+(\gamma)$ and $R = R_+(\gamma) \sqcup R_-(\gamma)$. The sets $R_+(\gamma)$ and $R_-(\gamma)$ are called the sets of positive and negative roots respectively.

Definition 4.32. We say that an element $\alpha \in R_+(\gamma)$ is a simple root if it can not be written as a sum of two positive roots. Denote the set of all simple roots by $\Pi(\gamma)$.

Lemma 4.33. Any positive root can be written as a sum of simple roots.

Proof. If $\alpha \in R_+(\gamma)$ is not simple, then $\alpha = \alpha' + \alpha''$ for some $\alpha', \alpha'' \in R_+(\gamma)$. This implies that $(\alpha', \gamma) < (\alpha, \gamma)$ and $(\alpha'', \gamma) < (\alpha, \gamma)$. Applying induction on (α, γ) (there is a finite set of values for $\alpha \in R_+(\gamma)$), we obtain representations of α' and α'' as sums of simple roots. Then we can represent α as a sum of simple roots. $\square$

Theorem 4.34. For any regular $\gamma \in E$, the set of simple roots $\Pi(\gamma)$ is a base of R . Any base of the root system R has this form.

Proof. 1. We have seen that any positive root is a linear combination of $\Pi = \Pi(\gamma)$. On the other hand R generates E . This implies that Π generates E .

2. We have $(\alpha, \beta) \leq 0$ for (distinct) $\alpha, \beta \in \Pi$. If $(\alpha, \beta) > 0$, then we have seen that $\beta' = \alpha - \beta \in R$. If $\beta \in R_+(\gamma)$, then $\alpha = \beta + \beta'$ is not simple. If $\beta' \in R_-(\gamma)$, then $\beta = \alpha + (-\beta')$ is not simple. This is a contradiction.

3. Let us show that elements of Π are linearly independent. Assume that $\sum_{\alpha \in \Pi} n_\alpha \alpha = 0$ for some $n_\alpha \in \mathbb{R}$. Let $S = \{ \alpha \mid n_\alpha > 0 \}$ and $T = \{ \alpha \mid n_\alpha < 0 \}$. Then $\lambda := \sum_{\alpha \in S} n_\alpha \alpha = \sum_{\beta \in T} t_\beta \beta$, where $t_\beta = -n_\beta$. Therefore

$$(\lambda, \lambda) = \sum_{\alpha \in S, \beta \in T} n_\alpha t_\beta (\alpha, \beta) \leq 0$$

as $n_\alpha > 0$, $t_\beta > 0$ and $(\alpha, \beta) \leq 0$. This implies that $\lambda = 0$. Therefore

$$0 = (\gamma, \lambda) = \sum_{\alpha \in S} n_\alpha (\gamma, \alpha).$$

But $n_\alpha > 0$ and $(\gamma, \alpha) > 0$ as $\alpha \in R_+$. We conclude that $S = \emptyset$ and similarly $T = \emptyset$. This implies that elements of Π are linearly independent.

4. We have seen that every $\beta \in R_+(\gamma)$ is a sum of simple roots, hence $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}_{\geq 0}$. Similarly, every $\beta \in R_-(\gamma)$ can be written in the form $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}_{\leq 0}$. This implies that $\Pi = \Pi(\gamma)$ is a base of R .

5. Assume that $\Pi \subset R$ is a base of R and let $\gamma \in E$ be such that $(\gamma, \alpha) > 0$ for all $\alpha \in \Pi$. We claim that $\Pi = \Pi(\gamma)$. We have $\Pi \subset R_+(\gamma)$, hence $R_+ \subset R_+(\gamma)$. Similarly $R_- \subset R_-(\gamma)$. This implies

$R_+ = R_+(\gamma)$. Any $\alpha \in \Pi$ is a simple root. Indeed, if $\alpha = \alpha' + \alpha''$, for $\alpha' = \sum_{\beta \in \Pi} n'_\beta \beta \in R_+$ and $\alpha'' = \sum_{\beta \in \Pi} n''_\beta \beta \in R_+$, then $n'_\beta + n''_\beta = 0$ for $\beta \neq 0$ and $n'_\alpha + n''_\alpha = 1$. This implies $\alpha' = 0$ or $\alpha'' = 0$, a contradiction. Therefore $\Pi \subset \Pi(\gamma)$. We conclude that $\Pi = \Pi(\gamma)$ as they are both bases of E and have the same number of elements. $\square$

Example 4.35. Let $R \subset E$ be the root system of $\mathfrak{sl}_n$. We identify E with $\{x \in \mathbb{R}^n \mid \sum x_i = 0\}$ (corresponds to the traceless diagonal matrices) and define

$$R = \{\alpha_{ij} = e_i - e_j \mid i \neq j\}.$$

Choose $\gamma = (n, n-1, \dots, 1) \in \mathbb{R}^n$. Then

$$(\gamma, \alpha_{ij}) = (\gamma, e_i) - (\gamma, e_j) = (n+1-i) - (n+1-j) = j-i > 0$$

if and only if $i < j$. Therefore

$$R_+ = \{e_i - e_j \mid i < j\}.$$

For any $i < j$, we can write

$$e_i - e_j = (e_i - e_{i+1}) + \dots + (e_{j-1} - e_j).$$

Therefore the simple roots are

$$\alpha_i = e_i - e_{i+1}, \quad 1 \leq i \leq n-1$$

and $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\}$ is a base of R . Note that Π is a basis of E . $\diamond$

4.7. Dynkin diagrams and classification. Let $r = \dim E$ (called the rank of the root system $R \subset E$) and let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be a base of R . For any simple root $\alpha_i \in \Pi$, let $s_i = s_{\alpha_i} \in W$ be the corresponding reflection, called a simple reflection.

Theorem 4.36. *We have*

- (1) *Simple reflections generate the Weyl group W .*
- (2) $W\Pi = R$.

Corollary 4.37. *R can be recovered from Π .*

Proof. We can recover W as the group generated by simple reflections. Then we recover R as $R = W\Pi$. $\square$

Definition 4.38. A Cartan matrix A of the base $\Pi \subset R$ is the $r \times r$ matrix with entries

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

Lemma 4.39. *We have*

- (1) $a_{ii} = 2$.
- (2) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$.
- (3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.
- (4) $a_{ij}a_{ji} \in \{0, 1, 2, 3\}$.

Proof. 1. Is obvious.

2. Was proved earlier.

3. $a_{ij} = 0$ if and only if $(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i) = 0$.

4. We proved more generally that $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$ for any roots α, β with $\beta \neq \pm\alpha$. $\square$

Example 4.40. Consider the root system R of $\mathfrak{sl}_n$ with simple roots

$$\alpha_i = e_i - e_{i+1}, \quad 1 \leq i \leq n-1.$$

Then

$$(\alpha_i, \alpha_i) = (e_i - e_{i+1}, e_i - e_{i+1}) = 2,$$

$$(\alpha_i, \alpha_{i+1}) = (e_i - e_{i+1}, e_{i+1} - e_{i+2}) = -1$$

and $(\alpha_i, \alpha_j) = 0$ for $|i - j| > 1$. This implies $a_{ii} = 2$,

$$a_{i,i+1} = a_{i+1,i} = \frac{2(\alpha_i, \alpha_{i+1})}{(\alpha_i, \alpha_i)} = -1$$

and $a_{ij} = 0$ for $|i - j| > 1$. $\diamond$

The integers a_{ij} and a_{ji} are both ≤ 0 for $i \neq j$. The property $a_{ij}a_{ji} \in \{0, 1, 2, 3\}$ implies that in order to determine a_{ij} and a_{ji} it is enough to know $a_{ij}a_{ji}$ and if

$$\frac{a_{ij}}{a_{ji}} = \frac{2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)}{2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)} = \frac{\|\alpha_j\|^2}{\|\alpha_i\|^2} \leq 1$$

or ≥ 1 . The first inequality means that the root α_j is shorter than the root α_i . Define the Dynkin diagram of (R, Π) as follows. It has vertices $1, \dots, r$ and the number of edges between i and j , for $i \neq j$, is equal to $a_{ij}a_{ji}$. Moreover, if α_i and α_j have different length, we draw an arrow in the direction of the shorter root. This diagram uniquely determines the Cartan matrix.

Remark 4.41. The Weyl group W has generators $s_1, \dots, s_r$ subject to relations

- (1) $s_i^2 = 1$,
- (2) $(s_i s_j)^{m_{ij}} = 1$, where m_{ij} are defined as follows

$a_{ij}a_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

Groups of this form (with arbitrary m_{ij}) are called Coxeter groups. They are not finite in general. Note, in particular, that W is determined by the Cartan matrix. $\diamond$

Example 4.42. If $R \subset E$ is the root system of $\mathfrak{sl}_n$ and the simple roots are

$$\alpha_i = e_i - e_{i+1}, \quad 1 \leq i \leq n-1,$$

then the simple reflections correspond to transpositions $s_i = (i, i+1)$. One can show that they generate $W = S_n$ and subject to the relations

$$s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad s_i s_j = s_j s_i \quad |i - j| > 1.$$

$\diamond$

Definition 4.43. A root system $R \subset E$ is called reducible if $R = R_1 \sqcup R_2$ for non-empty subset R_1, R_2 such that $(R_1, R_2) = 0$. Otherwise R is called irreducible.

Theorem 4.44. The Dynkin diagrams of irreducible root systems are

- (1) A_n :
- (2) B_n :
- (3) C_n :
- (4) D_n :
- (5) E_6 :
- (6) E_7 :
- (7) E_8 :
- (8) F_4 :
- (9) G_2 :

Theorem 4.45 (Serre relations). Let L be a semisimple Lie algebra, $H \subset L$ be a Cartan subalgebra, $R \subset H^*$ be the corresponding roots system, $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset R$ be a base of R , and $R_+ \subset R$ be the corresponding positive root system. For any simple root $\alpha_i \in \Pi$, let $x_i \in L_{\alpha_i}$, $y_i \in L_{-\alpha_i}$ and $h_i = h_{\alpha_i} \in H$ be such that x_i, y_i, h_i form a standard basis of $\mathfrak{sl}_2$. Then the elements x_i, y_i, h_i for $i = 1, \dots, r$ generate the Lie algebra L and subject to the relations

- (1) $[h_i, h_j] = 0$,
- (2) $[h_i, x_j] = a_{ij}x_j$ and $[h_i, y_j] = -a_{ij}y_j$,
- (3) $[x_i, y_j] = \delta_{ij}h_i$,
- (4) $(\text{ad } x_i)^{1-a_{ij}}x_j = 0$ and $(\text{ad } y_i)^{1-a_{ij}}y_j = 0$,

where $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. Conversely, given a root system R with a base $\Pi \subset R$ as above, the Lie algebra $L(R)$ with generators x_i, y_i, h_i as above and the above relations is a finite-dimensional semisimple Lie algebra with a root system R . Irreducible root systems correspond to simple Lie algebras.

5. HIGHEST WEIGHT REPRESENTATIONS

5.1. Universal enveloping algebra. Given a vector space V over a field $\mathbb{k}$, let

$$T^0V = \mathbb{k}, \quad T^1V = V, \quad T^2V = V \otimes V, \quad T^nV = V \otimes \dots \otimes V.$$

Define an associative product on $T(V) = \bigoplus_{i \geq 0} T^iV$ by

$$(v_1 \otimes \dots \otimes v_m) \cdot (w_1 \otimes \dots \otimes w_n) = v_1 \otimes \dots \otimes v_m \otimes w_1 \otimes \dots \otimes w_n.$$

Then $T(V)$ is an associative graded algebra with a unit $1 \in T^0V$. This algebra is called the tensor algebra of V . It has a universal property, that for any associative algebra A and a linear map $f : V \rightarrow A$, there exists a unique algebra homomorphism $\bar{f} : T(V) \rightarrow A$ making the following diagram commute

$$\begin{array}{ccc} V & \xrightarrow{i} & T(V) \\ & \searrow f & \swarrow \bar{f} \\ & A & \end{array}$$

where $i : V = T^1V \hookrightarrow T(V)$ is an embedding. $\bar{f} : T(V) \rightarrow A$ is defined on T^nV by the rule

$$\bar{f}(v_1 \otimes \dots \otimes v_n) = f(v_1)f(v_2)\dots f(v_n).$$

Now let us consider a Lie algebra L and the corresponding tensor algebra $T(L)$. Using an embedding $i : L = T^1L \hookrightarrow T(L)$, we identify L with a subspace of $T(L)$. Let $I \subset T(L)$ be an ideal generated by elements

$$x \otimes y - y \otimes x - [x, y] \quad \forall x, y \in L$$

and let $U(L) = T(L)/I$ be the corresponding quotient algebra, called the **universal enveloping algebra** of L .

Theorem 5.1. *For any associative algebra A and for any Lie algebra homomorphism $f : L \rightarrow A$ (meaning that f is linear and $f([x, y]) = f(x)f(y) - f(y)f(x)$), there exists a unique algebra homomorphism $\bar{f} : U(L) \rightarrow A$ (of associative algebras) such that the following diagram commutes*

$$\begin{array}{ccc} L & \xrightarrow{i} & U(L) \\ & \searrow f & \swarrow \bar{f} \\ & A & \end{array}$$

Proof. We can uniquely extend the linear map $f : L \rightarrow A$ to an algebra homomorphism $f' : T(L) \rightarrow A$. For any $x, y \in L$, we have

$$f'(x \otimes y - y \otimes x - [x, y]) = f(x) \otimes f(y) - f(y) \otimes f(x) - f([x, y]) = 0$$

as $f : L \rightarrow A$ is a homomorphism of Lie algebras. This implies that $x \otimes y - y \otimes x - [x, y] \in \ker f'$ and therefore $I \subset \ker f'$. We conclude that $f' : T(L) \rightarrow A$ can be factored through $\bar{f} : T(L)/I \rightarrow A$. Uniqueness of $\bar{f}$ follows from the uniqueness of f' . $\square$

Remark 5.2. Let $\rho : L \rightarrow \text{End}(V)$ be a representation of a Lie algebra L . By the above result this map extends to an associative algebra homomorphism $\bar{\rho} : U(L) \rightarrow \text{End}(V)$. This means that we can consider V as a module over the associative algebra $U(L)$, where multiplication is given by

$$av := \bar{\rho}(a)v, \quad a \in U(L), v \in V.$$

Conversely, if V is a module over the associative algebra $U(L)$ then it is automatically a module over the Lie algebra L , by using the embedding $i : L \hookrightarrow U(L)$. $\diamond$

Remark 5.3. Let V be a vector space equipped with an abelian Lie algebra structure. The ideal $I \subset T(V)$ from the above construction is generated by the elements

$$x \otimes y - y \otimes x, \quad x, y \in V.$$

The corresponding algebra $U(V) = T(V)/I$ is called a symmetric algebra of V and is denoted by $S(V)$. If V has a basis $(x_1, \dots, x_n)$, then $S(V)$ is isomorphic to the algebra of polynomials $\mathbb{k}[x_1, \dots, x_n]$. $\diamond$

The following important result tells us that one can similarly describe a basis of the universal enveloping algebra $U(L)$.

Theorem 5.4 (Poincaré–Birkhoff–Witt Theorem). *Let $(x_1, \dots, x_n)$ be a basis of a Lie algebra L . Then the universal enveloping algebra $U(L)$ has a basis*

$$i(x_1)^{k_1} \dots i(x_n)^{k_n}, \quad k_1, \dots, k_n \in \mathbb{N},$$

where $i : L \hookrightarrow T(L) \rightarrow U(L)$ is a canonical embedding.

Let $\pi : T(L) \rightarrow U(L)$ be the canonical projection. Define a filtration $0 = U_{-1} \subset U_0 \subset U_1 \subset \dots$ of $U(L)$ by

$$U_m = \pi(T^0 L \oplus \dots \oplus T^m L).$$

Then $U_m U_n \subset U_{m+n}$ and there is an induced associative algebra structure on

$$\text{Gr } U(L) = \bigoplus_{i \geq 0} U_i / U_{i-1}.$$

If $x, y \in L \subset U_1$ then $xy - yx = [x, y]_L \in U_1$. More generally, one can show that for any $x \in U_m$ and $y \in U_n$, we have $xy - yx \in U_{m+n-1}$, hence this element induces zero in U_{m+n}/U_{m+n-1} . This implies that the algebra $\text{Gr } U(L)$ is commutative. The natural embedding $L = U_1/U_0 \hookrightarrow \text{Gr } U(L)$ can be considered as a homomorphism of abelian Lie algebras. It can be extended to an (associative) algebra homomorphism $S(L) \rightarrow \text{Gr } U(L)$ by the universal property of $S(L)$. Then the PBW theorem can be reformulated as follows.

Theorem 5.5 (Poincaré–Birkhoff–Witt Theorem). *The natural algebra homomorphism $S(L) \rightarrow \text{Gr } U(L)$ is an isomorphism.*

Corollary 5.6. *Let L be a Lie algebra and $L_1, L_2 \subset L$ be subalgebras such that $L = L_1 \oplus L_2$ as a vector space. Then $U(L) \simeq U(L_1) \otimes U(L_2)$ as a vector space.*

Let L be a semisimple Lie algebra, $H \subset L$ be a Cartan subalgebra and

$$L = H \oplus \bigoplus_{\alpha \in R} L_\alpha$$

be its root space decomposition. Let $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset R$ be a base of R , and $R_+ \subset R$ be the corresponding positive root system. Then we can write

$$L = L_- \oplus H \oplus L_+, \quad L_- = \bigoplus_{\alpha \in R_-} L_\alpha, \quad L_+ = \bigoplus_{\alpha \in R_+} L_\alpha.$$

This implies

$$U(L) \simeq U(L_-) \otimes U(H) \otimes U(L_+).$$

Note that $U(H) = S(H)$ is isomorphic to $\mathbb{k}[h_1, \dots, h_r]$, where $h_i = \alpha_i^\vee$ are simple coroots. The algebra $U(L_+)$ is generated by the elements $x_1, \dots, x_r$ and the algebra $U(L_-)$ is generated by the elements $y_1, \dots, y_r$.

5.2. Weights. Let L be a semisimple Lie algebra with a choice of a Cartan algebra, root system and a base. Representations in this section are not necessarily finite-dimensional, unless otherwise stated.

For any (f.d.) L -representation $\rho : L \rightarrow \mathfrak{gl}(V)$, the image of H consists of semisimple endomorphisms and we have a weight decomposition

$$V = \bigoplus_{\lambda \in H^*} V_\lambda,$$

where V_λ are weight spaces defined by

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \ \forall h \in H\}.$$

Define the set of weights of V

$$P(V) = \{\lambda \in H^* \mid V_\lambda \neq 0\}.$$

For any root $\alpha \in R \subset H^*$, there is a coroot $h_\alpha = \alpha^\vee \in H$.

Lemma 5.7. *We have $L_\alpha V_\lambda \subset V_{\lambda+\alpha}$, for any $\alpha \in R$ and $\lambda \in P(V)$.*

Proof. Let $x \in L_\alpha$ and $v \in V_\lambda$. Then, for any $h \in H$, we have

$$h(xv) = ([h, x] + xh)v = \alpha(h)xv + x(\lambda(h)v) = (\alpha(h) + \lambda(h))xv = \langle \lambda + \alpha, h \rangle xv.$$

Therefore $xv \in V_{\lambda+\alpha}$. □

Definition 5.8. Let V be an L -representation. A vector $0 \neq v \in V_\lambda$ (for some $\lambda \in H^*$) is called a highest weight (or primitive) vector if $x_\alpha v = 0$ for all $\alpha \in R_+$. Representation V is called a highest weight representation if v generates V . In this case λ is called the highest weight of V .

Lemma 5.9. *Every f.d. irreducible L -representation is a highest weight representation.*

Proof. The set of weights $P(V)$ is finite, hence there exists $\lambda \in P(V)$ such that $\lambda + \alpha \notin P(V)$ for all $\alpha \in R_+$. Let $0 \neq v \in V_\lambda$. Then $x_\alpha v \in V_{\lambda+\alpha} = 0$, hence $x_\alpha v = 0$, for all $\alpha \in R_+$. The vector v generates V as V is irreducible. □

Theorem 5.10. *Let V be a highest weight representation with highest weight λ . Then*

- (1) *V has a weight decomposition and weights of V are of the form $\lambda - \sum k_i \alpha_i$ with $k_i \in \mathbb{Z}_{\geq 0}$.*
- (2) *$\dim V_\lambda = 1$ and $\dim V_\mu < \infty$ for $\mu \in H^*$.*
- (3) *If V is f.d. then V is irreducible.*

Proof. 1. Let $v \in V_\lambda$ be a highest weight vector generating V . We have $L_+ v = 0$ and $Hv \subset V_\lambda$. Therefore $U(L_+)v = \mathbb{k}v$, $U(H)v = \mathbb{k}v$ and

$$V = U(L)v = U(L_-)v.$$

The space $U(L_-)v$ is generated by

$$y_{\beta_1} \cdots y_{\beta_k} v, \quad \beta_1, \dots, \beta_k \in R_+.$$

The above element has weight

$$\lambda - \sum \beta_i = \lambda_i - \sum_{i=1}^r m_i \alpha_i$$

for some $m_i \in \mathbb{Z}$. This proves (1).

2. If $v' = y_{\beta_1} \cdots y_{\beta_k} v \in V_\lambda$, then $\lambda = \lambda - \sum \beta_i$, hence $\beta_1 = \cdots = \beta_k = 0$. This implies that $v' = v$ and V_λ is generated by v . If V_μ is generated by elements of the form $y_{\beta_1} \cdots y_{\beta_k} v$, then they satisfy $\mu = \lambda - \sum \beta_i$. There is a finite number of tuples (β_i) in R_+ satisfying this condition.

3. If V is not irreducible, then $V = V' \oplus V''$ for non-trivial V', V'' . Then $V_\mu = V'_\mu \oplus V''_\mu$ for any weight μ . In particular, $V_\lambda = V'_\lambda \oplus V''_\lambda$. As $\dim V_\lambda = 1$, we have $V'_\lambda = V_\lambda$ or $V''_\lambda = V_\lambda$. Then $V' = V$ or $V'' = V$ as $v \in V_\lambda$ generates V . □

Theorem 5.11. *Let V be an irreducible f.d. representation. Then*

- (1) *V contains only one highest weight vector (up to a scalar).*

- (2) *Two irreducible (f.d.) representations have the same highest weight if and only if they are isomorphic.*

Proof. 1. Assume that $v \in V_\lambda$ and $v' \in V_{\lambda'}$ are two highest weight vectors. Then by the previous theorem

$$\lambda - \lambda' = \sum m_i \alpha_i, \quad \lambda' - \lambda = \sum m'_i \alpha_i$$

for some $m_i, m'_i \in \mathbb{Z}_{\geq 0}$. This implies $m_i = m'_i = 0$ and $\lambda = \lambda'$.

2. Let V_1 and V_2 be irreducible representations having the same highest weight λ and having highest weight vectors v_1, v_2 . Then the representation $W \subset V_1 \oplus V_2$ generated by the vector $v = (v_1, v_2) \in V_1 \oplus V_2$ is a highest weight representation. By the previous theorem W is irreducible. The composition $W \hookrightarrow V_1 \oplus V_2 \rightarrow V_1$ is non-trivial, hence is an isomorphism (as W and V_1 are irreducible). Similarly, $W \simeq V_2$ and we conclude that $V_1 \simeq V_2$. $\square$

5.3. Construction of highest weight representations. Let V be a highest weight representation of L with a highest weight λ and a highest weight vector $v \in V_\lambda$. Then we have

$$hv = \lambda(h)v \quad \forall h \in H, \quad xv = 0 \quad \forall x \in L_+.$$

Let us construct a universal module with these properties. Consider the left ideal $J(\lambda) \subset U(L)$ generated by L_+ and elements $h - \lambda(h)$ for all $h \in H$. Define the module

$$M(\lambda) = U(L)/J(\lambda)$$

called Verma module. If V is the highest weight module with highest weight vector v as above, then there exists a surjective homomorphism of $U(L)$ (or L) modules

$$M(\lambda) \rightarrow V, \quad 1 \mapsto v.$$

One can construct $M(\lambda)$ also as follows. Let $\mathbb{k}_\lambda = \mathbb{k}1_\lambda$ be a one-dimensional module over $H \oplus L_+$, where L_+ acts trivially and H acts as

$$h1_\lambda = \lambda(h)1_\lambda \quad \forall h \in H.$$

Then $\mathbb{k}_\lambda$ is also a module over $U(H \oplus L_+)$ and we define

$$M(\lambda) = U(L) \otimes_{U(H \oplus L_+)} \mathbb{k}_\lambda.$$

It is isomorphic to $U(L_-)$ as a vector space.

Theorem 5.12. *We have*

- (1) $M(\lambda) = U(L_-)v_\lambda$.
- (2) *Weights are of the form $\lambda - \sum m_i \alpha_i$ for $m_i \in \mathbb{Z}_{\geq 0}$.*
- (3) *Each submodule has a weight space decomposition.*

Lemma 5.13. *For every $\lambda \in H^*$ there exists an irreducible module $V(\lambda)$ with highest weight λ .*

Proof. If $W \subset M(\lambda)$ is a submodule, then $W = \bigoplus W_\mu$ has a weight space decomposition. If $W \subset M(\lambda)$ is proper, then $W_\lambda = 0$ as otherwise $W_\lambda = M(\lambda)_\lambda$ generates $M(\lambda)$ and $W = M(\lambda)$. Taking the sum over all proper submodules, we obtain the maximal submodule $W \subset M(\lambda)$ (it is proper, as it satisfies $W_\lambda = 0$). Then the quotient $V(\lambda) = M(\lambda)/W$ is irreducible. $\square$