LECTURES ON LINEAR PROGRAMMING

DRAFT VERSION

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Conventions

(1) $\mathbb{R}^{m\times n}$ is the set (or vector space) of $m \times n$ matrices with entries in $\mathbb{R}$.
(2) $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the set (or vector space) of dimension n column vectors over $\mathbb{R}$.
(3) $\mathbb{R}^{1\times n}$ is the set (or vector space) of dimension n row vectors over $\mathbb{R}$.
(4) $\lambda^{m\times n} \in \mathbb{R}^{m\times n}$ is the matrix with all entries equal to $\lambda \in \mathbb{R}$.
(5) $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Similarly, $\mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$ and $\mathbb{N} = \mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.
(6) $[n] = \{1, \dots, n\}$ for $n \geq 0$. In particular, $[0] = \varnothing$. For $J \subset [n]$, let $\bar{J} = [n]\backslash J$.
(7) For $A \in \mathbb{R}^{m\times n}$, let A_{i*} denote the i-th row and $A_j = A_{*j}$ denote the j-th column.
(8) For $A \in \mathbb{R}^{m\times n}$ and $J \subset [n]$, let A_J be the matrix consisting of columns A_j with $j \in J$.
(9) For $x \in \mathbb{R}^n$ and $J \subset [n]$, let $x_J = (x_j)_{j\in J} \in \mathbb{R}^J$.
(10) A matrix $A \in \mathbb{R}^{m\times n}$ can be identified with the linear map $A\colon \mathbb{R}^n \to \mathbb{R}^m$

$$x \mapsto Ax = \left(\sum\nolimits_{j=1}^{n} a_{ij}x_j\right)_{i\in[m]}$$

For $J \subset [n]$, the matrix A_J corresponds to the restriction $A_J\colon \mathbb{R}^J \to \mathbb{R}^m$ of A.
(11) For $x, y \in \mathbb{R}^n$, let $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. Equivalently, $x \leq y \iff y - x \in \mathbb{R}^n_+$. This defines a partial order on $\mathbb{R}^n$. We write $x < y$ if $x \leq y$ and $x \neq y$.
(12) For $x, y \in \mathbb{R}^n$, we define the *dot product* $x \cdot y = x^\mathsf{T} y = \sum_{i=1}^n x_i y_i$.
(13) For $a_1, \dots, a_n \in \mathbb{R}^m$, let $\operatorname{rk}\{a_1, \dots, a_n\} = \operatorname{rk} A = \dim \operatorname{span}\{a_1, \dots, a_n\}$, where A is the matrix with columns a_i.
(14) For a subset $X \subset \mathbb{R}$, we define the maximum $\max X$ (if it exists) to be an element $y \in X$ such that $x \leq y$ for all $x \in X$. Similarly, we define the minimum $\min X$.
(15) For a subset $X \subset \mathbb{R}$, we define the *supremum* (or the least upper bound) $\sup X$ to be an element $y \in \mathbb{R} \sqcup \{\pm\infty\}$ such that
   (1) $x \leq y$ for all $x \in X$,
   (2) if $z \in \mathbb{R}$ is such that $x \leq z$ for all $x \in X$, then $y \leq z$.

   The supremum always exists and is unique. For example, $\sup \mathbb{R} = +\infty$ and $\sup \varnothing = -\infty$. If X has the maximum, then $\max X = \sup X$. Similarly, we define the *infimum* (or the greatest lower bound) $\inf X$.

1. Optimization problems

1.1. **Optimization problems.** An *optimization problem* is a problem of maximizing (or minimizing) a function $f\colon X \to \mathbb{R}$ on a set X. We will write such problem in the form

$$\begin{aligned} &\text{maximize} && f(x) \\ &\text{subject to} && x \in X \end{aligned} \tag{1}$$

or just

$$\max\{f(x) \,|\, x \in X\}.$$

(1) The set X is called the *feasible set* of the problem. Usually, the set X is given by a collection of equalities or inequalities, so it is unknown a priori if $X = \varnothing$.
(2) The elements of X are called *feasible solutions* (or feasible points).
(3) The problem is called *feasible* if $X \neq \varnothing$, and *infeasible* otherwise.
(4) The function f is called the *objective function* of the problem (or sometimes loss/cost function for minimization problems).
(5) The value $p^* = \sup\{f(x) \,|\, x \in X\} \in \mathbb{R} \cup \{\pm\infty\}$ is called the *optimal value*.
(6) We have $p^* = -\infty \iff X = \varnothing$ and $p^* = +\infty \iff f(x)$ is unbounded above.
(7) An element $x^* \in X$ such that $f(x^*) = p^*$ is called an *optimal solution* (or *optimal point*) of the problem. Equivalently, $f(x) \leq f(x^*)$ for all $x \in X$ (or $\max_{x\in X} f(x) = f(x^*)$).
(8) We say that the optimal value is *attained* or *achieved* if x^* as above exists.

Note that

$$-\max\{f(x) \,|\, x \in X\} = \min\{-f(x) \,|\, x \in X\},$$

hence the problems of maximization of f and minimization of $-f$ are equivalent.

Example 1.1. Consider the optimization problem $\max\{1 - 1/x \,|\, x \geq 1\}$. Here the feasible set is $X = [1, +\infty)$ and the objective function is $f(x) = 1 - 1/x$. The optimal value is $p^* = \sup_{x\geq 1} f(x) = 1$ and it is not attained.

Example 1.2. Consider the optimization problem $\max\{x \,|\, x \geq 1\}$. The optimal value is $p^* = +\infty$ and it is not attained.

Example 1.3. Consider the optimization problem $\max\{x_1 \,|\, x \in \mathbb{R}^2,\ x \leq 0\}$. Here the feasible set is $X = \{x = (x_1, x_2) \in \mathbb{R}^2 \,|\, x_1,\ x_2 \leq 0\}$ and the objective function is $f(x_1, x_2) = x_1$. The optimal value is $p^* = 0$ and the set of optimal points is $X_0 = \{(x_1, x_2) \in X \,|\, x_1 = 0\} = \{(0, t) \,|\, t \leq 0\}$.

A *linear programming problem* (to be studied later) is an optimization problem of the form

$$\max\bigl\{c^{\mathsf{T}} x \,\bigm|\, Ax \leq b\bigr\},$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Note that the objective function $f(x) = c^{\mathsf{T}} x$ is linear and the feasible set $P = \{x \in \mathbb{R}^n \,|\, Ax \leq b\}$ (called a *polyhedron*) is given by linear inequalities.

Example 1.4. The optimization problem $\max\{x_1 \,|\, x_1 + x_2 \leq 1,\ x_1, x_2 \geq 0\}$ is a linear programming problem. The optimal point is $(x_1, x_2) = (1, 0)$.

Example 1.5. The optimization problem $\max\{1 - x^2 \,|\, \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) x \leq \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\}$ is not linear as the objective function $f(x) = 1 - x^2$ is not linear. We have $P = \{x \in \mathbb{R} \,|\, \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) x \leq \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\} = [-1, 1]$. To obtain $\max\{f(x) \,|\, x \in P\}$ we need to check the endpoints $x = \pm 1$ and the local extrema

of f (points $x \in P$ where $f'(x) = 0$). We have $f'(x) = -2x = 0$ only for $x = 0 \in [-1, 1]$. The corresponding values are $f(\pm 1) = 1 - 1 = 0$ and $f(0) = 1 - 0 = 1$. Therefore $\max\{f(x) \,|\, x \in [-1, 1]\} = 1$, attained at $x = 0$.

1.2. **Least squares problem.** The least squares problem is the problem of finding

$$\min\{f(x) \mid x \in \mathbb{R}^n\}, \qquad f(x) = \|Ax - b\|,$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $\|x\| = (\sum_i x_i^2)^{1/2}$. If A_{i*} are the rows of A, then

$$f^2(x) = \|Ax - b\|^2 = \sum_{i=1}^{m} (A_{i*}x - b_i)^2$$

and we have to mimimize this sum of squares. We can interpret A as a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^m$. If A has rank n, then the above map is injective and defines an n-dimensional subspace $V = A(\mathbb{R}^n) \subset \mathbb{R}^m$ (if A is not injective, we can decompose $\mathbb{R}^n = \mathrm{Ker}(A) \oplus V$ and consider an injective restriction $A|_V\colon V \hookrightarrow \mathbb{R}^m$). The above optimization problem can be interpreted as finding the minimal distance between the points of V and b. It is attained at $v = Ax$ that is the orthogonal projection of b onto V. This implies $V \perp (Ax - b)$, hence

$$(Ay)^\mathsf{T}(Ax - b) = 0 \quad \forall y \in \mathbb{R}^n \implies A^\mathsf{T}(Ax - b) = 0 \implies A^\mathsf{T}Ax = A^\mathsf{T}b.$$

By our assumption $\operatorname{rk} A = n$, hence $A^\mathsf{T}A$ is invertible, and we obtain

$$x = (A^\mathsf{T}A)^{-1}A^\mathsf{T}b.$$

Example 1.6. Let $A = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$ and $f(x) = \|Ax - b\|$ for $x \in \mathbb{R}$. Then $A^\mathsf{T}A = \left(\begin{smallmatrix}1 & -1\end{smallmatrix}\right)\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) = 2$, hence the minimum of $f(x)$ is attained at $x = (A^\mathsf{T}A)^{-1}A^\mathsf{T}b = \frac{1}{2}(2 - 1) = \frac{1}{2}$. We have $f(x) = \left\|\left(\begin{smallmatrix}-3/2\\-3/2\end{smallmatrix}\right)\right\| = \sqrt{2}\frac{3}{2}$. Alternatively, we can write $f^2(x)$ explicitly (as a quadratic polynomial) and find its minimum.

1.2.1. *Fitting a line.* Assume that we have measurements $(x_i, y_i) \in \mathbb{R}^2$ of an experiment, for $1 \le i \le n$, and we want to find a line $y = w_1x + w_2$ that best fits the given points. We need to minimize (by choosing appropriate $w_1, w_2 \in \mathbb{R}$)

$$\sum_{i=1}^{n} (w_1x_i + w_2 - y_i)^2.$$

Consider the matrix $A \in \mathbb{R}^{n\times 2}$ with entries $a_{i1} = x_i$ and $a_{i2} = 1$. Then we can formulate the above problem as an optimization problem

$$\min\big\{\|Aw - y\| \;\big|\; w \in \mathbb{R}^2\big\}$$

This is the least squares problem considered earlier.

Problem 1: *Consider two points $(x_1, y_1) \in \mathbb{R}^2$ and $(x_2, y_2) \in \mathbb{R}^2$. The best fitting line will be the line that goes through these points. Apply the above approach to derive an explicit formula for the best fitting line.*

1.2.2. *Linear alternative.* Alternatively, we can look for $w \in \mathbb{R}^2$ that minimizes

$$\sum_{i=1}^{n} |w_1x_i + w_2 - y_i|.$$

This problem can be reduced to a linear problem using the following trick. We introduce auxiliary variables $u_i = |w_1x_i + w_2 - y_i|$ for $1 \le i \le n$. Then the problem is equivalent to

$$\begin{aligned} \text{minimize} \quad & u_1 + \cdots + u_n \\ \text{subject to} \quad & -u_i \le w_1x_i + w_2 - y_i \le u_i \quad \forall 1 \le i \le n \end{aligned}$$

1.3. **Lagrangian multipliers.** Consider an optimization problem

$$\begin{aligned} &\text{maximize} && f(x) \\ &\text{subject to} && g(x) = 0 \end{aligned}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}^m$ are two C^1-functions.

Theorem 1.7 (Lagrange multiplier theorem)**.** *Let x^* be an optimal solution such that the Jacobian matrix $Dg(x^*) = \left(\frac{\partial g_i}{\partial x_j}(x^*)\right)_{i,j} \in \mathbb{R}^{m\times n}$ has rank m. Then there exists a unique $\lambda \in \mathbb{R}^m$ (called a Lagrange multiplier) such that*

$$Df(x^*) = \lambda^{\mathsf{T}} Dg(x^*).$$

Sketch of the proof. By the Fredholm alternative 4.27, for arbitrary $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$,

$$\exists \lambda \in \mathbb{R}^m\colon\ b^{\mathsf{T}} = \lambda^{\mathsf{T}} A \iff b^{\mathsf{T}} v = 0\ \forall v \in \operatorname{Ker} A.$$

For $A = Dg(x^*)$ and $b^{\mathsf{T}} = Df(x^*)$, the statement $Av = 0$ means that v is a tangent vector of $M = \{x \in \mathbb{R}^n \mid g(x) = 0\}$ at x^* and the statement $b^{\mathsf{T}} v = 0$ means that the directional derivative $D_v f(x^*) = Df(x^*)v$ is zero. If x^* is an optimal solution, then all directional derivatives of f along M should be zero, hence we can apply the Fredholm alternative. □

We conclude that to find potential optimal solutions we need to solve the system of equations

$$g(x^*) = 0, \qquad Df(x^*) = \lambda^{\mathsf{T}} Dg(x^*). \tag{2}$$

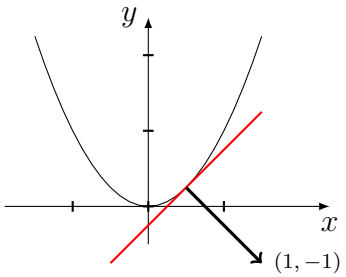
Remark 1.8. The above problem can be formulated using the *Lagrangian function*

$$\mathcal{L}\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \qquad \mathcal{L}(x, \lambda) = f(x) - \lambda^{\mathsf{T}} g(x).$$

Its critical (or stationary) points, meaning points where all derivatives of $\mathcal{L}$ vanish, are exactly the points $(x^*, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ such that (2) is satisfied.

Example 1.9. Consider the problem

$$\begin{aligned} &\text{maximize} && x - y = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \\ &\text{subject to} && y = x^2 \end{aligned}$$



Let $f(x, y) = x - y$ and $g(x, y) = y - x^2$. The required system of equations is

$$y = x^2, \qquad (1, -1) = \lambda(-2x, 1).$$

We obtain $\lambda = -1$, $x = 1/2$ and $y = 1/4$. The optimal solution is $(x, y) = (1/2, 1/4)$.

1.4. **Gradient descent.** Assume that we need to find a local minimum of a C^1-function $f\colon \mathbb{R}^n \to \mathbb{R}_+$. We have

$$f(x') = f(x) + \nabla f(x) \cdot (x' - x) + o(x' - x),$$

where $\nabla f(x) = (Df(x))^{\mathsf{T}} = \left(\frac{\partial f}{\partial x_i}(x)\right)_i \in \mathbb{R}^n$ is the *gradient* of f.

Remark 1.10. Here $o(x)$ means the class of functions $f\colon U \to \mathbb{R}$ (where $U \subset \mathbb{R}^n$ is a neighborhood of 0) such that $f(x)/\left\|x\right\| \to 0$ as $x \to 0$.

Assuming that the gradient is nonzero, we substitute x by

$$x' = x - \varepsilon \nabla f(x),$$

where $\varepsilon > 0$ is a small parameter called the *learning rate*. Then $x' - x = -\varepsilon \nabla f(x)$ and

$$f(x') \sim f(x) - \varepsilon \left\|\nabla f(x)\right\|^2 < f(x).$$

Similarly, we can substitute x' to decrease $f(x')$ and so on. This method of finding a local minimum (or its approximation) is called *gradient descent*.

This approach is used in machine learning as follows. Assume that we have a loss function

$$L\colon W \times Z \to \mathbb{R}_+,$$

where $W = \mathbb{R}^k$ (the set of weights or parameters) and $L(-, z)\colon W \to \mathbb{R}$ is a C^1 function for $z \in Z$. Given *training data* $z = (z_i)_{i=1}^N \in Z^N$, we need to find $w \in W$ that minimizes the *empirical risk* $\frac{1}{N}\sum_{i=1}^N L(w, z_i)$. We choose random subsets $I \subset [N]$ of batch size $b \geq 1$ and apply gradient descent to the function

$$f\colon \mathbb{R}^k \to \mathbb{R}, \qquad w \mapsto \frac{1}{b}\sum\nolimits_{i\in I} L(w, z_i). \tag{3}$$

After repeating this process several times we find $w \in W$ that minimizes $L(w, -)$ on the training data. This method is called *stochastic gradient descent* (SGD).

Example 1.11. Assume that we have points $z_i = (x_i, y_i) \in \mathbb{R}^2$ for $1 \leq i \leq N$ and we want to find a line $y = w_1 x + w_0$ that best fits these points. For $w = (w_0, w_1) \in W = \mathbb{R}^2$ and $z = (x, y) \in Z = \mathbb{R}^2$, we define

$$L(w, z) = (y - w_1 x - w_0)^2$$

and apply SGD to find $w \in W$ that minimizes $L(w, -)$ on the training data $(z_i)_{i=1}^N \in Z^N$.

2. Linear programming

2.1. **Linear programming problems.** A *linear programming problem* (or *linear program*) is a problem of maximizing (or minimizing) a linear function subject to a finite number of linear inequalities (constraints), meaning an optimization problem

$$\begin{aligned} &\text{maximize} && c^{\mathsf{T}}x \\ &\text{subject to} && Ax \le b \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. We also write it in the form

$$\max\{c^{\mathsf{T}}x \,|\, Ax \le b\}.$$

The objective function $f(x) = c^{\mathsf{T}}x$ is linear. The feasible set of the problem is

$$P = \{x \in \mathbb{R}^n \,|\, Ax \le b\}.$$

Such set is called a *polyhedron* and is given by linear inequalities. It is the intersection of the half-spaces $\{x \in \mathbb{R}^n \,|\, A_{i*}x \le b_i\}$, where A_{i*} is the i-th row of A. The linear program is called *feasible* if the feasible set is nonempty (we also say that the system $Ax \le b$ has solutions or is feasible). Otherwise the linear program is called *infeasible*. The elements of P are called *feasible solutions* and the vertices of P are called *basic feasible solutions*.

Example 2.1. Consider the problem

$$\begin{aligned} &\text{maximize} && 2x_1 + x_2 \\ &\text{subject to} && x_1 + 2x_2 \le 2 \\ & && x_1, x_2 \ge 0 \end{aligned}$$

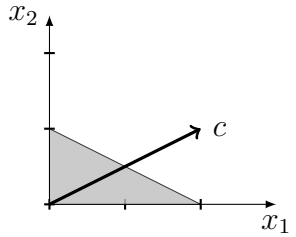
We can write it in the form $\max\{c^{\mathsf{T}}x \,|\, Ax \le b\}$ where

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The feasible set is

$$P = \{x \,|\, Ax \le b\} = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 + 2x_2 \le 2,\ x_1 \ge 0,\ x_2 \ge 0\}$$

which is a triangle with vertices $(0,0)$, $(2,0)$, $(0,1)$.



The maximum of $c^{\mathsf{T}}\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = 2x_1 + x_2$ is attained at one of the vertices, namely, the vertex $(2,0)$.

Example 2.2. Consider the problem $\max\{2x_1 + x_2 \,|\, x_1 + 2x_2 \le 2,\ x_1 = 0\}$. This is a linear problem as we can rewrite the constraint $x_1 = 0$ as two constraints $x_1 \le 0$ and $-x_1 \le 0$. We require $x_1 = 0$ and $x_2 \le 1$, hence the optimal point is $(0,1)$. The optimal value is $2\cdot 0 + 1 = 1$.

2.2. **Converting constraints.** We have

$$-\max\{c^\mathsf{T}x \mid Ax \le b\} = \min\{-c^\mathsf{T}x \mid Ax \le b\},$$

hence maximization and minimization problems are equivalent (can be reduced to each other). The following problem types are also equivalent

(1) $\max\{c^\mathsf{T}x \mid Ax \le b\}$.
(2) $\max\{c^\mathsf{T}x \mid Ax \le b, x \ge 0\}$ (Canonical form).
(3) $\max\{c^\mathsf{T}x \mid Ax = b, x \ge 0\}$ (Standard form).

We can reduce (2) to (1) by considering $\begin{bmatrix} A \\ -I \end{bmatrix} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$.

We can reduce (1) to (2) by writing $x \in \mathbb{R}^n$ in the form $x = x' - x''$, where $x', x'' \ge 0$ are defined by

$$x' = [x]_+ = (\max\{x_i, 0\})_i, \qquad x'' = [x]_- = (\max\{-x_i, 0\})_i.$$

The new problem is (of the form (2))

$$\max\left\{ c^\mathsf{T}x' - c^\mathsf{T}x'' \;\middle|\; \begin{bmatrix} A & -A \end{bmatrix} \begin{pmatrix} x' \\ x'' \end{pmatrix} \le b,\ x' \ge 0,\ x'' \ge 0 \right\}.$$

We can reduce (3) to (2) by considering $\begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{pmatrix} b \\ -b \end{pmatrix}$.

We can reduce (2) to (3) by adding new variables $s \in \mathbb{R}^m$ (slack variables) such that

$$\begin{bmatrix} A & I \end{bmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = b, \qquad x \ge 0, \qquad s \ge 0.$$

Definition 2.3. A linear program

$$\min\{c^\mathsf{T}x \mid Ax = b, x \ge 0\}$$

for $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, is called a linear program in *standard form*. Note that it is a minimization problem.

Later we will introduce an algorithm (simplex method) for solving linear programs in standard form. We have seen above that every linear program can be reduced to standard form.

Example 2.4. Let us reduce the linear program

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 0 \\ & x_1 - x_2 = 5 \end{aligned}$$

to standard form. We can write $x_i = x_i' - x_i''$ with $x_i', x_i'' \ge 0$ (and one of them is zero). Then the problem reduces to

$$\begin{aligned} \text{maximize} \quad & x_1' - x_1'' + 2x_2' - 2x_2'' \\ \text{subject to} \quad & x_1' - x_1'' + x_2' - x_2'' \le 0 \\ & x_1' - x_1' - x_2' + x_2'' = 5 \\ & x_i',\ x_i'' \ge 0 \end{aligned}$$

Introducing a slack variable $s \geq 0$, we obtain a problem

$$\begin{aligned} \text{maximize} \quad & x_1' - x_1'' + 2x_2' - 2x_2'' \\ \text{subject to} \quad & x_1' - x_1'' + x_2' - x_2'' + s = 0 \\ & x_1' - x_1' - x_2' + x_2'' = 5 \\ & x_i',\, x_i'',\, s \geq 0 \end{aligned}$$

For $x = (x_1', x_1'', x_2', x_2'', s) \in \mathbb{R}^5$, we obtain the problem

$$\max\bigl\{c^{\mathsf{T}}x \,\big|\, Ax = b, x \geq 0\bigr\}, \qquad A = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 0 \end{pmatrix}.$$

It is equivalent to the linear program in standard form

$$\min\bigl\{-c^{\mathsf{T}}x \,\big|\, Ax = b,\, x \geq 0\bigr\}.$$

Remark 2.5. Linear constraints of the form

$$\{x \,|\, Ax \leq a, Bx = b\}$$

can be converted to inequality form by considering

$$\{x \,|\, A'x \leq b'\}, \qquad A' = \begin{bmatrix} A \\ B \\ -B \end{bmatrix}, \qquad b' = \begin{pmatrix} a \\ b \\ -b \end{pmatrix}.$$

2.3. **Dual linear programs.**

Lemma 2.6 (Weak LP duality)**.** *For $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, we have*

$$\sup\{c^\mathsf{T}x \mid Ax \le b\} \le \inf\{b^\mathsf{T}y \mid A^\mathsf{T}y = c,\ y \ge 0\}.$$

Proof. If $Ax \le b$ and $A^\mathsf{T}y = c$, $y \ge 0$, then $c^\mathsf{T}x = (y^\mathsf{T}A)x = y^\mathsf{T}(Ax) \le y^\mathsf{T}b = b^\mathsf{T}y$. □

Remark 2.7. Later we will prove the Linear Programming Duality Theorem 4.29 stating that if both of the above problems are feasible, then there exist optimal solutions $x^* \in P = \{x \mid Ax \le b\}$ and $y^* \in D = \{y \mid A^\mathsf{T}y = c,\ y \ge 0\}$ satisfying $c^\mathsf{T}x^* = b^\mathsf{T}y^*$. In particular, the above inequality is an equality (if $P \ne \varnothing$ and $D \ne \varnothing$). Actually, we have an equality even if just one of the sets P, D is nonempty (see Lemma 4.31).

Definition 2.8. The linear program $\max\{c^\mathsf{T}x \mid Ax \le b\}$ is called the *primal* linear program and the linear program $\min\{b^\mathsf{T}y \mid A^\mathsf{T}y = c,\ y \ge 0\}$ is called its *dual*.

Lemma 2.9 (Complementary slackness)**.** *Feasible solutions x and y of the above problems are optimal if and only if*

$$(Ax - b)_i \cdot y_i = 0 \qquad \forall i \in [m].$$

Equivalently, either $(Ax)_i = b_i$ or $y_i = 0$ for all $i \in [m]$.

Proof. If the latter condition is satisfied, then $(Ax-b)^\mathsf{T}y = 0 \implies b^\mathsf{T}y = x^\mathsf{T}A^\mathsf{T}y = x^\mathsf{T}c = c^\mathsf{T}x$. Therefore x and y are optimal solutions by the above lemma. Conversely, if solutions x, y are optimal, then $b^\mathsf{T}y = x^\mathsf{T}c = x^\mathsf{T}A^\mathsf{T}y$ (by Duality Theorem), hence $(b - Ax)^\mathsf{T}y = 0$. We have $Ax \le b$ and $y \ge 0$, hence $(b - Ax)_i \cdot y_i = 0$ for all $i \in [m]$. □

Example 2.10. Consider the problem $\max\{c^\mathsf{T}x \mid Ax \le b,\ x \ge 0\}$. It can be written in the form $\max\{c^\mathsf{T}x \mid \left(\begin{smallmatrix} A \\ -I \end{smallmatrix}\right)x \le \left(\begin{smallmatrix} b \\ 0 \end{smallmatrix}\right)\}$. The dual problem has the form (for $(y, z) \in \mathbb{R}^m \times \mathbb{R}^n$) $\min\{b^\mathsf{T}y \mid A^\mathsf{T}y - z = c,\ y, z \ge 0\}$. It is equivalent to $\min\{b^\mathsf{T}y \mid A^\mathsf{T}y \ge c,\ y \ge 0\}$. Therefore we have the primal and the dual linear programs

$$\max\{c^\mathsf{T}x \mid Ax \le b,\ x \ge 0\}, \qquad \min\{b^\mathsf{T}y \mid A^\mathsf{T}y \ge c,\ y \ge 0\}.$$

Complementary slackness here is the statement that feasible solutions x and y are optimal if and only if

$$(Ax - b)_i \cdot y_i = 0 \quad \forall i \in [m], \qquad (A^\mathsf{T}y - c)_j \cdot x_j = 0 \quad \forall j \in [n].$$

Similarly, one can show that there are the primal and the dual problems

$$\min\{c^\mathsf{T}x \mid Ax \le b,\ x \ge 0\}, \qquad \min\{b^\mathsf{T}y \mid -A^\mathsf{T}y \le c,\ y \ge 0\}.$$

It is clear from this presentation that the dual of the dual problem is the primal problem.

Example 2.11. Consider the problem

$$\max\{c^\mathsf{T}x \mid Ax \le b,\ x \ge 0\}, \qquad A = \left(\begin{smallmatrix} 1 & 4 & 0 \\ 3 & -1 & 1 \end{smallmatrix}\right), \quad b = \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right), \quad c = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}.$$

Let p^* be the optimal value. For example, $x = (0, 0, 3)$ is feasible, hence $p^* \ge c^\mathsf{T}x = 9$. To find some upper bound for p^*, consider the dual problem $\min\{b^\mathsf{T}y \mid A^\mathsf{T}y \ge c,\ y \ge 0\}$. For a feasible solution, we require $y_1, y_2 \ge 0$ such that $y_1 + 3y_2 \ge 4$, $4y_1 - y_2 \ge 1$, $y_2 \ge 3$. For example, $y = (1, 3)$ is a feasible solution and $p^* \le b^\mathsf{T}y = 10$. Therefore $9 \le p^* \le 10$. See Example 3.17 for an optimal solution of both problems.

2.4. **Linear programming models.** In this section we will discuss various applications of linear programming.

2.4.1. *Transportation problem.* Assume that

(1) Some commodity is transported from m suppliers to n recipients.
(2) The i-th supplier has s_i units of the commodity.
(3) The j-th recipient demands d_j units of the commodity.
(4) The cost of transporting a unit of commodity from i to j is c_{ij}.

We call

(1) The vector $s = (s_1, \dots, s_m) \in \mathbb{R}^m_+$ – the *supply vector*.
(2) The vector $d = (d_1, \dots, d_n) \in \mathbb{R}^n_+$ – the *demand vector*.
(3) The matrix $C = (c_{ij}) \in \mathbb{R}^{m\times n}$ – the *cost matrix*.

Let x_{ij} be the amount of commodity transported from i to j. Then the total transport cost is

$$f(x) = \sum_{i,j} c_{ij}x_{ij} = \operatorname{tr}(C^\mathsf{T}X), \qquad X = (x_{ij}) \in \mathbb{R}^{m\times n}.$$

The constraints are

$$\sum\nolimits_j x_{ij} \le s_i \quad \forall i \in [m], \qquad \sum\nolimits_i x_{ij} \ge d_j \quad \forall j \in [n].$$

For a feasible problem we require

$$\sum_i s_i \ge \sum_{i,j} x_{ij} \ge \sum_j d_j.$$

Lemma 2.12. *If $\sum_i s_i = \sum_j d_j$ (supply is equal to demand), then a feasible solution satisfies*

$$\sum_j x_{ij} = s_i \quad \forall i \in [m], \qquad \sum_i x_{ij} = d_j \quad \forall j \in [n].$$

Assuming that supply is equal to demand, we obtain the linear program

$$\begin{aligned} \text{minimize} \quad & \sum\nolimits_{i,j} c_{ij}x_{ij} \\ \text{subject to} \quad & \sum\nolimits_j x_{ij} = s_i \quad \forall i \in [m] \\ & \sum\nolimits_i x_{ij} = d_j \quad \forall j \in [n] \\ & x_{ij} \ge 0 \quad \forall i \in [m],\, j \in [n] \end{aligned}$$

Remark 2.13. Using the vector $\mathbb{1}_n = 1^{n\times 1} \in \mathbb{R}^n$, we can write the above constraints as $X\mathbb{1}_n = s$, $X^\mathsf{T}\mathbb{1}_m = d$ and $X \ge 0$.

2.4.2. *Production management.* Assume that

(1) A company produces products of type $j \in [n]$ out of raw materials of type $i \in [m]$.
(2) To produce a unit of $j \in [n]$ one requires a_{ij} units of $i \in [m]$. Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$.
(3) The price of a unit of $i \in [m]$ is ρ_i. Let $\rho = (\rho_i) \in \mathbb{R}^m$.
(4) The price of a unit of $j \in [n]$ is σ_j. Let $\sigma = (\sigma_j) \in \mathbb{R}^n$.
(5) We have b_i units of $i \in [m]$. Let $b = (b_i) \in \mathbb{R}^m$.

The net revenue for producing a unit of $j \in [n]$ is

$$c_j = \sigma_j - \sum_i \rho_i a_{ij}.$$

We can write this using vectors as $c = \sigma - A^{\mathsf{T}}\rho \in \mathbb{R}^n$. Let x_j be the amount of produced units of type $j \in [n]$. Then we want to maximize the total net revenue

$$f(x) = \sum_j c_j x_j = c^{\mathsf{T}} x$$

subject to the constrains

$$\sum\nolimits_j a_{ij} x_j \le b_i \quad \forall i \in [m], \qquad x_j \ge 0 \quad \forall j \in [n].$$

We obtain the linear program

$$\begin{aligned} \text{maximize} \quad & c^{\mathsf{T}} x \\ \text{subject to} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

Note that we can omit products $j \in [n]$ with $c_j \le 0$ and assume that $c_j > 0$ for all $j \in [n]$. It is also clear that in the above optimization problem we can require $Ax = b$ so that we utilize all resources.

Remark 2.14. Assume that we can buy raw material $i \in [m]$ only at a price $w_i \ge \rho_i$. Then the total price for raw materials is $\sum_i b_i w_i$. The revenue for producing $j \in [n]$ is $\sigma_j - \sum_i w_i a_{ij}$ and we can assume that it is ≤ 0 as otherwise someone else could produce the same product. Therefore $\sum_i w_i a_{ij} \ge \sigma_j$. Using the variables $y_i = w_i - \rho_i \ge 0$, we obtain $\sum_i y_i a_{ij} \ge \sigma_j - \sum_i \rho_i a_{ij} = c_j$. Minimization of $\sum_i b_i w_i$ is equivalent to minimization of $\sum_i b_i y_i = b^{\mathsf{T}} y$. Therefore we obtain the problem

$$\begin{aligned} \text{minimize} \quad & b^{\mathsf{T}} y \\ \text{subject to} \quad & A^{\mathsf{T}} y \ge c \\ & y \ge 0 \end{aligned}$$

Note that this problem is dual to the original problem.

2.4.3. *Assignment problem.* Assume that

(1) We have people $i \in [m]$ and tasks $j \in [n]$.
(2) The costs of assigning person i to task j is c_{ij}. Let $C = (c_{ij})$ be the cost matrix.
(3) One person can be assigned to only one task and each task should be covered.

Note that we require $m \ge n$. Let us assume that $m = n$. Let $x_{ij} = 1$ if person i is assigned to task j and let $x_{ij} = 0$ otherwise. Then we want to minimize the cost $\sum_{ij} c_{ij} x_{ij}$ subject to the constraints

$$\sum\nolimits_j x_{ij} \le 1 \ \forall i, \qquad \sum\nolimits_i x_{ij} \ge 1 \ \forall j.$$

Note that the second inequality can be assumed to be an equality. But then the first inequality also has to be an equality since a strict inequality would imply $m > \sum_{ij} x_{ij} = n$ as we assume that $m = n$. Therefore we obtain an optimization problem

$$\begin{aligned} \text{minimize} \quad & \sum\nolimits_{i,j} c_{ij} x_{ij} \\ \text{subject to} \quad & \sum\nolimits_j x_{ij} = 1 \ \forall i \\ & \sum_i x_{ij} = 1 \ \forall j \\ & x_{ij} \in \{0,1\} \ \forall i, j \end{aligned}$$

This is an example of an integer programming problem as x_{ij} are required to be integers. We can substitute the last constraint by $x_{ij} \geq 0$ and obtain a linear program, called an LP relaxation. Note that it is equivalent to a transportation problem.

2.4.4. *Chicken and egg problem.* Assume that a hen can lay 12 eggs or hatch 4 eggs in two weeks. What is the best laying and hatching program if at the end of the 4-th period all hens and chicks are sold at 60c apiece and eggs at 10c apiece? Initially one has 100 hens and 0 eggs.

Let y_i be the number of chicken after period i, z_i be the number of eggs after period i and x_i be the number of chicken that hatch during period $i+1$. Then

(1) $0 \leq x_i \leq y_i$.
(2) $4x_i \leq z_i$
(3) $y_{i+1} = y_i + 4x_i$.
(4) $z_{i+1} = (z_i - 4x_i) + 12(y_i - x_i) = z_i + 12y_i - 16x_i$.

We have $y_0 = 100$, $z_0 = 0$ and we need to maximize $60y_4 + 10z_4$. We can decrease the number of variables by using $y_k = y_0 + 4\sum_{i=0}^{k-1} x_i$ and $z_k = z_0 + 4\sum_{i=0}^{k-1}(3y_i - 4x_i)$.

2.4.5. *Matrix games.* A matrix game is a two-person game, where the first person selects an action $i \in [m]$ and the second person selects an action $j \in [n]$. Then the first person receives from the second a_{ij} units (it can be negative). The matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is called the *payoff matrix*.

Example 2.15 (Rock-Paper-Scissors)**.** Every player selects rock, paper, or scissors (1, 2, 3). Then rock beats scissors, scissors beats paper, and paper beats rock. The payoff matrix is

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Let $x = (x_i) \in \mathbb{R}^m$ be a vector such that x_i is the probability that the first player selects action $i \in [m]$. Then we require

$$\mathbb{1}^\mathsf{T} x = \sum\nolimits_{i\in[m]} x_i = 1, \qquad x_i \geq 0 \quad \forall i \in [m],$$

where $\mathbb{1} = (1, \dots, 1)^\mathsf{T} \in \mathbb{R}^m$. Such vector x is called a *stochastic vector*. Similarly, let $y = (y_j) \in \mathbb{R}^n$ be a vector such that y_j is the probability that the second player selects action $j \in [n]$. For the outcome (i, j), the payoff is a_{ij} and the probability is $x_i y_j$. Therefore the expected payoff from the second to the first player is

$$\sum\nolimits_{i,j} x_i a_{ij} y_j = x^\mathsf{T} A y.$$

If the second player adopts strategy $y \in \mathbb{R}^n$ (plays in accordance with the stochastic vector y), then the first player has to solve the problem

$$\max\{x^\mathsf{T} A y \mid \mathbb{1}^\mathsf{T} x = 1,\ x \geq 0\}.$$

The optimal solution of the problem is attained at a vertex of the polyhedron $\{x \mid \mathbb{1}^\mathsf{T} x = 1, x \geq 0\}$, hence it is equal to some basis vector $e_i \in \mathbb{R}^m$ for $i \in [m]$. Therefore

$$\max\{x^\mathsf{T} A y \mid \mathbb{1}^\mathsf{T} x = 1,\ x \geq 0\} = \max_i e_i^\mathsf{T} A y = \max_i (Ay)_i.$$

If the second player assumes that the first player will choose the above strategy, then the second player should adopt a strategy $y \in \mathbb{R}^n$ solving the problem

$$\min_y \max_x x^\mathsf{T} A y = \min_y \max_i (Ay)_i$$

Introducing a new variable v for $\max_i (Ay)_i$, we obtain a linear problem

$$\begin{aligned} \text{minimize} \quad & v \\ \text{subject to} \quad & (Ay)_i \le v \ \forall i \in [m] \\ & \sum y_j = 1 \\ & y_j \ge 0 \ \forall j \in [n] \end{aligned}$$

Similarly, the first player has to adopt a strategy x that solves the problem $\max_x \min_y x^\mathsf{T} A y$. We will see below that the optimal values are the same for both problems, meaning that

$$\max_x \min_y x^\mathsf{T} A y = \min_y \max_x x^\mathsf{T} A y.$$

Remark 2.16 (Minimax)**.** Let X, Y be compact topological spaces and $f\colon X \times Y \to \mathbb{R}$ be continuous. We have $f(x,y) \le \max_x f(x,y)$, hence $\min_y f(x,y) \le \min_y \max_x f(x,y)$. Therefore

$$\max_x \min_y f(x,y) \le \min_y \max_x f(x,y).$$

Theorem 2.17 (Minimax theorem, von Neumann)**.** *Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ be compact and convex and let $f\colon X \times Y \to \mathbb{R}$ be continuous and concave-convex:*

(1) $f_x = f(x,-)\colon Y \to \mathbb{R}$ is convex: $f_x(ty + (1-t)y') \le t f_x(y) + (1-t) f_x(y')$ for $t \in [0,1]$.

(2) $f_y = f(-,y)\colon X \to \mathbb{R}$ is concave, meaning that $-f_y$ is convex.

Then

$$\max_x \min_y f(x,y) = \min_y \max_x f(x,y).$$

In our case we have a function $f(x,y) = x^\mathsf{T} A y$ which is linear in both variables, hence concave-convex. The sets are

$$X = \Big\{ x \in \mathbb{R}^m \;\Big|\; \sum x_i = 0,\ x \ge 0 \Big\}, \qquad Y = \Big\{ x \in \mathbb{R}^n \;\Big|\; \sum y_j = 0,\ y \ge 0 \Big\},$$

which are compact and convex. Therefore we can apply the minimax theorem.

Example 2.18. Let $f(x,y) = xy - x = x(y-1)$ for $x \in [0,1]$, $y \in [0,2]$. Then $\min_y f(x,y) = x0 - x = -x$, hence $\max_x \min_y f(x,y) = 0$. On the other hand $\max_x f(x,y) = y - 1$ if $y \ge 1$ and 0 if $y \le 1$. Therefore $\min_y \max_x f(x,y) = 0$.

3. Simplex method

The simplex method is an effective algorithm to solve linear programs. It was developed by G. Dantzig in 1947 for US military. While there exist nowadays more efficient algorithms for solving linear programs, the simplex method remains the most popular. It has wide applications in optimizing various processes like transportation, scheduling and resource allocation.

3.1. **Simplex method: Vertices.** Consider a linear program

$$\max\{c^\mathsf{T}x \mid Ax \le b\}$$

for $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The maximum of the objective function $f(x) = c^\mathsf{T}x$ is attained (if it exists) at one of the vertices of the polyhedron

$$P = \{x \in \mathbb{R}^n \mid Ax \le b\}.$$

The points of P are called *feasible solutions* and the vertices of P are called *basic feasible solutions* of the problem.

Definition 3.1. A point x of a polyhedron $P \subset \mathbb{R}^n$ is called a *vertex* of P if there exists $0 \ne v \in \mathbb{R}^n$ such that $v^\mathsf{T}x > v^\mathsf{T}y$ for all $y \in P\backslash\{x\}$. This means that the hyperplane $\{y \in \mathbb{R}^n \mid v^\mathsf{T}y = v^\mathsf{T}x\}$ touches P exactly at x.

Lemma 3.2. *If $x \in P$ is a unique optimal solution, then it is a vertex of P.*

Proof. By the uniqueness of the optimal solution, we have $c^\mathsf{T}x > c^\mathsf{T}y$ for all $y \in P\backslash\{x\}$. □

The idea of the simplex method, which solves linear programs, is to make a trip on P from vertex to vertex along the edges, until the optimal vertex is reached. As P has finitely many vertices, the algorithm will terminate. Assume that we know a vertex $x_0 \in P$ and that it is not a maximum point of the objective function. Then there exists an edge $[x_0, x_1]$ of P (where x_1 is another vertex of P) such that the objective function strictly increases along the edge $[x_0, x_1]$. Now we take the vertex x_1 and repeat the process.

The first problem we need to resolve is to find a simple method to describe the vertices of P. For this, we will assume that we have a linear program in standard form

$$\min\{c^\mathsf{T}x \mid Ax = b,\ x \ge 0\}.$$

Remark 3.3. For $A \in \mathbb{R}^{m\times n}$ and $J \subset [n]$, let A_J be the matrix consisting of columns $A_j = A_{*j}$ for $j \in J$. Similarly, for $x \in \mathbb{R}^n$, let $x_J = (x_j)_{j\in J} \in \mathbb{R}^J$.

Lemma 3.4. *A point x of $P = \{x \mid Ax = b,\ x \ge 0\}$ is a vertex $\iff$ $\operatorname{rk} A_J = |J|$ for $J = \{j \in [n] \mid x_j > 0\}$ (the support of x).*

Remark 3.5. Intuitively, a vertex of $P \subset \mathbb{R}^n$ should be determined by n equations. We have equations $x_j = 0$ for $j \in [n]\backslash J$ and we require an additional collection of equations (of size $|J|$) induced by $Ax = A_Jx_J = b$. The number of independent equations in this system is equal to $\operatorname{rk} A_J$ and we always have $\operatorname{rk} A_J \le |J|$ as A_J has $|J|$ columns.

Proof. Let $x \in P$ be a vertex and let $v \in \mathbb{R}^n$ be such that $v \cdot x > v \cdot y$ for all $y \in P\backslash\{x\}$. If the map $A_J\colon \mathbb{R}^J \to \mathbb{R}^m$ is not injective, then there exists $0 \ne z \in \operatorname{Ker} A_J$ such that $v_J \cdot z \ge 0$ (if $v_J \cdot z < 0$, we can take $-z$) and $\pm z \le x_J$ (we have $x_j > 0$ for all $j \in J$, hence we can take z close to 0 so that $|z_j| < x_j$ for all $j \in J$). We extend z to $\mathbb{R}^n$ by $z_j = 0$ for $j \notin J$. Then

$y = x + z \geq 0$ and $Ay = Ax + A_J z = Ax$, hence $y \in P$. But $v \cdot y = v \cdot x + v_J \cdot z \geq v \cdot x$, a contradiction. This implies that A_J is injective, hence $\operatorname{rk} A_J = |J|$.

Conversely, let $A_J \colon \mathbb{R}^J \to \mathbb{R}^m$ be injective. Consider $v \in \mathbb{R}^n$ with $v_j = 0$ for $j \in J$ and $v_j = -1$ for $j \in [n] \backslash J$. Then $v \cdot x = 0$. If $y \in P$ and $v \cdot y \geq 0$, then $y_j = 0$ for all $j \in [n] \backslash J$, hence $A_J y_J = Ay = b = Ax = A_J x_J$ and $y = x$. Therefore $v \cdot y < v \cdot x$ for all $y \in P \backslash \{x\}$. $\square$

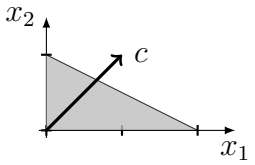
We can perform elementary row operations on $[A \mid b]$ and assume that A has rank m. For every vertex $x \in P$, we can extend the set $J = \{j \in [n] \mid x_j > 0\}$ (with $|J| = \operatorname{rk} A_J \leq m$) to a subset $\mathcal{B} \subset [n]$ of size m such that $\operatorname{rk} A_{\mathcal{B}} = m$ (hence $A_{\mathcal{B}}$ is non-singular, meaning an invertible square matrix). Such extension is not unique, but it determines the vertex x uniquely as $x_{\mathcal{B}} = A_{\mathcal{B}}^{-1} b$ and $x_{\bar{\mathcal{B}}} = 0$ for $\bar{\mathcal{B}} = [n] \backslash \mathcal{B}$.

Definition 3.6. A subset $\mathcal{B} \subset [n]$ is called a *basis* if $|\mathcal{B}| = m = \operatorname{rk} A_{\mathcal{B}}$ (meaning that $A_{\mathcal{B}}$ is non-singular). A basis $\mathcal{B} \subset [n]$ is called *feasible* if $A_{\mathcal{B}}^{-1} b \geq 0$. Equivalently, the point $x \in \mathbb{R}^n$ defined by $x_{\mathcal{B}} = A_{\mathcal{B}}^{-1} b$ and $x_{\bar{\mathcal{B}}} = 0$ is a vertex of $P = \{x \mid Ax = b,\ x \geq 0\}$.

In what follows, we will encode the vertices of P by feasible bases and move between them.

Example 3.7. A vertex can be encoded by several feasible bases. For example, for $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $b = 0$, we have $P = \{x \mid Ax = b, x \geq 0\} = \{(0,0)\}$ and the vertex $x = (0,0)$ is encoded by $\mathcal{B} = \{1\}$ and $\mathcal{B} = \{2\}$.

Example 3.8. Consider the problem $\max\{x_1 + x_2 \mid x_1 + 2x_2 \leq 2,\ x_1, x_2 \geq 0\}$ with the feasible set $P = \{x \in \mathbb{R}^2 \mid x_1 + 2x_2 \leq 2,\ x_1, x_2 \geq 0\}$ (a triangle)



We add a slack variable $x_3 = 2 - x_1 - 2x_2 \geq 0$ so that the problem is equivalent to $\max\{c^{\mathsf{T}} x \mid Ax = b,\ x \geq 0\}$ for $A = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$, $b = 2$ and $c^{\mathsf{T}} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$. The feasible set is $P' = \{x \in \mathbb{R}^3 \mid Ax = 2,\ x \geq 0\}$. A feasible basis $\mathcal{B} \subset [3]$ is a subset (with one element) such that $A_{\mathcal{B}}$ is non-singular and $A_{\mathcal{B}}^{-1} b \geq 0$. We have three feasible bases

(1) $\mathcal{B} = \{1\}$ with the corresponding vertex $x = (2,0,0)$ of P'.
(2) $\mathcal{B} = \{2\}$ with the corresponding vertex $x = (0,1,0)$ of P'.
(3) $\mathcal{B} = \{3\}$ with the corresponding vertex $x = (0,0,2)$ of P'.

The corresponding vertices of P are $(2,0)$, $(0,1)$ and $(0,0)$.

3.2. **Simplex method: Phase II.** Consider a linear program in standard form

$$\min\{c^\mathsf{T}x \mid Ax = b,\ x \geq 0\} \tag{4}$$

and assume that we have some feasible basis $\mathcal{B} \subset [n]$.

Example 3.9. Consider a linear program

$$\min\{c^\mathsf{T}x \mid Ax \leq b,\ x \geq 0\}$$

such that $b \geq 0$ (this is the crucial assumption). We add slack variables $x_{n+1}, \ldots, x_{n+m}$ and consider an equivalent linear program (for $x \in \mathbb{R}^{n+m}$)

$$\min\{\begin{bmatrix}c^\mathsf{T} & 0^{1\times m}\end{bmatrix} x \mid \begin{bmatrix}A & I\end{bmatrix} x = b,\ x \geq 0\}.$$

Then $\mathcal{B} = \{n+1, \ldots, n+m\}$ is a feasible basis since the point $x \in \mathbb{R}^{m+n}$ with $x_\mathcal{B} = b$ and $x_{\bar{\mathcal{B}}} = 0$ satisfies $\begin{bmatrix}A & I\end{bmatrix} x = b$ and $x \geq 0$.

We encode problem (4) using the matrix, called a *simplex tableau*,

A	b
c^T	d

where we set initially $d = 0$. The above problem can be written as

$$\min\{c^\mathsf{T}x - d \mid Ax = b,\ x \geq 0\}$$

We can perform elementary row operations that don't add/subtract the row $\begin{bmatrix}c^\mathsf{T} & d\end{bmatrix}$ to/from other rows. These operations preserve the optimal solution and the optimal value of the problem. Assuming that we have a feasible basis $\mathcal{B} = \{j_1 < \cdots < j_m\} \subset [n]$, we perform row operations to make $A_\mathcal{B} = I$ and $c_\mathcal{B} = 0$ (then $b \geq 0$) and repeat the following steps

(1) Choose $j \in [n]$ with $c_j < 0$ (minimal c_j or minimal j with $c_j < 0$). If $c_j \geq 0$ for all j, then we stop (the vertex corresponding to $\mathcal{B}$ is optimal).
(2) Choose $k \in [m]$ with $a_{kj} > 0$ and minimal b_k/a_{kj}. If $a_{ij} \leq 0$ for all i, then we stop (the minimum is not attained).
(3) Consider the new feasible basis $\mathcal{B}' = \mathcal{B} \cup \{j\} \backslash \{j_k\}$ and pivot at a_{kj} (to make $A_{\mathcal{B}'} = I$ and $c_{\mathcal{B}'} = 0$).

Remark 3.10. *Pivoting* at a nonzero entry a_{kj} of a matrix consists of dividing the k-th row by a_{kj} (to make the new entry $a'_{kj} = 1$) and using elementary row operations to make $a'_{ij} = 0$ for all $i \neq k$. When we substitute the basis $\mathcal{B}$ by the basis $\mathcal{B}' = \mathcal{B} \cup \{j\} \backslash \{j_k\}$, we substitute the k-th column of $A_\mathcal{B} = I$ by the column A_j. To make $A_{\mathcal{B}'} = I$ and $c_{\mathcal{B}'} = 0$, we pivot at a_{kj}.

The element j above is called entering and the element $j_k \in \mathcal{B}$ (the k-th element of $\mathcal{B}$) is called leaving. At the end of the algorithm we obtain the optimal vertex $x \in \mathbb{R}^n$ with $x_\mathcal{B} = A_\mathcal{B}^{-1}b = b$ and $x_{\bar{\mathcal{B}}} = 0$ and $c \in \mathbb{R}^n$ with $c_\mathcal{B} = 0$, so that $c^\mathsf{T}x = 0$. The optimal value of the problem is $c^\mathsf{T}x - d = -d$.

Theorem 3.11.

(1) If $c_j \geq 0$ for all j, then the corresponding vertex is optimal.
(2) If $c_j < 0$ and $a_{ij} \leq 0$ for all i, then the minimum is not attained.
(3) The new basis $\mathcal{B}' = \mathcal{B} \cup \{j\} \backslash \{j_k\}$ is feasible.
(4) If x, x' are the the old and the new vertices, then $c^\mathsf{T}x \geq c^\mathsf{T}x'$. If $x \neq x'$, then $c^\mathsf{T}x > c^\mathsf{T}x'$.

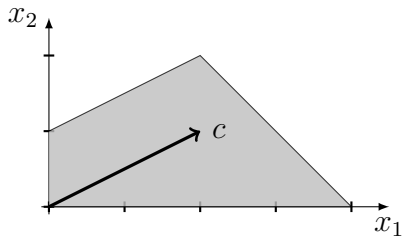
Proof. **(1)** For every $x \geq 0$ we have $c^\mathsf{T} x \geq 0$, hence $\min\{c^\mathsf{T} x \mid Ax = b, x \geq 0\} \geq 0$. On the other hand, for the vertex x we have $c_\mathcal{B} = 0$ and $x_{\bar{\mathcal{B}}} = 0$, hence $c^\mathsf{T} x = 0$.
(2) Consider the vector $y \in \mathbb{R}^n$ with $y_j = 1$, $y_\mathcal{B} - A_j$ and other coordinates 0. Then $y \geq 0$ and $Ay = A_\mathcal{B} y_\mathcal{B} + A_j = y_\mathcal{B} + A_j = 0$. For any $x \in P$ and $t \geq 0$, we have $A(x + ty) = Ax = b$ and $x + ty \geq 0$, hence $x + ty \in P$. We have $c^\mathsf{T} y = c_j < 0$, hence $\lim_{t\to\infty} c^\mathsf{T}(x + ty) = -\infty$.
(3) Without loss of generality, let $\mathcal{B} = [m]$. We have $A_\mathcal{B} = I$, hence $b = x_\mathcal{B} \geq 0$. The vertex x' for $\mathcal{B}' = \mathcal{B} \cup \{j\} \backslash \{k\}$ has coordinates (this is the right column after pivoting at a_{kj})

$$x'_j = \tfrac{b_k}{a_{kj}}, \qquad x'_i = b_i - a_{ij}\tfrac{b_k}{a_{kj}} \qquad \forall i \in \mathcal{B}\backslash\{k\},$$

and all other coordinates zero. If $a_{ij} \leq 0$, then clearly $x'_i \geq 0$. If $a_{ij} > 0$, then $x'_i = a_{ij}\left(\frac{b_i}{a_{ij}} - \frac{b_k}{a_{kj}}\right) \geq 0$ by the minimality of $\frac{b_k}{a_{kj}}$. Therefore $x' \geq 0$ and $\mathcal{B}'$ is a feasible basis.
(4) We have $c^\mathsf{T} x = 0$ and $c^\mathsf{T} x' = c_j x'_j = c_j b_k / a_{kj} \leq 0$. If $x \neq x'$, then $b_k \neq 0$, hence $c^\mathsf{T} x' = c_j b_k / a_{kj} < 0$ (here $c_j < 0$, $b_k > 0$, $a_{kj} > 0$). □

Example 3.12. Consider the linear program

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 4 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1,\, x_2 \geq 0 \end{aligned}$$



The maximum is obtained by moving a line orthogonal to $c = (2, 1)$ (with the equation $2x_1 + x_2 = C$ for some $C \in \mathbb{R}$) along c until it touches the gray polyhedron at only one point. One can see that the maximum is attained at $x = (4, 0)$ and is equal to 8. To solve the problem using simplex method, we introduce slack variables x_3, x_4 and consider

$$\min\{-2x_1 - x_2 \mid Ax = b, x \geq 0\}, \qquad A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

The simplex tableau is

$\boxed{1}$	1	1	0	4
-1	2	0	1	2
-2	-1	0	0	0

(1) We choose a feasible basis $\mathcal{B}_0 = \{3, 4\} = \{j_1 < j_2\}$.
(2) We choose $j \in [n] = [4]$ with minimal $c_j < 0$, namely $j = 1$.
(3) We choose $k \in [m] = [2]$ with $a_{kj} > 0$ and minimal b_k/a_{kj}, namely $k = 1$.
(4) We remove $j_k = 3$ and add $j = 1$ to $\mathcal{B}_0$, hence obtain $\mathcal{B}_1 = \{1, 4\}$.
(5) We make $A_{\mathcal{B}_1}$ the identity matrix (and annihilate $c_{\mathcal{B}_1}$) by pivoting at $a_{kj} = a_{11}$

1	1	1	0	4
0	3	1	1	6
0	1	1	0	4

(6) All $c_j \geq 0$, hence we stop. We have $\mathcal{B} = \mathcal{B}_1 = \{1, 4\}$. The corresponding vertex satisfies $x_\mathcal{B} = b$ (hence $x_1 = 4$, $x_4 = 6$) and $x_{\bar{\mathcal{B}}} = 0$ (hence $x_2 = x_3 = 0$).

In particular, we obtain $(x_1, x_2) = (4, 0)$ as before. The minimal value is $-d = -4$, hence the maximal value of the original problem is 4. Note that for the original feasible basis $\mathcal{B}_0 = \{3, 4\}$ we obtain $x_1 = x_2 = 0$. This means that in the simplex method we moved from the vertex $(0, 0)$ to the vertex $(4, 0)$ along the corresponding edge of the polyhedron.

3.3. **Reduced simplex tableau.** The variables x_j for $j \in \mathcal{B}$ are called *basic* and the variables x_j for $j \in \bar{\mathcal{B}}$ are called *non-basic*. If we have a simplex tableau with $A_{\mathcal{B}} = I$ and $c_{\mathcal{B}} = 0$, we can drop these matrices and write the (reduced) simplex tableau

Basic $\downarrow$	$x_j : j \notin \mathcal{B}$	
x_{j_1} $\dots$ x_{j_m}	$A_{\bar{\mathcal{B}}}$	b
z	$c_{\bar{\mathcal{B}}}^{\mathsf{T}}$	d

where basic variables are written on the left and non-basic variables at the top. This tableau can be used to express basic variables

$$x_{j_k} = b_k - \sum\nolimits_{j \in \bar{\mathcal{B}}} a_{kj} x_j, \quad j_k \in \mathcal{B}, \qquad z := d - c^{\mathsf{T}} x = d - \sum\nolimits_{j \in \bar{\mathcal{B}}} c_j x_j \tag{5}$$

as linear combinations of non-basic variables (note that a_{kj} and c_j appear with the minus sign). Let $\mathcal{B}' = \mathcal{B} \cup \{j\} \backslash \{j_k\}$, where $j \in \bar{\mathcal{B}}$ (with $c_j < 0$) enters the basis and $j_k \in \mathcal{B}$ (with $a_{kj} > 0$ and minimal b_k / a_{kj}) leaves the basis. We call x_j the *entering variable* and x_{j_k} the *leaving variable*. For the simplex method we swap x_j and x_{j_k} in the tableau and we pivot at a_{kj} (to make $A_{\mathcal{B}'} = I$ and $c_{\mathcal{B}'} = 0$, see Remark 3.10). But additionally we need to write the content of the column A_{j_k} (from the non-reduced tableau) under x_{j_k}. We call these two steps the *full pivot* at a_{kj}. Explicitly, the new entries of the column are

$$a'_{kj} = 1/a_{kj}, \qquad a'_{ij} = -a_{ij}/a_{kj} \quad \forall i \neq k, \qquad c'_j = -c_j / a_{kj}. \tag{6}$$

After the algorithm is stopped, the optimal vertex x is determined by $x_{\mathcal{B}} = b$ and $x_{\bar{\mathcal{B}}} = 0$.

Remark 3.13. Another way to think about this procedure is that we can use (5) to express

$$x_j = \frac{b_k}{a_{kj}} - \sum\nolimits_{l \in \bar{\mathcal{B}} \backslash \{j\}} \frac{a_{kl}}{a_{kj}} x_l - \frac{1}{a_{kj}} x_{j_k}$$

which is a linear combination of new non-basic variables. This formula can be used to express other basic variables in terms of new non-basic variables. As a result, we obtain a new simplex tableau which is the same tableau as the one obtained by full pivoting.

Remark 3.14. Consider the primal and the dual problems (for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$)

$$\min\bigl\{c^{\mathsf{T}} x \,\big|\, Ax \leq b,\, x \geq 0\bigr\}, \qquad \min\bigl\{b^{\mathsf{T}} y \,\big|\, -A^{\mathsf{T}} y \leq c,\, y \geq 0\bigr\}$$

We introduce the usual slack variables (m variables for the first problem and n variables for the second problem) and the bases, so that the corresponding reduced simplex tableaux are (with $d = 0$)

A	b
c^{T}	d

$-A^{\mathsf{T}}$	c
b^{T}	d

One can check that full pivots of both tableaux correspond to each other. Therefore by solving one problem, we automatically obtain a solution of the second problem. More precisely, if the first tableau solves the primal problem with a basis $\mathcal{B} \subset [n]$, then the optimal solution $y \in \mathbb{R}^m$ of the dual problem has coordinates $y_i = c_{n+i}$ (the bottom row entry under x_{n+i}) if x_{n+i} is non-basic and $y_i = 0$ otherwise. One can use slack variables y_i instead of x_{n+i} to keep track of both problems simultaneously.

Example 3.15. Consider the simplex tableau

1	1	1	0	4
−1	2	0	1	2
−2	−1	0	0	0

and the basis $\mathcal{B}_0 = \{3, 4\}$ from Example 3.12. We consider basic variables x_3, x_4 and the tableau

Basic ↓	x_1	x_2	
x_3	$\boxed{1}$	1	4
x_4	−1	2	2
z	−2	−1	0

corresponding to equations $x_4 - x_1 + 2x_2 = 2$ etc.

Choose the entering variable $x_j = x_1$ (with $c_1 = -2 < 0$) and the leaving variable $x_k = x_3$ (with the entry $a_{kj} > 0$ at the row of x_k and the column of x_j, and the minimal b_k/a_{kj}). After the full pivot at a_{kj} (swapping of x_k and x_j, pivot at a_{kj} and application of (6)), we obtain

Basic ↓	x_3	x_2	
x_1	1	1	4
x_4	1	3	6
z	2	1	8

All $c_j \geq 0$, hence we stop. The optimal solution $x \in \mathbb{R}^4$ has coordinates $x_1 = 4$, $x_4 = 6$ and $x_2 = x_3 = 0$. Alternatively, we can write the original system of equations as

$$\begin{aligned} x_3 &= 4 - x_1 - x_2 \\ x_4 &= 2 + x_1 - 2x_2 \\ z &= 0 + 2x_1 + x_2 \end{aligned}$$

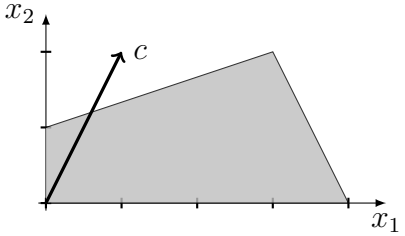
which corresponds to the first tableau. After entering x_1 and leaving x_3, we express x_1, x_4 and z in terms of non-basic variables

$$\begin{aligned} x_1 &= 4 - x_3 - x_2 \\ x_4 &= 6 - x_3 - 3x_2 \\ z &= 8 - 2x_3 - x_2 \end{aligned}$$

which corresponds to the last tableau.

Example 3.16. Consider the linear program

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 8 \\ & -x_1 + 3x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$



The maximum is obtained by moving a line orthogonal to $c = (1, 2)$ (with the equation $x_1 + 2x_2 = C$ for some $C \in \mathbb{R}$) along c until it touches the gray polyhedron at only one point. One can see that the maximum is attained at $x = (3, 2)$ and is equal to 7.

Now let us solve the problem using simplex method. Using slack variables, we express the problem in standard form for $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$\min\{-x_1 - 2x_2 \,|\, Ax = b, x \geq 0\}, \qquad A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 8 \\ 3 \end{pmatrix}.$$

We have a feasible basis $\mathcal{B}_0 = \{3, 4\}$ with the corresponding basic variables x_3, x_4 and the reduced simplex tableau

Basic ↓	x_1	x_2	
x_3	2	1	8
x_4	-1	$\boxed{3}$	3
z	-1	-2	0

Choose the entering variable $x_j = x_2$ (with minimal $c_j < 0$) and the leaving variable $x_k = x_4$ (with $a_{kj} > 0$ and minimal b_k/a_{kj}). The new basis is $\mathcal{B}_1 = \{2, 3\}$. After the full pivot at $a_{kj} = 3$ (swapping of x_k and x_j, pivot at a_{kj} and application of (6)), we obtain the tableau

Basic ↓	x_1	x_4	
x_3	$\boxed{7/3}$	$-1/3$	7
x_2	$-1/3$	$1/3$	1
z	$-5/3$	$2/3$	2

Choose the entering variable $x_j = x_1$ and the leaving variable $x_k = x_3$. The new basis is $\mathcal{B}_2 = \{1, 2\}$. After the full pivot at a_{kj}, we obtain the tableau

Basic ↓	x_3	x_4	
x_1	$3/7$	$-1/7$	3
x_2	$1/7$	$2/7$	2
z	$5/7$	$3/7$	7

All $c_j \geq 0$, hence we stop. The optimal point satisfies $x_{\mathcal{B}_2} = b$, hence $x_1 = 3$, $x_2 = 2$. The maximal value of the original problem is 7.

Note that for the original feasible basis $\mathcal{B}_0 = \{3, 4\}$ we obtain $(x_1, x_2) = (0, 0)$. For the basis $\mathcal{B}_1 = \{2, 3\}$ we obtain $x_2 = 1$, $x_3 = 7$, $x_1 = x_4 = 0$, hence $(x_1, x_2) = (0, 1)$. For the basis $\mathcal{B}_2 = \{1, 2\}$ we have seen that $(x_1, x_2) = (3, 2)$. This means that by applying the simplex method we moved from the vertex $(0, 0)$ to the vertex $(0, 1)$ and then to the vertex $(3, 2)$ along the corresponding edges of the polyhedron.

Example 3.17. Consider the linear program

$$\max\{c^\mathsf{T} x \,|\, Ax \leq b,\, x \geq 0\}, \qquad A = \left(\begin{smallmatrix} 1 & 4 & 0 \\ 3 & -1 & 1 \end{smallmatrix}\right), \quad b = \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right), \quad c = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}.$$

and the dual problem $\min\{b^\mathsf{T} y \,|\, A^\mathsf{T} y \geq c,\, y \geq 0\}$ from Example 2.11. We introduce slack variables x_4, x_5 (original basic variables) and write the simplex tableau

Basic ↓	x_1	x_2	x_3	
x_4	1	4	0	1
x_5	3	-1	$\boxed{1}$	3
z	-4	-1	-3	0

Choose the entering variable x_3 and the leaving variable x_5. After the full pivot we get

Basic ↓	x_1	x_2	x_5	
x_4	1	$\boxed{4}$	0	1
x_3	3	-1	1	3
z	5	-4	3	9

Choose the entering variable x_2 and the leaving variable x_4. After the full pivot we get

Basic ↓	x_1	x_4	x_5	
x_2	$\frac{1}{4}$	1	0	$\frac{1}{4}$
x_3	$\frac{13}{4}$	$\frac{1}{4}$	1	$\frac{13}{4}$
z	6	1	3	10

All $c_j \geq 0$, hence we obtain the optimal point $x \in \mathbb{R}^5$ with $x_{\mathcal{B}} = (\frac{1}{4}, \frac{13}{4})$ and $x_{\bar{\mathcal{B}}} = 0$ (for $\mathcal{B} = \{2, 3\}$). Therefore $(x_1, x_2, x_3) = (0, \frac{1}{4}, \frac{13}{4})$ and the maximal value is 10. We conclude from the above tableau that the optimal point $y \in \mathbb{R}^2$ of the dual problem has coordinates $y_1 = 1$, $y_2 = 3$ (coordinates of the bottom row under x_{3+1} and x_{3+2}). The minimal value is $y_1 + 3y_2 = 10$.

Example 3.18. Consider the linear program

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 4,\ 2x_1 + 3x_2 \leq 3 \\ & 4x_1 + x_2 \leq 5,\ x_1 + 5x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Adding slack variables x_3, x_4, x_5, x_6 (original basic variables), we obtain the simplex tableau

Basic ↓	x_1	x_2	
x_3	2	1	4
x_4	2	3	3
x_5	4	1	5
x_6	$\boxed{1}$	5	1
z	-2	-1	0

We choose the entering variable $x_j = x_1$ with $c_j < 0$ and the leaving variable $x_k = x_6$ with $a_{kj} > 0$ and the minimal b_k/a_{kj} (the values are $\frac{4}{2}, \frac{3}{2}, \frac{5}{4}, 1$). After the full pivot at a_{kj}, we obtain

Basic ↓	x_6	x_2	
x_3	-2	-9	2
x_4	-2	-7	1
x_5	-4	-19	1
x_1	1	5	1
z	2	9	2

All $c_j \geq 0$, hence we obtain the optimal point with $x_1 = 1$ and $x_2 = 0$. The maximal value of the problem is 2. The dual problem has the optimal point $y \in \mathbb{R}^4$ with $y_4 = 2$ (coordinate of the bottom row under x_{2+4}) and all other coordinates equal 0.

3.4. **Simplex method: Phase I.** Consider a linear program in standard form

$$\min\{c^\mathsf{T} x \mid Ax = b,\ x \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$ has rank m. We can always find a basis (so that $A_\mathcal{B}$ is invertible),

$$\mathcal{B} = \{j_1 < \cdots < j_m\} \subset [n]$$

but $\mathcal{B}$ is not necessarily feasible as we don't have $A_\mathcal{B}^{-1} b \geq 0$ in general.

To handle this issue, we first apply row operations to $[A \mid b]$ to make $A_\mathcal{B} = I$. If $b \geq 0$, then we are done. Otherwise, we introduce a new variable x_0 and the auxiliary problem

$$\begin{aligned} \text{minimize} \quad & x_0 \\ \text{subject to} \quad & -x_0 + \sum\nolimits_j a_{ij} x_j = b_i \quad \forall i \in [m] \\ & x_j \geq 0 \quad \forall 0 \leq j \leq n \end{aligned}$$

If the original problem has a feasible solution, then the auxiliary problem has a feasible solution with $x_0 = 0$ and this is the optimal value. An optimal solution of the auxiliary problem is a vertex of the auxiliary polyhedron satisfying $x_0 = 0$ (this is the optimal value), hence it is also a vertex of the original polyhedron and it corresponds to a feasible basis of the original problem. If $\tilde{\mathcal{B}} \subset \{0\} \cup [n]$ is an optimal feasible basis (of size m) of the auxiliary problem and $0 \in \tilde{\mathcal{B}}$, then $\tilde{\mathcal{B}} \backslash \{0\} \cup \{j\} \subset [n]$ is a basis for some $j \in [n] \backslash \tilde{\mathcal{B}}$, hence it is automatically a feasible basis of the original problem (the vertex satisfies $x_0 = x_j = 0$). If $\tilde{\mathcal{B}}$ is a feasible basis and $0 \notin \tilde{\mathcal{B}}$, then $\tilde{\mathcal{B}} \subset [n]$ is automatically a feasible basis of the original problem.

To solve the auxiliary problem, we construct its feasible basis and apply Phase II of the simplex method. Choose $k \in [m]$ with minimal b_k (we have $b_k < 0$ as $b \not\geq 0$). For $\mathcal{B} = \{j_1 < \cdots < j_m\}$, we claim that

$$\tilde{\mathcal{B}} = \mathcal{B} \backslash \{j_k\} \cup \{0\}$$

is a feasible basis. The matrix of the auxiliary problem is $\begin{bmatrix} -\mathbb{1} & A \end{bmatrix}$, where $\mathbb{1} = 1^{m\times 1}$. Therefore $A_{\tilde{\mathcal{B}}}$ is obtained from $\begin{bmatrix} -\mathbb{1} & A_\mathcal{B} \end{bmatrix} = \begin{bmatrix} -\mathbb{1} & I \end{bmatrix}$ by removing the k-th column of I, and is invertible, hence $\tilde{\mathcal{B}}$ is a basis. The point $x \in \mathbb{R}^{n+1}$ corresponding to $\tilde{\mathcal{B}}$ has coordinates

$$x_0 = -b_k, \qquad x_{j_i} = b_i + x_0 = b_i - b_k \quad \forall i \in [m], \qquad x_j = 0 \quad \forall j \in [n] \backslash \mathcal{B}.$$

All of these coordinates are non-negative, hence $\tilde{\mathcal{B}}$ is a feasible basis.

Example 3.19. Consider the linear program

$$\begin{aligned} \text{maximize} \quad & x_1 - 3x_2 \\ \text{subject to} \quad & -x_1 - x_2 - x_3 \leq -2 \\ & 2x_1 - x_2 + x_3 \leq 1 \\ & x_i \geq 0 \ \forall i \end{aligned}$$

Introducing slack variables x_4, x_5 we obtain the linear program in standard form

$$\min\left\{ -x_1 + 3x_2 \;\middle|\; \begin{pmatrix} -1 & -1 & -1 & 1 & 0 \\ 2 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ x \geq 0 \right\}$$

One can show that $\mathcal{B} = \{1, 2\}$ is a feasible basis (with $x_1 = x_2 = 1$), but we will apply the above method to find one. We have $b_1 < 0$, hence we introduce an auxiliary problem with a

new variable x_0

$$\min\left\{x_0 \;\middle|\; \begin{pmatrix} -1 & -1 & -1 & -1 & 1 & 0 \\ -1 & 2 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ x \geq 0\right\}$$

The simplex tableau (for the non-feasible basis $\mathcal{B}_0 = \{4, 5\} = \{j_1 < j_2\}$) is

Basic ↓	x_0	x_1	x_2	x_3	
x_4	$\boxed{-1}$	-1	-1	-1	-2
x_5	-1	2	-1	1	1
z	1	0	0	0	0

(1) We choose $k = 1$ with minimal b_k so that $j_k = j_1 = 4$. The feasible basis is $\mathcal{B}_1 = \mathcal{B}_0 \backslash \{j_k\} \cup \{0\} = \{0, 5\}$. This corresponds to the entering variable x_0 and the leaving variable $x_{j_k} = x_4$. We swap x_4 and x_0 and perform full pivot at $a_{40} = -1$

Basic ↓	x_4	x_1	x_2	x_3	
x_0	-1	1	1	1	2
x_5	-1	$\boxed{3}$	0	2	3
z	1	-1	-1	-1	-2

(2) Choose the entering variable $x_j = x_1$ with minimal $c_j < 0$ and the leaving variable $x_k = x_5$ with $a_{kj} > 0$ and minimal b_k/a_{kj}. The new basis is $\mathcal{B}_2 = \{0, 1\}$. We swap x_5 and x_1 and perform full pivot at $a_{51} = 3$

Basic ↓	x_4	x_5	x_2	x_3	
x_0	$-\frac{2}{3}$	$-\frac{1}{3}$	$\boxed{1}$	$\frac{1}{3}$	1
x_1	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	1
z	$\frac{2}{3}$	$\frac{1}{3}$	-1	$-\frac{1}{3}$	-1

(3) Choose the entering variable $x_j = x_2$ with minimal $c_j < 0$ and the leaving variable $x_k = x_0$ with $a_{kj} > 0$ and minimal b_k/a_{kj}. The new basis is $\mathcal{B}_3 = \{1, 2\}$. We don't need to continue as $0 \notin \mathcal{B}_3$, hence $\mathcal{B}_3$ is also a feasible basis of the original problem.

The simplex tableau for the original problem is

-1	-1	-1	1	0	-2
2	-1	1	0	1	1
-1	3	0	0	0	0

(1) We consider the feasible basis $\mathcal{B}_0 = \{1, 2\}$ and make $A_{\mathcal{B}_0} = I$ and $c_{\mathcal{B}_0} = 0$.

1	0	$2/3$	$-1/3$	$1/3$	1
0	1	$1/3$	$-2/3$	$-1/3$	1
0	0	$-1/3$	$5/3$	$4/3$	-2

Basic ↓	x_3	x_4	x_5	
x_1	$\boxed{2/3}$	$-1/3$	$1/3$	1
x_2	$1/3$	$-2/3$	$-1/3$	1
z	$-1/3$	$5/3$	$4/3$	-2

(2) Choose entering $x_j = x_3$ with $c_j < 0$ and leaving $x_k = x_1$ with $a_{kj} > 0$ and minimal b_k/a_{kj}. The new basis is $\mathcal{B}_1 = \{2, 3\}$. We swap x_1 and x_3 and perform full pivot at $a_{13} = 2/3$

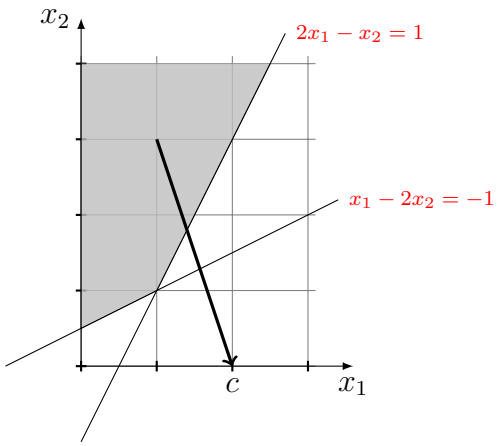
Basic ↓	x_1	x_4	x_5	
x_3	$3/2$	$-1/2$	$1/2$	$3/2$
x_2	$-1/2$	$-1/2$	$-1/2$	$1/2$
z	$1/2$	$3/2$	$3/2$	$-3/2$

(3) All $c_j \geq 0$, hence $\mathcal{B}_1 = \{2, 3\}$ is optimal. The corresponding optimal solution is $x_1 = 0$, $x_2 = 1/2$, $x_3 = 3/2$.

We could alternatively apply the FME (see below). The first two constraints imply

$$-x_1 - x_2 + 2 \leq x_3 \leq -2x_1 + x_2 + 1.$$

Therefore $x_1 - 2x_2 \leq -1$. We also have $2x_1 - x_2 \leq 1$ since $-2x_1 + x_2 + 1 \geq x_3 \geq 0$.



We can see that the maximum is attained at one of the vertices of the polyhedron, namely $(x_1, x_2) = (0, 1/2)$. The above inequalities for x_3 imply $3/2 \leq x_3 \leq 3/2$, hence $x_3 = 3/2$.

3.5. **Dual simplex method.** Consider a linear problem (for $A \in \mathbb{R}^{m\times n}$ of rank m)

$$\min\{c^\mathsf{T}x \,|\, Ax = b, x \geq 0\}$$

and its dual problem

$$\max\{b^\mathsf{T}y \,|\, A^\mathsf{T}y \leq c\}$$

Remark 3.20. If x, y are feasible, then $b^\mathsf{T}y = x^\mathsf{T}A^\mathsf{T}y \leq x^\mathsf{T}c = c^\mathsf{T}x$.

Consider the corresponding feasible sets

$$P = \{x \in \mathbb{R}^n \,|\, Ax = b, x \geq 0\}, \qquad D = \{y \in \mathbb{R}^n \,|\, A^\mathsf{T}y \leq c\}.$$

Recall that a subset $\mathcal{B} \subset [n]$ is called a basis if $|\mathcal{B}| = m = \operatorname{rk} A_\mathcal{B}$. It is called feasible if $A_\mathcal{B}^{-1}b \geq 0$ (so that the vector $x \in \mathbb{R}^n$ with $x_\mathcal{B} = A_\mathcal{B}^{-1}b$ and $x_{\bar{\mathcal{B}}} = 0$ is a vertex of P). We will say that a basis $\mathcal{B}$ is *dual feasible* if the vector $y \in \mathbb{R}^m$ defined by $A_\mathcal{B}^\mathsf{T}y = c_\mathcal{B}$ satisfies $A^\mathsf{T}y \leq c$ (so that y is a vertex of D, see Lemma 3.24). To verify if $\mathcal{B}$ is dual feasible, we consider the simplex tableau

A	b
c^T	d

with $d = 0$, apply row operations to make $A_\mathcal{B} = I$ and $c_\mathcal{B} = 0$, and check if $c \geq 0$.

Remark 3.21. Under the above operations c^T transforms to $c^\mathsf{T} - c_\mathcal{B}^\mathsf{T}A_\mathcal{B}^{-1}A$. The inequality $c^\mathsf{T} - c_\mathcal{B}^\mathsf{T}A_\mathcal{B}^{-1}A \geq 0$ is equivalent to $A^\mathsf{T}y \leq c$ for $y = (A_\mathcal{B}^\mathsf{T})^{-1}c_\mathcal{B}$.

The *dual simplex method* can be described as the primal simplex method applied to the dual problem. More precisely, assume that we have a dual feasible basis $\mathcal{B} = \{j_1 < \cdots < j_m\}$ and apply row operations to make $A_\mathcal{B} = I$, $c_\mathcal{B} = 0$ (so that $c \geq 0$). Then we repeat the steps

(1) Choose $k \in [m]$ with minimal $b_k < 0$ (if $b \geq 0$, then stop).
(2) Choose $j \in [n]$ with $a_{kj} < 0$ and maximal c_j/a_{kj} (if $a_{kj} \geq 0$ for all j, then stop).
(3) Consider $\mathcal{B}' = \mathcal{B} \cup \{j\}\backslash\{j_k\}$ and pivot at a_{kj} (to make $A_{\mathcal{B}'} = I$ and $c_{\mathcal{B}'} = 0$)

At the end we obtain a basis $\mathcal{B} \subset [n]$ and an optimal solution $x \in \mathbb{R}^n$ with $x_\mathcal{B} = b$ and $x_{\bar{\mathcal{B}}} = 0$.

Remark 3.22. After we make $A_\mathcal{B} = I$ and $c_\mathcal{B} = 0$, we can remove these matrices from A and c respectively and consider the basic variables as slack variables for the problem $\min\{c^\mathsf{T}x \,|\, Ax \leq b,\, x \geq 0\}$. Its dual problem is $\min\{b^\mathsf{T}y \,|\, -A^\mathsf{T}y \leq c,\, y \geq 0\}$ (see Remark 3.14) and the above algorithm is the (primal) simplex method applied to this problem. As in Remark 3.14, by applying the dual simplex method to the primal problem, we also obtain an optimal solution of the dual problem.

Remark 3.23*. The previous remark explains why the algorithm works, but it can also be proved directly in the same way as Theorem 3.11. Let us show that the basis $\mathcal{B}'$ is again dual feasible, meaning that $c \geq 0$ after the pivot. The new coordinates are of the form $c'_l = c_l - \frac{c_j}{a_{ij}}a_{il}$. If $a_{il} \geq 0$, then $\frac{c_j}{a_{ij}}a_{il} \leq 0$, hence $c'_l \geq c_l \geq 0$. If $a_{il} < 0$, then $c'_l = a_{il}\big(\frac{c_l}{a_{il}} - \frac{c_j}{a_{ij}}\big) \geq 0$ as $a_{il} < 0$ and $\frac{c_l}{a_{il}} \leq \frac{c_j}{a_{ij}}$. At the end we will obtain a simplex tableau with both $b \geq 0$ and $c \geq 0$. The vectors $x \in \mathbb{R}^n$ with $x_\mathcal{B} = b$ and $x_{\bar{\mathcal{B}}} = 0$ and $y \in \mathbb{R}^m$ with $A_\mathcal{B}^\mathsf{T}y = c_\mathcal{B}$ are feasible solutions of dual problems. They satisfy $x_j \cdot (A^\mathsf{T}y - c)_j = 0$ for all $j \in [n]$, hence they are both optimal (see Lemma 2.9).

Lemma 3.24. *Let $A \in \mathbb{R}^{m\times n}$ have rank m and $D = \{y \in \mathbb{R}^m \,|\, A^\mathsf{T}y \leq c\}$. Then a point $y \in D$ is a vertex of D if and only if $\operatorname{rk} A_J = m$ for $J = \{j \in [n] \,|\, (A^\mathsf{T}y)_j = c_j\}$.*

Proof. Let $y \in D$ be a vertex and $v \in \mathbb{R}^m$ be such that $v^\mathsf{T} y > v^\mathsf{T} z$ for all $z \in D \backslash \{y\}$. If $\operatorname{rk} A_J < m$, then $A_J^\mathsf{T} \colon \mathbb{R}^m \to \mathbb{R}^J$ is not injective. We can choose $0 \neq z \in \operatorname{Ker} A_J^\mathsf{T}$ such that $v \cdot z \geq 0$ (if $v \cdot z < 0$, we can take $-z$) and $\left|A_j^\mathsf{T} z\right| < (c - A^\mathsf{T} y)_j$ for all $j \in [n] \backslash J$. Then $A_J^\mathsf{T}(y+z) = A_J^\mathsf{T} y = c_J$ and $A_j^\mathsf{T}(y+z) < c_j$ for $j \in [n] \backslash J$, hence $y + z \in D$. On the other hand $v^\mathsf{T}(y+z) \geq v^\mathsf{T} y$, a contradiction. Conversely, if $z \in D$ satisfies $(A^\mathsf{T} z - c)_j = 0$ for all $j \in J$, then $(A^\mathsf{T} z)_J = A_J^\mathsf{T} z = c$. As A_J has rank m, the equation $A_J^\mathsf{T} z = A_J^\mathsf{T} y$ implies $z = y$. Therefore, if $y \neq z \in D$, then there exists $j \in J$ with $A_j^\mathsf{T} z = (A^\mathsf{T} z)_j < c_j$. Therefore, for the vector $v = \sum_{j \in J} A_j$, we have $v^\mathsf{T} z < \sum_{j \in J} c_j = v^\mathsf{T} y$. Hence y is a vertex of D. □

Remark 3.25*. The above lemma implies that if $y \in D$ is a vertex and $J = \left\{ j \,\middle|\, (A^\mathsf{T} y)_j = c_j \right\}$, then $|J| \geq \operatorname{rk} A_J = m$. Therefore we can find a subset $\mathcal{B} \subset J$ of size m such that $\operatorname{rk} A_\mathcal{B} = m$. This basis is dual feasible (since $A_\mathcal{B}^\mathsf{T} y = c_\mathcal{B}$ and $A^\mathsf{T} y \leq c$). Conversely, if $\mathcal{B} \subset [n]$ is a dual feasible basis and $y \in \mathbb{R}^m$ satisfies $A_\mathcal{B}^\mathsf{T} y = c_\mathcal{B}$, then $y \in D$ and $J = \left\{ j \,\middle|\, (A^\mathsf{T} y)_j = c_j \right\} \supset \mathcal{B}$, hence $\operatorname{rk} A_J = \operatorname{rk} A_\mathcal{B} = m$ and y is a vertex by the previous lemma. This means that the vertices of D can be encoded by dual feasible bases.

Example 3.26. Consider the linear program

$$\begin{aligned} \text{minimize} \quad & x_1 + 2x_2 + 3x_3 \\ \text{subject to} \quad & -x_1 + x_2 - 2x_3 \leq -3 \\ & x_1 - x_2 + x_3 \leq 1 \\ & x_i \geq 0 \end{aligned}$$

We introduce slack variables x_4, x_5 and consider the corresponding simplex tableau

Basic ↓	x_1	x_2	x_3	
x_4	$\boxed{-1}$	1	-2	-3
x_5	1	-1	1	1
z	1	2	3	0

We note that $b \not\geq 0$, hence we can not directly apply Phase II of the simplex method (we need first to apply Phase I). But we have $c \geq 0$, hence we can apply dual simplex method. Choose the leaving variable $x_k = x_4$ with $b_k = -3 < 0$ and choose entering variable $x_j = x_1$ with $a_{kj} < 0$ and maximal c_j / a_{kj}. After the full pivot at a_{kj} we get

Basic ↓	x_4	x_2	x_3	
x_1	-1	-1	2	3
x_5	1	0	$\boxed{-1}$	-2
z	1	3	1	-3

Choose the leaving variable $x_k = x_5$ with $b_k = -2 < 0$ and choose the entering variable $x_j = x_3$ with $a_{kj} < 0$ and maximal c_j / a_{kj}. After the full pivot at a_{kj} we get

Basic ↓	x_4	x_2	x_5	
x_1	1	$\boxed{-1}$	2	-1
x_3	-1	0	-1	2
z	2	3	1	-5

Choose the leaving variable $x_k = x_1$ and the entering variable $x_j = x_2$. After the full pivot at a_{kj} we get

Basic $\downarrow$	x_4	x_1	x_5	
x_2	-1	-1	-2	1
x_3	-1	0	-1	2
z	5	3	7	-8

We have $b \geq 0$, hence the optimal solution is $x_1 = 0$, $x_2 = 1$, $x_3 = 2$. The optimal solution of the dual problem is $y_1 = c_4 = 5$, $y_2 = c_5 = 7$.

4. Theoretical aspects of linear programming

4.1. **Fourier-Motzkin elimination.** Fourier-Motzkin elimination (FME) is similar to Gaussian elimination, but deals with systems of linear inequalities instead of equalities. Suppose that we want to find solutions of the system $Ax \le b$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. We will reduce this question to a system with $n-1$ variables. Multiplying the rows of $[A \mid b]$ with positive scalars, we can assume that all entries of the last column of A are $1, -1$ or 0. Let

$$I_\pm = \{i \in [m] \,|\, a_{in} = \pm 1\}, \qquad I_0 = \{i \in [m] \,|\, a_{in} = 0\},$$

so that $[m] = I_+ \sqcup I_- \sqcup I_0$. If $x \in \mathbb{R}^n$ is a solution of $Ax \le b$, then

$$\sum\nolimits_{j=1}^{n-1} a_{ij}x_j \pm x_n \le b_i \qquad\qquad i \in I_\pm.$$
$$\sum\nolimits_{j=1}^{n-1} a_{ij}x_j \le b_i \qquad\qquad i \in I_0.$$

Lemma 4.1. *For $x' = (x_1, \dots, x_{n-1})$ there exists a solution $x = (x', x_n)$ of $Ax \le b$ $\iff$*

$$\begin{aligned} \sum\nolimits_{j=1}^{n-1}(a_{ij} + a_{kj})x_j \le b_i + b_k \quad \forall i \in I_-,\ k \in I_+, \\ \sum\nolimits_{j=1}^{n-1} a_{ij}x_j \le b_i \qquad \forall i \in I_0. \end{aligned} \tag{7}$$

Proof. If $x = (x', x_n) \in \mathbb{R}^n$ is a solution, then

$$\sum\nolimits_{j=1}^{n-1} a_{ij}x_j - b_i \le x_n \le b_k - \sum\nolimits_{j=1}^{n-1} a_{kj}x_j, \qquad i \in I_-,\ k \in I_+ \tag{8}$$

Therefore $\sum_{j=1}^{n-1}(a_{ij} + a_{kj})x_j \le b_i + b_k$. Conversely, assume that this inequality is satisfied for all $i \in I_-, k \in I_+$. Then

$$u = \max_{i\in I_-}\Big(\sum\nolimits_{j=1}^{n-1} a_{ij}x_j - b_i\Big) \le v = \min_{k\in I_+}\Big(b_k - \sum\nolimits_{j=1}^{n-1} a_{kj}x_j\Big)$$

and we can choose any $u \le x_n \le v$. □

Let us denote the original system $Ax \le b$ by $A^{(n)}x \le b^{(n)}$ and the system from (7) by $A^{(n-1)}x \le b^{(n-1)}$, where $A^{(n-1)}$ has n columns with the last column equal zero. Repeating the above procedure, we obtain systems $A^{(k)}x \le b^{(k)}$ for $0 \le k \le n$, where $A^{(k)}$ has n columns with the last $n-k$ columns equal zero. The original problem reduces to finding a solution of $A^{(k)}x \le b^{(k)}$. Note that the rows of $[A^{(k)} \mid b^{(k)}]$ are non-negative linear combinations of the rows of $[A \mid b]$. More precisely,

$$[A^{(k)} \mid b^{(k)}] = B \cdot [A \mid b]$$

for some $B \in \mathbb{R}_+^{m_k\times m}$. We will see later several proofs and interpretations of the following important result, but the next proof is probably the most elementary.

Lemma 4.2 (Farkas lemma)**.** *$\{Ax \le b\}$ is feasible $\iff$ $y^\mathsf{T}A = 0$, $y \ge 0$ implies $y^\mathsf{T}b \ge 0$.*

Proof. If $\{Ax \le b\}$ has a solution x and $y^\mathsf{T}A = 0$, $y \ge 0$, then $y^\mathsf{T}b \ge y^\mathsf{T}(Ax) = 0$. Conversely, assume that $y^\mathsf{T}b \ge 0$ for all $y \ge 0$ such that $y^\mathsf{T}A = 0$ and assume that $\{Ax \le b\}$ is infeasible. Applying the FME, we conclude that the system $0 = A^{(0)}x \le b^{(0)}$ is infeasible, hence $b_i^{(0)} < 0$ for some i. We have $[0 \mid b^{(0)}] = [A^{(0)} \mid b^{(0)}] = B \cdot [A \mid b]$ for some $B \in \mathbb{R}_+^{m_0\times m}$. For $y^\mathsf{T} = B_{i*} \in \mathbb{R}_+^m$, we obtain $y^\mathsf{T}A = 0$ and $y^\mathsf{T}b = b_i^{(0)} < 0$, which is a contradiction. □

Remark 4.3. Recall the LP duality (to be proved later)

$$\sup\{c^\mathsf{T}x \mid Ax \le b\} = \inf\{b^\mathsf{T}y \mid A^\mathsf{T}y = c,\ y \ge 0\}$$

For $c = 0$, this means that $\sup\{0 \mid Ax \le b\} = \inf\{b^\mathsf{T}y \mid A^\mathsf{T}y = 0,\ y \ge 0\}$ (note that the dual problem is feasible). The first condition of the Farkas lemma means that $\sup\{0 \mid Ax \le b\} = 0$. The second condition of the Farkas lemma means that $\inf\{b^\mathsf{T}y \mid A^\mathsf{T}y = 0,\ y \ge 0\} = 0$. Therefore Farkas lemma can be interpreted as LP duality for $c = 0$. We will see later that it is crucial for the proof of general LP duality.

Example 4.4. We can apply FME to solve linear programs. The idea is that we have

$$\max\{c^\mathsf{T}x \mid Ax \le b\} = \max\{\lambda \mid Ax \le b,\ \lambda - c^\mathsf{T}x \le 0\}$$

Indeed, the inequality $\lambda \le c^\mathsf{T}x$ implies that the maximum is attained at $\lambda = c^\mathsf{T}x$, hence the second problem is equivalent to the first one. We consider λ as a new variable and eliminate all other variables using FME. Then we obtain several inequalities satisfied by λ and we easily determine its maximal value. For example, consider

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

We add another inequality $\lambda - 2x_1 - x_2 \le 0$. To eliminate x_2, we note that

$$0 \le x_2, \qquad \lambda - 2x_1 \le x_2, \qquad x_2 \le 1 - x_1$$

Therefore elimination of x_2 leads to

$$0 \le 1 - x_1, \qquad \lambda - 2x_1 \le 1 - x_1, \qquad x_1 \ge 0$$

We obtain $0 \le x_1 \le 1$ and $\lambda - 1 \le x_1$. Elimination of x_1 leads to $\lambda - 1 \le 1$, hence the maximal λ is $\lambda = 2$. Then $\lambda - 1 = 1 \le x_1 \le 1$, hence $x_1 = 1$. Similarly, $0 \le x_2 \le 1 - x_1 = 0$, hence $x_2 = 0$. We conclude that the optimal solution is $x = (1, 0)$.

Alternatively, we can use matrices. We first write $\begin{bmatrix} 0 & A & b \\ 1 & -c^\mathsf{T} & 0 \end{bmatrix}$ (for variables λ, x_1, x_2)

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -2 & -1 & 0 \end{pmatrix}$$

We make all $a_{i3} = \pm 1, 0$ (automatic in our case) and add rows with $a_{i3} = 1$ and rows with $a_{i3} = -1$. Also all rows with $a_{i3} = 0$ remain in the matrix.

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

We add rows with $a_{i2} = 1$ and rows with $a_{i2} = -1$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

We conclude that $\lambda = 2$ is the maximal value. From the previous matrices we conclude that $x_1 = 1$ and $x_2 = 0$.

4.2. **Cones and polyhedra.** Let E be a vector space over $\mathbb{R}$ (usually $E = \mathbb{R}^n$).

Definition 4.5. Let $X \subset E$ be a subset. We define

(1) The *linear span* (or the linear hull)
$$\operatorname{span}(X) = \Big\{ \sum\nolimits_{i=1}^{n} t_i x_i \,\Big|\, x_i \in X,\, t_i \in \mathbb{R},\, n \geq 0 \Big\}.$$
A point in $\operatorname{span}\{x_1, \dots, x_n\}$ is called a *linear combination* of $x_1, \dots, x_n$.

(2) The *affine hull*
$$\operatorname{aff}(X) = \Big\{ \sum\nolimits_{i=1}^{n} t_i x_i \,\Big|\, x_i \in X,\, t_i \in \mathbb{R},\, \sum\nolimits_i t_i = 1,\, n \geq 1 \Big\}.$$
A point in $\operatorname{aff}\{x_1, \dots, x_n\}$ is called an *affine combination* of $x_1, \dots, x_n$.

(3) The (convex) *cone generated* by X
$$\operatorname{cone}(X) = \Big\{ \sum\nolimits_{i=1}^{n} t_i x_i \,\Big|\, x_i \in X,\, t_i \geq 0,\, n \geq 0 \Big\}.$$
A point in $\operatorname{cone}\{x_1, \dots, x_n\}$ is called a *conic combination* of $x_1, \dots, x_n$.

(4) The *convex hull*
$$\operatorname{conv}(X) = \Big\{ \sum\nolimits_{i=1}^{n} t_i x_i \,\Big|\, x_i \in X,\, t_i \geq 0,\, \sum\nolimits_i t_i = 1,\, n \geq 1 \Big\}.$$
A point in $\operatorname{conv}\{x_1, \dots, x_n\}$ is called a *convex combination* of $x_1, \dots, x_n$.

(5) The *interval* $[x, y] = \operatorname{conv}\{x, y\} = \{ tx + (1-t)y \,|\, t \in [0,1] \}$ for $x, y \in E$.

Definition 4.6. Elements $x_1, \dots, x_n \in E$ are called

(1) *linearly independent* if $x_i \notin \operatorname{span}\{x_j \,|\, j \neq i\}$ for all $i \in [n]$.
(2) *affinely independent* if $x_i \notin \operatorname{aff}\{x_j \,|\, j \neq i\}$ for all $i \in [n]$.

Lemma 4.7. *For $x_1, \dots, x_n \in E$ the following are equivalent*

(1) $x_1, \dots, x_n \in E$ are affinely independent.
(2) If $\sum_{i=1}^n t_i x_i = 0$ and $\sum_{i=1}^n t_i = 0$, then $t_1 = \dots = t_n = 0$.
(3) The map $\{ t \in \mathbb{R}^n \mid \sum_i t_i = 1 \} \to E$, $t \mapsto \sum_i t_i x_i$ is injective.
(4) The vectors $x_2 - x_1, \dots, x_n - x_1$ are linearly independent.

Definition 4.8. Let $C \subset E$ be a subset. Then

(1) C is called a *convex set* if $[x, y] = \{ tx + (1-t)y \,|\, t \in [0,1] \} \subset C \ \forall x, y \in C$.
(2) $C \neq \varnothing$ is called a *convex cone* if $\operatorname{cone}\{x, y\} = \{ sx + ty \,|\, s, t \geq 0 \} \subset C \ \forall x, y \in C$.
(3) $C \neq \varnothing$ is called an *affine space* if $\operatorname{aff}\{x, y\} = \{ tx + (1-t)y \,|\, t \in \mathbb{R} \} \subset C \ \forall x, y \in C$.

Lemma 4.9. *Let $C \subset E$ be a subset. Then*

(1) C is a convex set $\iff \operatorname{conv}(C) = C$.
(2) C is a convex cone $\iff \operatorname{cone}(C) = C \iff C$ is convex and $tx \in C \ \forall x \in C$, $t \geq 0$.
(3) C is an affine space $\iff \operatorname{aff}(C) = C \iff C - x = \{ y - x \,|\, y \in C \} \subset E$ is a vector space for some (all) $x \in C$.

Example 4.10. For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$

(1) The set $P = \{ x \in \mathbb{R}^n \,|\, Ax \leq b \}$ is a convex set.
(2) The set $C = \{ x \in \mathbb{R}^n \,|\, Ax \leq 0 \}$ is a convex cone.
(3) The set $D = \{ x \in \mathbb{R}^n \,|\, Ax = b \}$ is an affine space.

Consider the dual space

$$E^* = \{f\colon E \to \mathbb{R} \mid f \text{ is linear}\}.$$

The elements of E^* are called linear functionals (linear forms) on E. For $E = \mathbb{R}^n$ we identify a linear functional $f\colon E \to \mathbb{R}$ with $a \in \mathbb{R}^n$ such that $f(x) = a^\mathsf{T} x$ for all $x \in \mathbb{R}^n$. In this way we obtain an isomorphism $E^* \simeq \mathbb{R}^n$.

Definition 4.11. A subset $P \subset E$ is called

(1) a *polyhedron* if it is an intersection of finitely many half-spaces

$$\{x \in E \mid f_i(x) \le b_i\}, \qquad i \in [m],$$

for some $0 \ne f_i \in E^*$ and $b_i \in \mathbb{R}$.

(2) a *polytope* if $P = \operatorname{conv}\{x_1, \dots, x_n\}$ for finitely many $x_i \in E$.

(3) a *simplex* if $P = \operatorname{conv}\{x_1, \dots, x_n\}$ for some affinely independent $x_i \in E$.

(4) a *polyhedral cone* if it is an intersection of finitely many half-spaces

$$\{x \in E \mid f_i(x) \le 0\}, \qquad i \in [m],$$

for some $0 \ne f_i \in E^*$.

(5) a *finitely generated cone* if $P = \operatorname{cone}\{x_1, \dots, x_n\}$ for finitely many $x_i \in E$.

(6) a *primitive* (or simplicial) cone if $P = \operatorname{cone}\{x_1, \dots, x_n\}$ for some linearly independent $x_i \in E$.

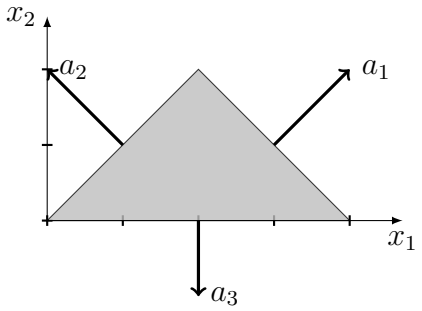
Remark 4.12. For $E = \mathbb{R}^n$ we identify $f_i\colon E \to \mathbb{R}$ with $a_i \in \mathbb{R}^n$ such that $f_i(x) = a_i^\mathsf{T} x$. Let $A \in \mathbb{R}^{m\times n}$ be the matrix with the rows a_i^T and let $b = (b_i)_{i=1}^m \in \mathbb{R}^m$. Then the above polyhedron and polyhedral cone are

$$P = \{x \in \mathbb{R}^n \mid Ax \le b\}, \qquad C = \{x \in \mathbb{R}^n \mid Ax \le 0\}.$$

Generally, we can consider the linear map $A\colon E \to \mathbb{R}^m$, $x \mapsto (f_i(x))_{i=1}^m$, so that the polyhedron is $P = \{x \in E \mid Ax \le b\}$ and the polyhedral cone is $C = \{x \in E \mid Ax \le 0\}$.

Example 4.13. Polytopes are not simplexes in general. For example, the square with the vertices $(0,0), (0,1), (1,0), (1,1)$ is a polytope, but not a simplex since the vertices are not affinely independent.

Example 4.14. For $0 \ne a \in \mathbb{R}^n$ and $c \in \mathbb{R}$, consider the *half-space* $\left\{x \in \mathbb{R}^n \,\middle|\, a^\mathsf{T} x \le c\right\}$. Its boundary is the *hyperplane* $\left\{x \in \mathbb{R}^n \,\middle|\, a^\mathsf{T} x = c\right\}$. The vector a is normal to the hyperplane and is directed outside of the half-space. For example, consider the polyhedron given by $\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \binom{x_1}{x_2} \le \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$.



Remark 4.15. We will see later that $P \subset E$ is a polytope if and only if it is a bounded polyhedron. A convex cone is polyhedral if and only if it is finitely generated.

Lemma 4.16.
(1) A polyhedron is a convex set.
(2) A polyhedral cone is a convex cone.
(3) A polytope is a compact set (bounded and closed).

Proof. **(1)** Let $P = \{x \mid Ax \le b\}$ be a polyhedron and $x, y \in P$. For every $t \in [0,1]$, we have $A(tx + (1-t)y) \le tb + (1-t)b = b$, hence $tx + (1-t)y \in P$.
(2) Let $C = \{x \mid Ax \le 0\}$ be a polyhedral cone and $x, y \in C$. For $s, t \ge 0$, we have $A(sx + ty) = sAx + tAy \le s0 + t0 = 0$, hence $sx + ty \in C$.
(3) Let $P = \operatorname{conv}\{x_1, \dots, x_k\}$ for $x_i \in \mathbb{R}^n$. The set $\Delta = \left\{t \in \mathbb{R}^k_+ \;\middle|\; \sum_i t_i = 1\right\}$ is compact and the map $\phi\colon \Delta \to \mathbb{R}^n$, $t \mapsto \sum_i t_i x_i$, is continuous. Therefore $P = \phi(\Delta)$ is compact. □

Lemma 4.17. *For $X \subset \mathbb{R}^n$, let $P = \operatorname{conv}(X)$ and $C' = \operatorname{cone}\{(x,1) \mid x \in X\} \subset \mathbb{R}^{n+1}$. Then $x \in P \iff (x,1) \in C'$. Equivalently, $P \times \{1\} = C' \cap (\mathbb{R}^n \times \{1\})$.*

Proof. If $x \in P$, then $x = \sum_{i=1}^k t_i x_i$ for some $x_i \in X$ and $t_i \ge 0$ satisfying $\sum_i t_i = 1$. Therefore $\sum_i t_i(x_i, 1) = (x, 1)$, hence $(x,1) \in C'$. Conversely, if $(x,1) \in C'$, then $(x,1) = \sum_{i=1}^k t_i(x_i, 1)$ for some $x_i \in X$ and $t_i \ge 0$. Then $\sum_i t_i = 1$ and $x = \sum_i t_i x_i$, hence $x \in \operatorname{conv}(X) = P$. □

Theorem 4.18 (Carathéodory's theorem)**.** *Let $X \subset E = \mathbb{R}^n$ be a non-empty subset. Then*
(1) If $x \in \operatorname{cone}(X)$, then $x \in \operatorname{cone}\{x_1, \dots, x_k\}$ for $k \le n$ points $x_i \in X$.
(2) If $x \in \operatorname{conv}(X)$, then $x \in \operatorname{conv}\{x_1, \dots, x_k\}$ for $k \le n+1$ points $x_i \in X$.

Proof. **(1)** We have $x \in \operatorname{cone}\{x_1, \dots, x_m\}$ for some $x_i \in X$. Let $I \subset [m]$ be the minimal finite set such that $x \in \operatorname{cone}\{x_i \mid i \in I\}$. Then $x = \sum_{i \in I} t_i x_i$ for some $t_i > 0$. Assume that $\sum_{i \in I} s_i x_i = 0$ for some $0 \ne s \in \mathbb{R}^I$. There exists $\varepsilon \in \mathbb{R}$ such that $t'_i = t_i - \varepsilon s_i \ge 0$ for all $i \in I$ and $t'_i = 0$ for some $i \in I$. Then $\sum_{i \in I} t'_i x_i = \sum_i t_i x_i = x$, contradicting to the minimality of I. Therefore x_i for $i \in I$ are linearly independent, hence $|I| \le n$.
(2) Let $C' = \operatorname{cone}\{(x,1) \mid x \in X\} \subset \mathbb{R}^{n+1}$. Then $x \in \operatorname{conv}(X) \iff (x,1) \in C'$. We obtain from (1) that there exist elements $x_1, \dots, x_k \in X$, $k \le n+1$, such that $(x,1) \in \operatorname{cone}\{(x_i, 1) \mid 1 \le i \le k\}$. Then $x \in \operatorname{conv}\{x_1, \dots, x_k\}$. □

Remark 4.19. We proved actually that if $x \in \operatorname{cone}(X)$, then $x \in \operatorname{cone}\{x_1, \dots, x_k\}$ for k linearly independent vectors $x_i \in X$ (then $k \le n$ automatically). Similarly, if $x \in \operatorname{conv}(X)$, then $x \in \operatorname{conv}\{x_1, \dots, x_k\}$ for k affinely independent points $x_i \in X$ (then $k \le n+1$).

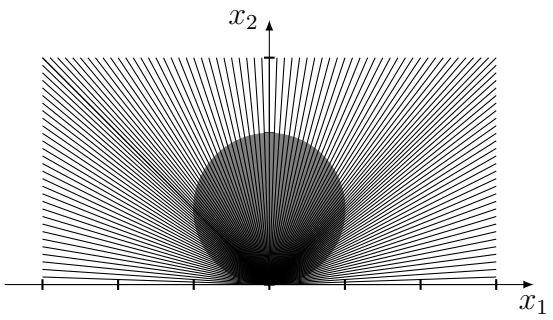
Lemma 4.20. *If $X \subset \mathbb{R}^n$ is compact, then $\operatorname{conv}(X)$ is also compact.*

Proof. For $\Delta = \left\{t \in \mathbb{R}^{n+1}_+ \;\middle|\; \sum_i t_i = 1\right\}$, the map $\phi\colon \Delta \times X^{n+1} \to \mathbb{R}^n$, $(t, x) \mapsto \sum_i t_i x_i$, has the image $\operatorname{conv}(X)$ by Carathéodory's theorem. This map is continuous and both Δ and X^{n+1} are compact, hence $\operatorname{Im}(\phi) = \operatorname{conv}(X)$ is compact. □

Lemma 4.21. *A finitely generated convex cone is closed.*

Proof. First assume that a cone $C \subset \mathbb{R}^n$ is primitive, meaning that it is generated by linearly independent vectors $x_1, \dots, x_k \in \mathbb{R}^n$. Then $V = \operatorname{span}\{x_1, \dots, x_k\} \simeq \mathbb{R}^k$ and $C \simeq \operatorname{cone}\{e_1, \dots, e_k\}$ under this homeomorphism, hence C is closed in V and in $\mathbb{R}^n$. For an arbitrary cone $C = \operatorname{cone}\{x_1, \dots, x_k\}$ let us show that it is a finite union of primitive cones. For any $x \in C$ we proved that $x \in \operatorname{cone}\{x_i \mid i \in I\}$ for some $I \subset [k]$ such that x_i, for $i \in I$, are linearly independent. But this means that x lies in the primitive cone $\operatorname{cone}\{x_i \mid i \in I\}$. We conclude that C is a finite union of closed sets, hence C is closed. □

Example 4.22. There exist compact (and convex) sets $X \subset \mathbb{R}^n$ such that $\operatorname{cone}(X)$ is not closed. For example, consider the disc $D = \{x \in \mathbb{R}^2 \mid \|x - y\| \leq 1\}$, where $y = (0, 1)$. Then D is compact and $\operatorname{cone}(D) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0 \text{ or } x_1 = x_2 = 0\}$ is not closed.



Problem 2: *Show that if $X \subset E$ is convex, then* $\operatorname{cone}(X) = \{tx \mid x \in X, t \geq 0\}$.

4.3. **Separation theorems.**

Theorem 4.23 (Hyperplane separation theorem)**.** *Let A, $B \subset \mathbb{R}^n$ be non-empty disjoint convex sets. Then there exists nonzero $v \in \mathbb{R}^n$ such that*

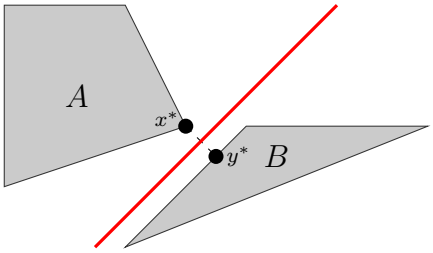
$$\inf\{v \cdot x \,|\, x \in A\} \geq \sup\{v \cdot y \,|\, y \in B\}.$$

If A is compact and B is closed, then there exists $v \in \mathbb{R}^n$ with a strict inequality. This means, for $c \in \mathbb{R}$ between the infimum and the supremum, that A lies on one side of the hyperplane $\{x \in \mathbb{R}^n \,|\, v \cdot x = c\}$ and B lies on the other side.

Proof. We assume that A is compact and B is closed. The idea is to find the closest points $x^* \in A$ and $y^* \in B$ and to define $v = x^* - y^*$. Let $d(x, y) = \|x - y\|$ for $x, y \in \mathbb{R}^n$. Let

$$f \colon \mathbb{R}^n \to \mathbb{R}, \qquad f(x) = d(x, B) = \inf\{d(x, y) \,|\, y \in B\}.$$

This function is continuous, hence there exists $x^* \in A$ such that $\inf\{f(x) \,|\, x \in A\} = f(x^*)$. There exists $y^* \in B$ such that $f(x^*) = d(x^*, B) = d(x^*, y^*)$: let $y' \in B$, $r = d(x^*, y')$ and $B' = \{y \in B \,|\, d(x^*, y) \leq r\}$; then $d(x^*, B) = d(x^*, B')$ and B' is compact. We conclude that $d(x, y) \geq d(x, B) \geq d(x^*, B) = d(x^*, y^*)$ for all $x \in A$, $y \in B$.



Let $v = x^* - y^* \in \mathbb{R}^n$. Then $v \neq 0$, hence $v \cdot v > 0$ and $v \cdot x^* > v \cdot y^*$. We claim that $v \cdot x \geq v \cdot x^*$ for all $x \in A$ and $v \cdot y^* \geq v \cdot y$ for all $y \in B$ (this proves the theorem).

For $t \in [0, 1]$ and $z = tx + (1 - t)x^* = t(x - x^*) + x^* \in A$, we have $d(z, y^*) \geq d(x^*, y^*)$, hence

$$(t(x - x^*) + v) \cdot (t(x - x^*) + v) \geq v \cdot v \implies 2tv \cdot (x - x^*) + t^2(x - x^*)^2 \geq 0.$$

Dividing by $t > 0$, we conclude that $v \cdot (x - x^*) \geq 0$. Similarly, $v \cdot (y^* - y) \geq 0$. □

Theorem 4.24 (Geometric Farkas lemma)**.** *Let $a_1, \ldots, a_n, b \in \mathbb{R}^m$. Then $b \notin \operatorname{cone}\{a_1, \ldots, a_n\}$ $\iff$ there exists $y \in \mathbb{R}^m$ such that $y^\mathsf{T} a_i \geq 0$ for all $1 \leq i \leq n$ and $y^\mathsf{T} b < 0$. This means that the vectors a_i lie on one side of the hyperplane $\left\{x \in \mathbb{R}^m \,\middle|\, y^\mathsf{T} x = 0\right\}$ and b lies on the other side.*

Proof. If there exists $y \in \mathbb{R}^m$ as above, then $y^\mathsf{T} x \geq 0$ for all $x \in C = \operatorname{cone}\{a_1, \ldots, a_n\}$. Therefore $b \notin C$. Conversely, assume that $b \notin C$. The cone C is closed by Lemma 4.21 and the set $B = \{b\}$ is compact. Since $b \notin C$, we have $B \cap C = \varnothing$. By the Hyperplane separation theorem there exists $y \in \mathbb{R}^m$ such that $\inf\left\{y^\mathsf{T} x \,\middle|\, x \in C\right\} > y^\mathsf{T} b$. If $\lambda = y^\mathsf{T} x < 0$ for some $x \in C$, then $t\lambda \in I = \left\{y^\mathsf{T} x \,\middle|\, x \in C\right\}$ for all $t > 0$ (as C is a cone). But this implies $\inf(I) = -\infty$, a contradiction. We conclude that $\inf I \geq 0$. Since $0 \in C$, we have $\inf(I) = 0$. Therefore $y^\mathsf{T} a_i \geq \inf(I) = 0 > y^\mathsf{T} b$ for all i. □

Lemma 4.25 (Farkas Lemma)**.** $\{Ax = b, x \geq 0\}$ *is feasible* $\iff$ $y^\mathsf{T} A \geq 0$ *implies* $y^\mathsf{T} b \geq 0$.

Proof. We will prove the backward direction. Let $a_1, \ldots, a_n \in \mathbb{R}^m$ be the columns of A. Then $P = \{Ax = b, x \geq 0\}$ is nonempty $\iff$ $b \in C = \operatorname{cone}\{a_1, \ldots, a_n\}$ (if $x \in P$, then $b = \sum_j x_j a_j \in C$). If $P = \varnothing$, then $b \notin C$, hence by the geometric Farkas lemma 4.24 there exists $y \in \mathbb{R}^m$ such that $y^\mathsf{T} a_i \geq 0$ $\forall i$ and $y^\mathsf{T} b < 0$. Then $y^\mathsf{T} A \geq 0$. □

4.4. **Versions of Farkas Lemma.**

Theorem 4.26 (Farkas lemma)**.** *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then*

(1) The system $\{Ax \le b\}$ is feasible $\iff$ $y^\mathsf{T}A = 0$, $y \ge 0$ implies $y^\mathsf{T}b \ge 0$.
(2) The system $\{Ax \le b, x \ge 0\}$ is feasible $\iff$ $y^\mathsf{T}A \ge 0$, $y \ge 0$ implies $y^\mathsf{T}b \ge 0$.
(3) The system $\{Ax = b, x \ge 0\}$ is feasible $\iff$ $y^\mathsf{T}A \ge 0$ implies $y^\mathsf{T}b \ge 0$.

Proof. (1) is proved in Lemma 4.2 and (3) is proved in Lemma 4.25. Here we will show that all three statements can be obtained from each other. The forward direction in all three cases is straightforward:

(1) If $Ax = b$, $x \ge 0$, $y^\mathsf{T}A \ge 0$, then $y^\mathsf{T}b = y^\mathsf{T}Ax \ge 0$.
(2) If $Ax \le b$, $y^\mathsf{T}A = 0$, $y \ge 0$, then $y^\mathsf{T}b \ge y^\mathsf{T}Ax = 0$.
(3) If $Ax \le b$, $x \ge 0$, $y^\mathsf{T}A \ge 0, y \ge 0$, then $y^\mathsf{T}b \ge y^\mathsf{T}Ax \ge 0$.

(1) $\implies$ (2). The system $\{Ax \le b, x \ge 0\}$ is equivalent to $\{A'x \le b'\}$ for $A' = \left[\begin{smallmatrix} A \\ -I \end{smallmatrix}\right]$ and $b' = \left(\begin{smallmatrix} b \\ 0 \end{smallmatrix}\right) \in \mathbb{R}^{m+n}$. If $v = \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) \in \mathbb{R}^{m+n}$ satisfies $v^\mathsf{T}A' = 0$, $v \ge 0$, then $y^\mathsf{T}A = z^\mathsf{T} \ge 0$ and $y \ge 0$, hence $v^\mathsf{T}b' = y^\mathsf{T}b \ge 0$ by our assumption. Therefore the conditions of (1) are satisfied.
(2) $\implies$ (3). The system $\{Ax = b, x \ge 0\}$ is equivalent to $\{A'x \le b', x \ge 0\}$ for $A' = \left[\begin{smallmatrix} A \\ -A \end{smallmatrix}\right]$ and $b' = \left(\begin{smallmatrix} b \\ -b \end{smallmatrix}\right)$. If $v = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \in \mathbb{R}^{m+m}$ satisfies $v^\mathsf{T}A' \ge 0$ (and $v \ge 0$), then $y = y_1 - y_2$ satisfies $y^\mathsf{T}A \ge 0$, hence $v^\mathsf{T}b' = y^\mathsf{T}b \ge 0$ by our assumption. Therefore the conditions of (2) are satisfied.
(3) $\implies$ (1). To show feasibility of $\{Ax \le b\}$, we will show that $\{A'x' = b, x' \ge 0\}$ is feasible for $A' = [A \mid -A \mid I]$. If $x' = \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \ge 0$ is a solution, then $A(x_1 - x_2) + x_3 = b$, hence $Ax \le b$ for $x = x_1 - x_2$. If $y^\mathsf{T}A' \ge 0$, then $y^\mathsf{T}A = 0$ and $y \ge 0$, hence $y^\mathsf{T}b \ge 0$ by our assumption. Therefore the conditions of (3) are satisfied. □

The above result is similar to the following.

Lemma 4.27 (Fredholm alternative)**.** *For $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, the system $Ax = b$ has a solution $\iff$ $A^\mathsf{T}y = 0$ implies $b^\mathsf{T}y = 0$. Equivalently, exactly one of the following is true*

(1) $Ax = b$ has a solution.
(2) $A^\mathsf{T}y = 0$ for some y with $b^\mathsf{T}y \ne 0$.

Proof. If $Ax = b$ and $A^\mathsf{T}y = 0$, then $b^\mathsf{T}y = x^\mathsf{T}A^\mathsf{T}y = 0$. Conversely, assume that $Ax \ne b$ for all x. Then $b \notin V = A(\mathbb{R}^n) \subset \mathbb{R}^m$. We can write $b = (v, y) \in V \oplus V^\perp = \mathbb{R}^m$ with $y \ne 0$ so that $b^\mathsf{T}y = b \cdot y = y \cdot y \ne 0$. We have $y^\mathsf{T}Ax = 0$ for all $x \in \mathbb{R}^n$, hence $y^\mathsf{T}A = 0$ and $A^\mathsf{T}y = 0$. □

4.5. **Linear programming duality.**

Theorem 4.28 (LP duality 1)**.** *For all $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ we have*

$$\max\{c^\mathsf{T}x \,|\, Ax \le b,\, x \ge 0\} = \min\{b^\mathsf{T}y \,|\, A^\mathsf{T}y \ge c,\, y \ge 0\},$$

provided that both sets $P = \{Ax \le b, x \ge 0\}$ and $D = \{A^\mathsf{T}y \ge c,\, y \ge 0\}$ are nonempty. In particular, there exist optimal solutions $x^* \in P$ and $y^* \in D$ such that $c^\mathsf{T}x^* = b^\mathsf{T}y^*$.

Proof. If $x \in P$ and $y \in D$, then $c^\mathsf{T}x \le (y^\mathsf{T}A)x = y^\mathsf{T}(Ax) \le y^\mathsf{T}b$. To prove the equality, we need to show that there exist $x \in P$ and $y \in D$ satisfying $c^\mathsf{T}x \ge b^\mathsf{T}y$. Equivalently, we require

$$\begin{bmatrix} A & 0 \\ 0 & -A^\mathsf{T} \\ -c^\mathsf{T} & b^\mathsf{T} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} b \\ -c \\ 0 \end{pmatrix}$$

for some $x, y \ge 0$. To prove that this system is feasible, we apply Farkas lemma (2). We need to show that if $(u, v, \lambda) \ge 0$ satisfies

$$u^\mathsf{T}A - \lambda c^\mathsf{T} \ge 0, \qquad -v^\mathsf{T}A^\mathsf{T} + \lambda b^\mathsf{T} \ge 0,$$

then $u^\mathsf{T}b - v^\mathsf{T}c \ge 0$. If $\lambda > 0$, then $x = v/\lambda \ge 0$ and $y = u/\lambda \ge 0$ satisfy $Ax \le b$, $A^\mathsf{T}y \ge c$. Therefore $c^\mathsf{T}x \le b^\mathsf{T}y$ and $v^\mathsf{T}c \le u^\mathsf{T}b$. If $\lambda = 0$, then $u^\mathsf{T}A \ge 0$ and $v^\mathsf{T}A^\mathsf{T} \le 0$. For arbitrary $x \in P$, $y \in D$, we obtain $u^\mathsf{T}b \ge u^\mathsf{T}Ax \ge 0 \ge v^\mathsf{T}A^\mathsf{T}y \ge v^\mathsf{T}c$. □

Theorem 4.29 (LP duality 2)**.** *For all $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ we have*

$$\max\{c^\mathsf{T}x \,|\, Ax \le b\} = \min\{b^\mathsf{T}y \,|\, A^\mathsf{T}y = c,\, y \ge 0\},$$

provided that both sets $P = \{Ax \le b\}$ and $D = \{A^\mathsf{T}y = c,\, y \ge 0\}$ are nonempty. In particular, there exist optimal solutions $x^* \in P$ and $y^* \in D$ such that $c^\mathsf{T}x^* = b^\mathsf{T}y^*$.

Proof. If $x \in P$ and $y \in D$, then $c^\mathsf{T}x = y^\mathsf{T}Ax \le y^\mathsf{T}b$. The problem $\max\{c^\mathsf{T}x \,|\, Ax \le b\}$ is equivalent to $\max\{(c')^\mathsf{T}x' \,|\, A'x' \le b,\, x' \ge 0\}$, where $A' = \begin{bmatrix} A & -A \end{bmatrix}$, $c' = \left(\begin{smallmatrix} c \\ -c \end{smallmatrix}\right)$ and $x' = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$: if $x_1 = [x]_+$ and $x_2 = [x]_-$, then $A'x' = A(x_1 - x_2) = Ax$ and $(c')^\mathsf{T}x' = c^\mathsf{T}x$. The problem $\min\{b^\mathsf{T}y \,|\, A^\mathsf{T}y = c,\, y \ge 0\}$ is equivalent to $\min\{b^\mathsf{T}y \,|\, (A')^\mathsf{T}y \le c',\, y \ge 0\}$. Now we apply the previous theorem. □

Remark 4.30. If both problems are infeasible, then we have $\sup\{c^\mathsf{T}x \,|\, Ax \le b\} = -\infty$ and $\inf\{b^\mathsf{T}y \,|\, A^\mathsf{T}y = c,\, y \ge 0\} = +\infty$. If at least one of the problems if feasible, then

$$\sup\{c^\mathsf{T}x \,|\, Ax \le b\} = \inf\{b^\mathsf{T}y \,|\, A^\mathsf{T}y = c,\, y \ge 0\}.$$

This follows from Theorem 4.29 if both problems are feasible. If the primal problem is feasible and the dual problem is infeasible, then both sides are equal to $+\infty$ by Lemma 4.31. The other case is proved in the same way.

Lemma 4.31. *If the problem $\max\{c^\mathsf{T}x \,|\, Ax \le b\}$ is feasible, then it is bounded $\iff$ the dual problem $\min\{b^\mathsf{T}y \,|\, A^\mathsf{T}y = c,\, y \ge 0\}$ is feasible.*

Proof. Let $P = \{x \,|\, Ax \le b\}$ and $D = \{y \,|\, A^\mathsf{T}y = c,\, y \ge 0\}$. If $D \ne \varnothing$ and $y \in D$, then we know that $c^\mathsf{T}x \le b^\mathsf{T}y$ for all $x \in P$, hence the problem is bounded (that is, $\sup\{c^\mathsf{T}x \,|\, x \in P\} < +\infty$). If $D = \varnothing$, then by Farkas lemma (3) there exists $z \in \mathbb{R}^n$ such that $Az \ge 0$ and $c^\mathsf{T}z < 0$. If $x \in P$ and $t > 0$, then $A(x - tz) \le Ax \le b$, hence $x - tz \in P$. We have $\lim_{t\to+\infty} c^\mathsf{T}(x - tz) = +\infty$. □

Corollary 4.32. *If the problem* $\max\{c^\mathsf{T}x \mid Ax \le b\}$ *is feasible and bounded, then the optimal value is attained.*

Proof. By the previous lemma, we have $D = \{y \mid A^\mathsf{T}y = c,\ y \ge 0\} \ne \varnothing$. Therefore we can apply Theorem 4.29 and conclude that the optimal solution exists. □

Example 4.33. The above statement is automatically true for any problem of the form $\max\{f(x) \mid x \in P\}$, where P is a compact topological space and $f\colon P \to \mathbb{R}$ is a continuous function. But it is not true in general for a non-compact polyhedron P and a non-linear objective function f. For example, consider the polyhedron $P = [1, +\infty)$ and the objective function $f(x) = 1 - 1/x$. Then the optimal value is $p^* = \sup\{f(x) \mid x \ge 1\} = 1$, hence the problem is feasible and bounded above. But the optimal value is not attained since $f(x) < 1$ for all $x \in P$.

4.6. **Minkowski-Weyl theorem.**

Theorem 4.34 (MW 1)**.** *A convex cone is finitely generated if an only if it is polyhedral.*

Proof. $\Longrightarrow$ Let $C = \operatorname{cone}\{b_1, \dots, b_k\} \subset \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times k}$ be the matrix with the columns b_i. Then $C = \left\{By \,\middle|\, y \in \mathbb{R}^k_+\right\} = \{x \in \mathbb{R}^n \,|\, x - By = 0, y \geq 0\}$. We apply FM elimination to the system $x - By = 0$, $y \geq 0$ to eliminate all variables y_i. Then we obtain a system of the form $Ax \leq 0$ for some $A \in \mathbb{R}^{m\times n}$. We conclude that $C = \{x \in \mathbb{R}^n \,|\, Ax \leq 0\}$.

$\Longleftarrow$ Consider a polyhedral cone $C = \left\{x \in \mathbb{R}^n \,\middle|\, a_i^\mathsf{T} x \leq 0 \ \forall i\right\}$ for some $a_i \in \mathbb{R}^n$. We proved that $C' = \operatorname{cone}\{a_1, \dots, a_m\}$ is a polyhedral cone, hence $C' = \left\{y \in \mathbb{R}^n \,\middle|\, b_i^\mathsf{T} y \leq 0 \ \forall i\right\}$ for some $b_i \in \mathbb{R}^n$. We claim that $C = \operatorname{cone}\{b_1, \dots, b_k\}$. We have $b_i^\mathsf{T} a_j \leq 0$, hence $b_i \in C$ and $\operatorname{cone}\{b_1, \dots, b_k\} \subset C$. Let $x \in C$ and $x \notin \operatorname{cone}\{b_1, \dots, b_k\}$. By the Farkas lemma, there exists $y \in \mathbb{R}^n$ with $y^\mathsf{T} b_i \leq 0$ for all i and $y^\mathsf{T} x > 0$. This implies $y \in C'$. As $x \in C$, we have $x^\mathsf{T} a_i \leq 0$ for all i, hence $x^\mathsf{T} y \leq 0$, a contradiction. □

Theorem 4.35 (MW 2, Decomposition theorem)**.** *A set $P \subset \mathbb{R}^n$ is a polyhedron $\iff$ $P = C + Q = \{x + y \,|\, x \in Q,\, y \in C\}$ for a polyhedral cone C and a polytope Q.*

Proof. **(1)** Let $P = \{x \in \mathbb{R}^n \,|\, Ax \leq b\}$ be a polyhedron. Consider the polyhedral cone

$$C' = \{(x, c) \in \mathbb{R}^n \oplus \mathbb{R} \,|\, Ax - cb \leq 0,\, c \geq 0\}.$$

It is generated by finitely many vectors $(x_1, c_1), \dots, (x_m, c_m)$. We can assume that $c_i = 1$ or $c_i = 0$ for all i. Let $Q = \operatorname{conv}\{x_i \,|\, c_i = 1\}$ and $C = \operatorname{cone}\{x_i \,|\, c_i = 0\}$. We have $x \in P \iff (x, 1) \in C'$. If $x \in Q$ and $y \in C$, then $(x, 1) \in C'$, $(y, 0) \in C'$, hence $(x + y, 1) \in C'$ and $x + y \in P$. Conversely, if $x \in P$, then $(x, 1) \in C'$, hence $x = \sum_i t_i x_i$ and $1 = \sum_i t_i c_i$ for some $t_i \geq 0$. We obtain $\sum_{i\in I} t_i = 1$ for $I = \{i \,|\, c_i = 1\}$, hence $y = \sum_{i\in I} t_i x_i \in Q$, $z = \sum_{i\notin I} t_i x_i \in C$ and $x = y + z \in Q + C$.

(2) Let $Q = \operatorname{conv}\{x_1, \dots, x_m\}$, $C = \operatorname{cone}\{y_1, \dots, y_r\}$ and $P = Q + C$. Then $x \in P \iff$

$$(x, 1) \in C' = \operatorname{cone}\{(x_1, 1), \dots, (x_m, 1), (y_1, 0) \dots, (y_r, 0)\}.$$

Indeed, if $x \in P$, then $x = \sum_i s_i x_i + \sum_j t_j y_j$ for some $s_i, t_j \geq 0$ with $\sum_i s_i = 1$. Then $(x, 1) = \sum_i s_i (x_i, 1) + \sum_j t (y_j, 0) \in C'$. The converse is similar. The cone C' is polyhedral by Theorem 4.34, hence it can be written as $C' = \{(x, c) \in \mathbb{R}^n \oplus \mathbb{R} \,|\, Ax + cb \leq 0\}$ for some $[A\ b] \in \mathbb{R}^{k\times(n+1)}$. Therefore $x \in P \iff Ax + b \leq 0$, hence P is a polyhedron. □

Theorem 4.36 (MW 3)**.** *A set $P \subset \mathbb{R}^n$ is a polytope $\iff$ P is a bounded polyhedron.*

Proof. If P is a polytope, then it is bounded. It is a polyhedron by the decomposition theorem. if P is a bounded polyhedron, then $P = Q + C$ for a polytope Q and a polyhedral cone C. Then C is bounded, hence $C = \{0\}$ and $P = Q$. □

Lemma 4.37. *If $P \subset \mathbb{R}^n$ is a polyhedron and $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $f(P)$ is a polyhedron.*

Proof. For any $X \subset \mathbb{R}^n$, we have $f(\operatorname{conv}(X)) = \operatorname{conv}(f(X))$ and $f(\operatorname{cone}(X)) = \operatorname{cone}(f(X))$. This implies that if $P = \operatorname{conv}\{x_1, \dots, x_k\} \subset \mathbb{R}^n$ is a polytope, then so is $f(P)$ and if $C = \operatorname{cone}\{x_1, \dots, x_k\} \subset \mathbb{R}^n$ is a finitely generated cone, then so is $f(C)$. Using the decomposition $P = Q + C$, where P is a polytope C is a finitely generated cone, we obtain that $f(P) = f(Q) + f(C)$ is a polyhedron. □

Remark 4.38. For a polyhedron $P \subset \mathbb{R}^n$ and a vector $c \in \mathbb{R}^n$, the set $I = \left\{c^\mathsf{T} x \,\middle|\, x \in P\right\} \subset \mathbb{R}$ is a polyhedron. If I is bounded above, then the maximum is attained (cf. Corollary 4.32).

5. Integer linear programming

5.1. **Integer linear programming.** An integer linear program (ILP) is a problem

$$\begin{aligned} \text{maximize} \quad & c^{\mathsf{T}}x \\ \text{subject to} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

where $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$. Note that we can multiply A, b and c by a positive integer and assume that $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. A mixed ILP is a problem

$$\begin{aligned} \text{maximize} \quad & c^{\mathsf{T}}x \\ \text{subject to} \quad & Ax \leq b \\ & x \in \mathbb{Z}^k \times \mathbb{Q}^{n-k} \end{aligned}$$

for some $1 \leq k \leq n$. A binary (or 0, 1) linear program is a problem

$$\begin{aligned} \text{maximize} \quad & c^{\mathsf{T}}x \\ \text{subject to} \quad & Ax \leq b \\ & x \in \{0,1\}^n \end{aligned}$$

The linear programming relaxation of the above problems is the linear program

$$\begin{aligned} \text{maximize} \quad & c^{\mathsf{T}}x \\ \text{subject to} \quad & Ax \leq b \\ & x \in \mathbb{Q}^n \end{aligned}$$

Example 5.1 (Assignment problem)**.** There are n workers that should perform n jobs. The cost of assigning worker i to job j is c_{ij}. We need to find the cheapest way of assigning one job to each worker. Alternatively, let c_{ij} be the enthusiasm level of worker i doing job j. We need to maximize the total enthusiasm level. The number of possible assignments is equal to the number of permutations (with $\sigma \in S_n$ indicating that job $\sigma(i)$ is assigned to worker i) which is $n!$. For $n = 70$ the value of $n!$ is approximately 10^{100} which exceeds the number of atoms in the universe (approximately 10^{80}). A binary programming formulation of the problem is the following. Let $x_{ij} = 1$ if worker i is assigned to job j and 0 otherwise. The problem is

$$\begin{aligned} \text{minimize} \quad & \sum\nolimits_{i,j} c_{ij}x_{ij} \\ \text{subject to} \quad & \sum\nolimits_{j} x_{ij} = 1 \quad \forall i \in [n] \\ & \sum\nolimits_{i} x_{ij} = 1 \quad \forall j \in [n] \\ & x_{i,j} \in \{0,1\} \quad \forall i,\, j \in [n] \end{aligned}$$

5.2. **Integral polyhedra.** Assume that we have a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ with rational A and b and we want to solve an ILP $\max\{c^\mathsf{T}x \mid x \in P \cap \mathbb{Z}^n\}$. It is equal to $\max\{c^\mathsf{T}x \mid x \in \operatorname{conv}(P \cap \mathbb{Z}^n)\}$ (see below). We say that P is an integral polyhedron if $P = \operatorname{conv}(P \cap \mathbb{Z}^n)$. In this case we can apply simplex method to find an optimal solution of the LP $\max\{c^\mathsf{T}x \mid Ax \le b\}$ (a vertex of P) which is automatically an optimal solution of the ILP. In this section we will investigate properties of integral polyhedra.

Lemma 5.2. *Let $S \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n$. Then*

$$\sup\{c^\mathsf{T}x \mid x \in S\} = \sup\{c^\mathsf{T}x \mid x \in \operatorname{conv}(S)\}.$$

The supremum is attained over S if and only if it is attained over $\operatorname{conv}(S)$.

Proof. **(1)** We have $S \subset P = \operatorname{conv}(S)$, hence $z = \sup\{c^\mathsf{T}x \mid x \in S\} \le \sup\{c^\mathsf{T}x \mid x \in P\}$. Let $x = t_1x_1 + \cdots + t_kx_k \in P$ be a convex combination of points in S. Then $c^\mathsf{T}x = \sum_i t_i(c^\mathsf{T}x_i) \le \sum_i t_i z = z$. Therefore $\sup\{c^\mathsf{T}x \mid x \in P\} \le z$.
(2) If the supremum of $\{c^\mathsf{T}x \mid x \in P\}$ is attained at $x^* = t_1x_1 + \cdots + t_kx_k$, then $c^\mathsf{T}x^* \le \max_i c^\mathsf{T}x_i$, hence there exists i such that $c^\mathsf{T}x_i = c^\mathsf{T}x^*$ and we have $x_i \in S$. □

Definition 5.3.
(1) The *integral hull* of $P \subset \mathbb{R}^n$ is $P_I = \operatorname{conv}_\mathbb{Z}(P) = \operatorname{conv}(P \cap \mathbb{Z}^n)$.
(2) A convex set $P \subset \mathbb{R}^n$ is called *integral* if $\operatorname{conv}(P \cap \mathbb{Z}^n) = P$.
(3) A polyhedron $P \subset \mathbb{R}^n$ is called *rational* if $P = \{x \mid Ax \le b\}$ for some $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$. Note that we can choose $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$.

Example 5.4. The polyhedron $P = [0, \frac{1}{2}] = \{x \in \mathbb{R} \mid \binom{-1}{2}x \le \binom{0}{1}\}$ is rational. We have $P_I = \operatorname{conv}(P \cap \mathbb{Z}) = \{0\} \ne P$, hence P is not integral. The polyhedron $P = [0, 1]$ is integral, since $P \cap \mathbb{Z} = \{0, 1\}$ and $P_I = \operatorname{conv}\{0, 1\} = P$.

Problem 3: *A polytope P is integral $\iff$ all of its vertices are integral (contained in $\mathbb{Z}^n$).*

Remark 5.5. If $P \subset \mathbb{R}^n$ is a polytope, then it is bounded, hence the set $P \cap \mathbb{Z}^n$ is finite. If P is an integral polytope, then all of its vertices are contained in $\mathbb{Z}^n \subset \mathbb{Q}^n$, hence P is a rational polyhedron by Minkowski-Weyl Theorem 4.36 (its rational version).

Remark 5.6. We define a *perfect formulation* of a set $S \subset \mathbb{R}^n$ to be a linear system of inequalities $Ax \le b$ such that $\operatorname{conv}(S) = \{x \in \mathbb{R}^n \mid Ax \le b\}$. We have $\sup\{c^\mathsf{T}x \mid x \in S\} = \sup\{c^\mathsf{T}x \mid Ax \le b\}$, hence we can apply LP methods to find an optimal solution over S. If $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ is an integral polyhedron, then $S = P \cap \mathbb{Z}^n$ has a perfect formulation, since $\operatorname{conv}(S) = P = \{x \mid Ax \le b\}$.

Lemma 5.7.
(1) If $S \subset \mathbb{Z}^n$, then $\operatorname{conv}(S)$ *is an integral convex set.*
(2) If $P \subset \mathbb{R}^n$ is convex, then $P_I = \operatorname{conv}(P \cap \mathbb{Z}^n)$ is integral and $P_I \cap \mathbb{Z}^n = P \cap \mathbb{Z}^n$.

Proof. **(1)** For $P = \operatorname{conv}(S)$, we have $S \subset P \cap \mathbb{Z}^n$, hence $P = \operatorname{conv}(S) \subset \operatorname{conv}(P \cap \mathbb{Z}^n)$.
(2) For $S = P \cap \mathbb{Z}^n$, the set $P_I = \operatorname{conv}(S)$ is integral by (1). We have $S \subset P$, hence $P_I \subset P$ and $P_I \cap \mathbb{Z}^n \subset P \cap \mathbb{Z}^n$. We have $P \cap \mathbb{Z}^n = S \subset P_I \cap \mathbb{Z}^n$. □

Lemma 5.8. *A rational polyhedral cone $C \subset \mathbb{R}^n$ is integral, meaning that $C = \operatorname{conv}(C \cap \mathbb{Z}^n)$.*

Proof. We have $C_I = \operatorname{conv}(C \cap \mathbb{Z}^n) \subset C$. Let us show that $C \subset C_I$. The cone C is finitely generated by rational vectors, hence we can write $C = \operatorname{cone}\{a_1, \ldots, a_k\}$ for some $a_i \in \mathbb{Z}^n$. Let

$x = \sum_i t_i a_i$ for some $t_i \geq 0$ and let $u = \sum_i t_i > 0$ and $v = \lceil u \rceil \in \mathbb{Z}$. Then $s_i = t_i/v \geq 0$ satisfy $\sum_i s_i = u/v \leq 1$, hence $s_0 = 1 - \sum_{i=1}^k s_k \geq 0$. We have $x = \sum_{i=1}^k t_i a_i = \sum_{i=1}^k s_i(va_i) + s_0 \cdot 0$, where $va_i \in C \cap \mathbb{Z}^n$, $0 \in C \cap \mathbb{Z}^n$, and $\sum_{i=0}^k s_i = 1$. Therefore $x \in \operatorname{conv}(C \cap \mathbb{Z}^n) = C_I$. □

Theorem 5.9. *A rational polyhedron $P \subset \mathbb{R}^n$ is integral $\iff$ $\max\{c^\mathsf{T} x \mid x \in P\} \in \mathbb{Z}$ for all $c \in \mathbb{Z}^n$ for which the problem is bounded.*

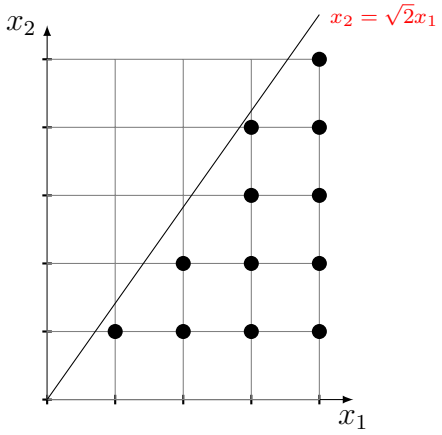
Proof. $\Longrightarrow$. We have $P = \operatorname{conv}(S)$ for $S = P \cap \mathbb{Z}^n$. Therefore

$$\max\{c^\mathsf{T} x \mid x \in P\} = \max\{c^\mathsf{T} x \mid x \in S\} \in \mathbb{Z}.$$

$\Longleftarrow$. We will assume that $P = \{x \mid Ax \leq b\}$ (for integral A, b) is a polytope. If $P \neq \operatorname{conv}(S)$ for $S = P \cap \mathbb{Z}^n$, then there exists a vertex $x_0 \in P$ not contained in $\mathbb{Z}^n$. There exists $I \subset [m]$ such that the matrix B consisting of rows A_{i*} for $i \in I$ is non-singular and x_0 is a unique solution of $Bx = b' = (b_i)_{i \in I}$. There exists $c \in \mathbb{Z}^n$ such that $c^\mathsf{T} x_0 \notin \mathbb{Z}$. If $c^\mathsf{T} = u^\mathsf{T} B$, then $u^\mathsf{T} b' = u^\mathsf{T} B x_0 = c^\mathsf{T} x_0 \notin \mathbb{Z}$. We can substitute u by $u - \lfloor u \rfloor = (u_i - \lfloor u_i \rfloor)_i$ and assume that $u \geq 0$, $c^\mathsf{T} = u^\mathsf{T} B \in \mathbb{Z}^n$ and $u^\mathsf{T} b' \notin \mathbb{Z}$. Let us extend u to $\mathbb{R}^m$ by zero. Then, for all $x \in P$, we have $c^\mathsf{T} x = u^\mathsf{T} Ax \leq u^\mathsf{T} b = c^\mathsf{T} x_0 \notin \mathbb{Z}$. We found $c \in \mathbb{Z}^n$ such that $\max\{c^\mathsf{T} x \mid x \in P\} \notin \mathbb{Z}$. □

Given a polyhedron P, we can ask if $P_I = \operatorname{conv}(P \cap \mathbb{Z}^n)$ is again a polyhedron.

Example 5.10. Let $P = \{x \in \mathbb{R}^2 \mid 1 \leq x_2 \leq \sqrt{2} x_1\}$ be a (non-rational) polyhedron. We claim that $P_I = \operatorname{conv}(P \cap \mathbb{Z}^2)$ is not a polyhedron.



If P_I is a polyhedron, then the problem $\min\{\sqrt{2}x_1 - x_2 \mid x \in P_I\}$ attains its minimum at some point of $S = P \cap \mathbb{Z}^2$. For every $N \geq 1$, there exists $k \geq 1$ such that $\sqrt{2}k - \lfloor \sqrt{2}k \rfloor < 1/N$ (exercise). Then $x = (k, \lfloor \sqrt{2}k \rfloor) \in S$ and $\sqrt{2}x_1 - x_2 < 1/N$, hence the infimum of the above problem is zero. But we have $\sqrt{2}x_1 - x_2 > 0$ for all $x \in S$ as $\sqrt{2}$ is irrational.

Theorem 5.11 (Meyer)**.** *If $P \subset \mathbb{R}^n$ is a rational polyhedron, then $\operatorname{conv}(P \cap \mathbb{Z}^n)$ is a (rational) polyhedron.*

Remark 5.12. The theorem is trivial if P is a polytope. In this case P is bounded, hence $S = P \cap \mathbb{Z}^n$ is finite and $\operatorname{conv}(S)$ is a (rational) polytope.

The above theorem implies that for $S = \{x \in \mathbb{Z}^n \mid Ax \leq b\}$ with rational A and b, we have $\operatorname{conv}(S) = \{x \mid A'x \leq b'\}$ for some (rational) A' and b', hence S has a perfect formulation. While the rational polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is not necessarily integral, the polyhedron $P_I = \operatorname{conv}(P \cap \mathbb{Z}^n) = \operatorname{conv}(S) = \{x \in \mathbb{R}^n \mid A'x \leq b'\}$ is integral and $S = P_I \cap \mathbb{Z}^n$. It can be difficult to determine the pair (A', b').

5.3. **Total unimodularity.** We would like to determine matrices $A \in \mathbb{Z}^{m\times n}$ such that the (rational) polyhedron $\{x \mid Ax \le b,\, x \ge 0\}$ is integral for all $b \in \mathbb{Z}^m$. The simplex method for the corresponding linear program produces solutions with integer coordinates.

Definition 5.13. A matrix $A \in \mathbb{Z}^{m\times n}$ is called totally unimodular if all of its minors (determinants of square submatrices) are $0, \pm 1$. This implies that all entries of A are $0, \pm 1$.

Example 5.14. The matrix $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is totally unimodular.

Theorem 5.15 (Hoffmann and Kruskal)**.** *For $A \in \mathbb{Z}^{m\times n}$ the polyhedron $\{x \mid Ax \le b,\, x \ge 0\}$ is integral for all $b \in \mathbb{Z}^m \iff$ A is totally unimodular.*

Proof. We will only prove the "if" part. Let A be totally unimodular. We need to show that every vertex $x' \in \mathbb{R}^n$ of $P = \{x \mid Ax \le b,\, x \ge 0\}$ is integral, for $b \in \mathbb{Z}^m$. We have $A'x' \le b'$ for $A' = \begin{bmatrix} A \\ -I \end{bmatrix}$ and $b' = \begin{bmatrix} b \\ 0 \end{bmatrix}$. Let A'' be the sumbatrix of A' consisting of rows where $A'x' \le b'$ is an equality. For x' to be a vertex one requires $\operatorname{rk} A'' = n$. Therefore there exists an invertible submatrix $B \in \mathbb{Z}^{n\times n}$ of A' such that $Bx' = d \in \mathbb{Z}^n$. We can assume that B has the form $B = \begin{bmatrix} C & D \\ 0 & -I \end{bmatrix}$, where C, D are submatrices of A. By our assumption $\det B = \pm \det C \in \{\pm 1\}$. By Cramer's rule $B^{-1} = \frac{1}{\det B}\tilde{B}$, where $\tilde{B}$ is the adjugate matrix of B (consisting of minors of B). Therefore $B^{-1} \in \mathbb{Z}^{n\times n}$ and $x' = B^{-1}d \in \mathbb{Z}^n$. □

Definition 5.16. We say that a matrix $A \in \mathbb{Z}^{m\times n}$ has an *equitable bicoloring* (EB) if there exists $J \subset [n]$ such that $\sum_{j\in J} A_j - \sum_{j\notin J} A_j \in \{0, 1, -1\}^m$.

Theorem 5.17. *A matrix $A \in \mathbb{Z}^{m\times n}$ is totally unimodular $\iff$ every column submatrix of A has an EB.*

Proof. We will only prove the "if" part. We will prove that all $k \times k$ submatrices of A have determinant in $\{0, \pm 1\}$ by induction on k. For $k = 1$ we note that every column of A has an EB, hence it has entries in $\{0, \pm 1\}$. Let B be a $(k+1) \times (k+1)$ submatrix of A and $\delta = \det B \ne 0$ (if $\delta = 0$, then we are done). By Cramer's rule and by induction, the adjugate matrix $\tilde{B} = \delta B^{-1}$ has entries in $\{0, \pm 1\}$. Let $a = \tilde{B}e_1 = \delta B^{-1}e_1$ be the first column of $\tilde{B}$, hence $Ba = \delta e_1$. We have $Ba = \sum_{j\in J} a_j B_j = \delta e_1$ for $J = \{j \mid a_j \ne 0\}$. The submatrix B_J has an EB, hence there exists $I \subset J$ such that $\sum_{j\in I} B_j - \sum_{j\in J\setminus I} B_j \in \{0, \pm 1\}^{k+1}$.

If δ is even, then the sum of elements in every row of B_J is even. This implies $\sum_{j\in I} B_j - \sum_{j\in J\setminus I} B_j = 0$. Therefore the columns of B_J are linearly dependent, hence $\det B = 0$, a contradiction.

If δ is odd, then $\sum_{j\in I} B_j - \sum_{j\in J\setminus I} B_j = \pm e_1$ (we can assume that it is e_1). Let $b \in \{0, \pm 1\}^{k+1}$ have coordinates $b_j = 1$ for $j \in I$, $b_j = -1$ for $j \in J\setminus I$ and $b_j = 0$ otherwise. Then $Bb = e_1$, $Ba = \delta e_1$, hence $\delta b = a$. We have $a, b \in \{0, \pm 1\}^{k+1}$, hence $\delta = \pm 1$. □

Corollary 5.18. *Let $A \in \mathbb{Z}^{m\times n}$ have entries in $\{0, \pm 1\}$ and have at most two nonzero elements in every row. Then A is totally unimodular $\iff$ A has an EB.*

Proof. Assume that A has an EB $J \subset [n]$. Then every column submatrix $B = A_I$ (with $I \subset [n]$), has the induces EB $I \cap J \subset J$. □

Corollary 5.19. *Let $A \in \mathbb{Z}^{m\times n}$ have entries in $\{0, \pm 1\}$ and have at most two nonzero elements in every column. Then A is totally unimodular $\iff$ A has a row EB, meaning that there exists $I \subset [m]$ such that $\sum_{i\in I} a_{ij} - \sum_{i\notin I} a_{ij} \in \{0, \pm 1\}$ for all $j \in [n]$.*

For a digraph $G = (V, E)$, we define the *incidence matrix* $A \in \mathbb{R}^{V \times E}$ with entries

$$a_{ie} = \begin{cases} -1 & e \in E(i, V) \\ 1 & e \in E(V, i) \\ 0 & \text{otherwise} \end{cases}$$

The corresponding linear map $A \colon \mathbb{R}^E \to \mathbb{R}^V$ is given by

$$A(v_{ij}) = v_j - v_i,$$

where $v_{ij} \in \mathbb{R}^E$ for $ij \in E$ and $v_i \in \mathbb{R}^V$ for $i \in V$ are standard basis vectors.

Theorem 5.20. *The incidence matrix of a digraph is totally unimodular.*

Proof. The matrix A has at most two nonzero elements in every row (for $e = ij$ we have $a_{ke} \neq 0$ only for $k = i$ or $k = j$). To see that A has a row EB, we consider $I = V$. Then $\sum_{i \in I} a_{ie} = 0$ for all $e \in E$. □

Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges (2-element subsets $e = \{i, j\} \subset V$ denoted as $e = ij$). We define the *incidence matrix* $A \in \mathbb{R}^{V \times E}$ of G with entries

$$a_{ie} = \begin{cases} 1 & i \in e \\ 0 & \text{otherwise} \end{cases}$$

Definition 5.21. A graph $G = (V, E)$ is called *bipartite* if there exists a decomposition $V = I \sqcup J$ such that every edge connects a vertex in I with a vertex in J.

Theorem 5.22. *The incidence matrix of G is totally unimodular $\iff$ G is bipartite.*

Proof. The matrix A has entries $\{0, 1\}$ and has at most two entries in every column. Therefore A totally unimodular $\iff$ A has a row EB, meaning $I \subset V$ (with $J = V \backslash I$) such that $\sum_{i \in I} a_{ie} - \sum_{j \in J} a_{je} \in \{0, \pm 1\}$ for all $e \in E$. The above sum is even, hence we require $\sum_{i \in I} a_{ie} = \sum_{j \in J} a_{je}$. This means that every edge $e \in E$ connects a vertex from I with a vertex from J, hence G is bipartite. □

5.4. **Combinatorial optimization.** Combinatorial optimization studies optimization problems of the form $\min\{f(x)\,|\,x\in X\}$ for objective functions $f\colon X\to\mathbb{R}$ on a finite set X. This is the area of discrete mathematics called polyhedral combinatorics. In some cases, we can embed $X\subset\mathbb{R}^n$ so that $f(x)=c^\mathsf{T}x$ for $x\in X$ and a fixed $c\in\mathbb{R}^n$. Then $\min\{f(x)\,|\,x\in X\}=\min\{c^\mathsf{T}x\,\big|\,x\in P\}$ for the polytope $P=\operatorname{conv}(X)\subset\mathbb{R}^n$ and we can apply methods of LP if we can find a polyhedral description of P.

5.4.1. *Perfect matchings.* Let $G=(V,E)$ be a graph. A *matching* in G is a set $M\subset E$ of edges such that every vertex of G is contained in at most one edge from M. A *perfect matching* in G is a set $M\subset E$ of edges such that every vertex of G is contained in exactly one edge from M. Let $\operatorname{Perf}(G)$ denote the set of all perfect matchings on G. Consider a function $w\colon E\to\mathbb{R}_+$, called a *weight function*. For every $M\in\operatorname{Perf}(G)$, we define its weight

$$w(M)=\sum_{e\in M}w(e)$$

and consider an optimization problem

$$\min\{w(M)\,|\,M\in\operatorname{Perf}(G)\}. \tag{9}$$

Example 5.23. Let $G=(V,E)$ be a graph with the set of vertices $V=I\sqcup J=[n]\sqcup[n]$ and with edges ij for all $i\in I$ and $j\in J$. Then a perfect matching $M\subset E$ corresponds to a permutation $\sigma\in S_n$ such that $ij\in M\iff\sigma i=j$. Such permutation can be interpreted as an assignment of worker i to job $j=\sigma i$. The weight $w(e)$ for $e=ij$ can be interpreted as a cost w_{ij} of assigning worker i to job j. The above optimization problem is equivalent to the assignment problem $\min\{\sum_{i=1}^n w_{i,\sigma i}\,|\,\sigma\in S_n\}$.

Equivalently, we can identify $M\subset E$ with its incidence vector (or characteristic function)

$$\chi^M\in\mathbb{R}^E,\qquad \chi^M_e=\chi^M(e)=\begin{cases}1 & e\in M,\\ 0 & e\notin M\end{cases}$$

and interpret w as a vector in $\mathbb{R}^E$. Then $w(M)=w^\mathsf{T}\chi^M$ and (9) is equivalent to

$$\min\{w^\mathsf{T}\chi^M\,\big|\,M\in\operatorname{Perf}(G)\}. \tag{10}$$

Remark 5.24. Let $A\in\mathbb{R}^{V\times E}$ be the incidence matrix of the graph $G=(V,E)$ (with $a_{ie}=1$ if $i\in e$ and zero otherwise). The vector $x=\chi^M\in\mathbb{R}^E$ has coordinates $x_e\in\{0,1\}$ and satisfies $(Ax)_i=\sum_e a_{ie}x_e=\sum_{e\ni i}x_e=1$ for all i since M is a perfect matching. Conversely, if $x\in\{0,1\}^E$ satisfies $Ax=\mathbb{1}=(1)_{i\in V}$, then $x=\chi^M$ for some perfect matching M. Therefore we can interpret the above optimization problem as a binary linear program

$$\begin{aligned}\text{minimize}\quad & w^\mathsf{T}x\\ \text{subject to}\quad & Ax=\mathbb{1}\\ & x\in\{0,1\}^E\end{aligned}$$

Using the *(perfect) matching polytope* of G

$$\operatorname{MP}(G)=\operatorname{conv}\{\chi^M\,\big|\,M\in\operatorname{Perf}(G)\}\subset\mathbb{R}^E,$$

we can write the optimization problem (10) in the form

$$\min\{w^\mathsf{T}x\,\big|\,x\in\operatorname{MP}(G)\}. \tag{11}$$

This is a linear problem which can be solved if we can represent $\operatorname{MP}(G)$ as a polyhedron.

Remark 5.25. All vectors χ^M for $M \in \mathrm{Perf}(G)$ are vertices of $\mathrm{MP}(G)$. Indeed, assume that $\chi^M = \sum_{i=1}^k t_i \chi^{M_i}$ for $M \neq M_i \in \mathrm{Perf}(G)$, $t_i > 0$, $\sum_i t_i = 1$. For $e \in M \backslash M_1$, we have $1 = \chi_e^M = \sum_{i=2}^k t_i \chi_e^{M_i} \leq \sum_{i=2}^k t_i = 1 - t_1 < 1$, a contradiction.

Theorem 5.26. *If $G = (V, E)$ is bipartite, then the matching polytope of G is*

$$\left\{ x \in \mathbb{R}^E \;\middle|\; x_e \geq 0 \ \forall e, \sum_{e \ni i} x_e = 1 \ \forall i \in V \right\}.$$

Proof. Let P be the matching polytope and Q be the above polyhedron. For $M \in \mathrm{Perf}(G)$, we have $\chi_e^M \geq 0 \ \forall e \in E$ and $\sum_{e \ni i} \chi_e^M = 1 \ \forall i \in V$. Therefore $\chi^M \in Q$ and $P \subset Q$.

Let us show that every $x \in Q$ is contained in P. Assume that we found $M \in \mathrm{Perf}(G)$ such that $x_e > 0$ for all $e \in M$. Let $\lambda = \min\{x_e \,|\, e \in M\}$ so that $0 < \lambda \leq 1$. The vector $y = x - \lambda \chi^M$ satisfies $y_e \geq 0 \ \forall e$ and $\sum_{e \ni i} y_e = 1 - \lambda \ \forall i$. If $\lambda = 1$, then $\sum_{e \ni i} y_e = 0$, hence $y = 0$ and $x = \chi_M$. If $\lambda < 1$, then $x' = y/(1-\lambda) \in Q$ has less nonzero coordinates than x, hence $x' \in P$ by induction. Therefore $x = \lambda \chi^M + (1-\lambda) x' \in P$ as P is convex.

Let us construct the required $M \in \mathrm{Perf}(G)$ with $x_e > 0$ for all $e \in M$. For a bipartite decomposition $V = I \sqcup J$ we have $|I| = \sum_{i \in I} \sum_{e \ni i} x_e = \sum_{e \in E} x_e = |J|$. Let $E' = \{e \in E \,|\, x_e > 0\}$ and $G' = (V, E')$. For $I' \subset I$, let $J' \subset J$ consist of vertices connected to I' in G'. We have $|I'| = \sum_{i \in I'} \sum_{e \ni i} x_e = \sum_{e \cap I' \neq \varnothing} x_e$ and similarly $|J'| = \sum_{e \cap J' \neq \varnothing} x_e$. If $e \cap I' \neq \varnothing$, then $e \cap J' \neq \varnothing$ (by definition of J'), hence $|I'| \leq |J'|$. By Hall's marriage theorem 5.29, there exists a perfect I-matching M in G' (which is a perfect matching since $|I| = |J|$). It satisfies $x_e > 0$ for all $e \in M$. □

Theorem 5.27 (Edmonds)**.** *For a graph $G = (V, E)$ the perfect matching polytope is equal to*

$$\left\{ x \in \mathbb{R}^E \;\middle|\; x_e \geq 0 \ \forall e, \sum_{e \ni i} x_e = 1 \ \forall i \in V, \sum_{e \in E(I, V \backslash I)} x_e \geq 1 \ \forall I \subset V \text{ with odd } |I| \right\}.$$

Example 5.28. Let $V = \mathbb{Z}/4\mathbb{Z}$ and $E = \{e_i = (i, i+1) \,|\, i \in V\}$. The required conditions for $x \in \mathbb{R}^4$ to be in the polytope are $x \geq 0$ and $x_i + x_{i+1} = 1$ for all i. Therefore $x = (t, 1-t, t, 1-t)$ for some $t \in [0, 1]$. The vertices are $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$. They correspond to the perfect matchings $\{e_1, e_3\}$ and $\{e_2, e_4\}$.

Theorem 5.29 (Hall's marriage theorem)**.** *Let $G = (V, E)$ be a bipartite graph with a bipartite decomposition $V = I \sqcup J$. Then there exists an I-perfect matching (a matching M such that every $i \in I$ is contained in exactly one edge from M) $\iff$ for every $I' \subset I$ the set $J' \subset J$ of vertices connected to I' satisfies $|I'| \leq |J'|$.*

Proof. The $\Longrightarrow$ direction is trivial. Let $M \subset E$ be a matching in G of maximal size. Assume that $i_0 \in I$ is unmatched by M (i.e. $i \notin e$ for all $e \in M$). Consider all paths that start at i_0 and use edges from $E \backslash M$ and M alternately. Let $I' \subset I$ and $J' \subset J$ be the corresponding sets of vertices of these paths. We claim that every $j \in J'$ is matched by M. Otherwise, we have a path $i_0 \xrightarrow{e_1} j_1 \xrightarrow{f_1} i_1 \to \cdots \to j_k \xrightarrow{f_k} i_k \xrightarrow{e_{k+1}} j$ with $e_i \in E \backslash M$, $f_i \in M$, and we can increase M by removing all f_i and adding all e_i. This contradicts to the maximality of M. We conclude that there is an injective map $J' \hookrightarrow I'$, $j \mapsto i$, where $ij \in M$. The element $i_0 \in I'$ is not in the image, hence $|I'| > |J'|$. We claim that every $j \in J$ connected to some $i \in I'$ is contained in J'. Indeed, if $e = ij \in E \backslash M$, then we have a path $i_0 \to \cdots \to i \xrightarrow{e} j$, hence $j \in J$. If $e \in M$, then we have a path $i_0 \to \cdots \to j \xrightarrow{e} i$, hence $j \in J'$. By our assumption, $|I'| \leq |J'|$, a contradiction. □

5.4.2. *Birkhoff theorem.* A matrix $X = (x_{ij}) \in \mathbb{R}^{n\times n}$ is called *doubly-stochastic* if $x_{ij} \geq 0$ and $\sum_j x_{ij} = \sum_i x_{ij} = 1$ for all $i, j \in [n]$. For example, for a permutation $\sigma \in S_n$, the matrix $P^\sigma = (\delta_{i,\sigma j})_{i,j}$ is a doubly-stochastic matrix, called a *permutation matrix*. The set

$$B_n = \left\{ (x_{ij}) \in \mathbb{R}^{n\times n} \,\middle|\, x_{ij} \geq 0,\ \sum\nolimits_k x_{ik} = \sum\nolimits_k x_{kj} = 1\ \forall i, j \in [n] \right\}$$

of all doubly-stochastic matrices is a polyhedron, called the *Birkhoff polytope*.

Theorem 5.30 (Birkhoff theorem)**.** *The Birkhoff polytope is equal to* $\operatorname{conv}\{P^\sigma \,|\, \sigma \in S_n\}$ *and its vertices are* P^σ *for* $\sigma \in S^n$.

Proof. This is a particular case of the theorem about matching polytopes. Consider the bipartite graph $G = (V, E)$ with $V = I \sqcup J = [n] \sqcup [n]$ and $E = \{ij \,|\, i \in I,\ j \in J\}$. A vector $x \in \mathbb{R}^E$ can be identified with the matrix $X = (x_{ij}) \in \mathbb{R}^{n\times n}$. The conditions $x_e \geq 0$ and $\sum_{e\ni i} x_e = 1$ mean exactly that the matrix X is doubly-stochastic, hence $X \in B_n$.

A perfect matching $M \subset E$ corresponds to the permutation $\sigma \in S_n$ such that $M = \{ij \,|\, i = \sigma(j)\}$. The corresponding permutation matrix P^σ has entries $\delta_{i,\sigma(j)} = \chi^M_{ij}$, hence it corresponds to the vector $\chi^M \in \mathbb{R}^E$. We conclude that $B_n = \operatorname{conv}\left\{\chi^M \,\middle|\, M \in \operatorname{Perf}(G)\right\} = \operatorname{conv}\{P^\sigma \,|\, \sigma \in S_n\}$ and $P^\sigma = \chi^M$ are the vertices of B_n. □

Example 5.31. We have seen in Example 5.23 that the assignment problem can be formulated as the optimization problem $\min\left\{\sum_{j=1}^n w_{\sigma j,j} \,\middle|\, \sigma \in S_n\right\}$ for fixed weights $w_{ij} \in \mathbb{R}$. For $X = (x_{ij}) \in \mathbb{R}^{n\times n}$, let $w(X) = \sum_{i,j} w_{ij}x_{ij}$. Then $w(P^\sigma) = \sum_{i,j} w_{ij}\delta_{i,\sigma j} = \sum_j w_{\sigma j,j}$. Therefore the above optimization problem is equivalent to

$$\min\{w(P^\sigma) \,|\, \sigma \in S_n\} = \min\{w(X) \,|\, X \in \operatorname{conv}\{P^\sigma \,|\, \sigma \in S_n\}\} = \min\{w(X) \,|\, X \in B_n\}.$$

This problem can be solved using LP methods since we have a polyhedral description of B_n. Note that the resulting linear program

$$\min\left\{ \sum\nolimits_{i,j} w_{ij}x_{ij} \,\middle|\, x_{ij} \geq 0,\ \sum\nolimits_k x_{ik} = \sum\nolimits_k x_{kj} = 1\ \forall i, j \in [n] \right\}$$

can be also interpreted as a transportation problem (where supply and demand equals 1 everywhere).

Appendix A. Graphs

Definition A.1. A *graph* is a pair $G = (V, E)$ of sets such that E consists of 2-elements subsets of V. The elements of V are called *vertices* of G and the elements of E are called *edges* of G. An edge $e = \{u, v\} \in E$ is called an edge between vertices u and v. It is sometimes written as uv, (u, v) or $u \overset{e}{\text{—}} v$. A vertex v is called *incident* with an edge e if $v \in e$.

Definition A.2. A collection of vertices and edges $i_0 \overset{e_1}{\text{—}} i_1 \overset{e_2}{\text{—}} \dots \overset{e_n}{\text{—}} i_n$ in a graph is called a *path* of length $n \geq 0$. Such path is called a *cycle* if $i_n = i_0$ and $n \geq 1$. A graph is called *acyclic* (or a forest) if it has no cycles. A graph is called *connected* if every two vertices are connected by a path. A graph is called a *tree* if it is acyclic and connected.

Problem 4: *If $T = (V, E)$ is a tree, then $|E| = |V| - 1$.*

Definition A.3. A *spanning tree* of a graph $G = (V, E)$ is a tree $T = (V, E')$, where $E' \subset E$.

Definition A.4. A *directed graph* (or digraph) is a tuple $G = (V, E, s, t)$, where V and E are sets and $s\colon E \to V$, $t\colon E \to V$ are maps. The elements of V are called *vertices* of G and the elements of E are called *arrows* (or directed edges) of G. For an arrow $a \in E$, we call $i = s(a)$ the source of a, $j = t(a)$ the target of a, and we denote this arrow as $a\colon i \to j$ or $i \overset{a}{\to} j$. An arrow $a\colon i \to j$ is called a *loop* if $i = j$. We usually write just (V, E) for a directed graph.

Remark A.5. Sometimes the above directed graph is called a directed multigraph and the name digraph is reserved for graphs with the injective map $(s, t)\colon E \to V \times V$. If additionally, the digraph has no loops (arrows $a\colon i \to j$ with $i = j$), then the digraph is called simple.

Definition A.6. A collection of vertices and arrows $i_0 \overset{a_1}{\to} i_1 \overset{a_2}{\to} \dots \overset{a_n}{\to} i_n$ in a directed graph is called a (directed) *path* of length $n \geq 0$. Such path is called a *cycle* if $i_n = i_0$ and $n \geq 1$. A directed graph is called *acyclic* if it has no cycles.

For $I, J \subset V$, let $E(I, J)$ be the set of arrows going from I to J

$$E(I, J) = \{a \in E \mid s(a) \in I,\, t(a) \in J\}.$$

We denote $E(\{i\}, J)$ by $E(i, J)$ and $E(J, \{i\})$ by $E(J, i)$ for $i \in V$. Let

$$\delta^-(i) = E(V, i) = \{a \in E \mid t(a) = i\}, \qquad \delta^+(i) = E(i, V) = \{a \in E \mid s(a) = i\}.$$

We define the input and output vertices (also called sources and sinks)

$$\mathbf{in}(G) = \{i \in V \mid E(V, i) = \varnothing\}, \qquad \mathbf{out}(G) = \{i \in V \mid E(i, V) = \varnothing\}.$$

Appendix B. Artificial neural networks

One can interpret artificial intelligence and machine learning as optimization problems. The resulting solutions are usually not optimal, but are sufficiently good approximations for practical needs. As an example, let us discuss artificial neural networks (also called deep learning neural networks). Assume that we need to approximate a function $F\colon X \to Y$, based on *training data* $(x_i, y_i) \in X \times Y$ such that $F(x_i) = y_i$ for $1 \le i \le N$.

Example B.1. Let $x \in X = \mathbb{R}^{100\times 100}$ encode a picture of a cat or a dog and $F(x) \in Y = \{0, 1\}$ be the corresponding label. Approximating the map F is at the core of image recognition.

Example B.2. Let $A = \mathbb{Z}/32\mathbb{Z}$ encode English alphabet and some additional symbols. Let $X = A^{100}$ be the set of length 100 texts and $Y = A$ (or $Y = \left\{t \in [0,1]^A \,\middle|\, \sum_i t_i = 1\right\}$ for probability distributions on A). For $x \in X$, let $F(x) \in Y$ denote the next (most probable) letter of the text x. Approximating the map F is at the core of large language models. If we have F, then for a text $x \in A^{100}$, we can construct the next letter $x_{101} = F(x)$, then the letter $x_{102} = F(x_2, \dots, x_{101})$ and so on. In this way we are able to continue a conversation.

Using some heuristics, one constructs maps

$$F_w\colon X \to Y, \qquad w \in W = \mathbb{R}^k,$$

for some large k (equivalently, $\bar{F}\colon \mathbb{R}^k \times X \to Y$, $(w, x) \mapsto F_w(x)$) and looks for $w \in \mathbb{R}^k$ that minimizes the loss function

$$f\colon \mathbb{R}^k \to \mathbb{R}, \qquad f(w) = \frac{1}{N}\sum\nolimits_{i=1}^{N} d(F_w(x_i), y_i),$$

where d is some metric on Y. This is our optimization problem. The vector $w \in \mathbb{R}^k$ is called the *weight vector* and is interpreted as the knowledge of our system. The training data is used to improve systems knowledge (optimize the weight vector). Artificial neural networks is a tool to encode the above maps $F_w\colon X \to Y$. Their widespread adoption can be attributed to the availability of effective optimization algorithms, such as back propagation (§B.1).

Definition B.3. A *neural network* (NN) is an acyclic digraph $G = (V, E)$ equipped with

(1) A set S_i of *states* (or signals) and a set W_i of *weights* for all $i \in V$.
(2) An *operation* $f_j\colon W_j \times S_j^- \to S_j$, where $S_j^- = \prod_{a\colon i\to j} S_i$, for all $j \in V$.

Let $W = \prod_{i\in V} W_i$, $S_{\mathbf{in}} = \prod_{i\in \mathbf{in}(G)} S_i$, $S_{\mathbf{out}} = \prod_{i\in \mathbf{out}(G)} S_i$. For $w = (w_i)_i \in W$, $(x_i)_i \in S_{\mathbf{in}}$, we define $(x_i)_i \in \prod_{i\in V} S_i$ inductively by *forward propagation*

$$x_j = f_j(w_j, (x_i)_{a\colon i\to j}).$$

Projecting to $S_{\mathbf{out}}$, we obtain the *transfer function* $F_w\colon S_{\mathbf{in}} \to S_{\mathbf{out}}$ for all $w \in W$.

Remark B.4*. The above neural network is called a *computational graph* if $W_i = \{*\}$ for all $i \in V$. We can interpret weights as states by adding arrows $i' \to i$ and defining $S_{i'} = W_i$ for all $i \in V$. In this way every neural network becomes a computational graph.

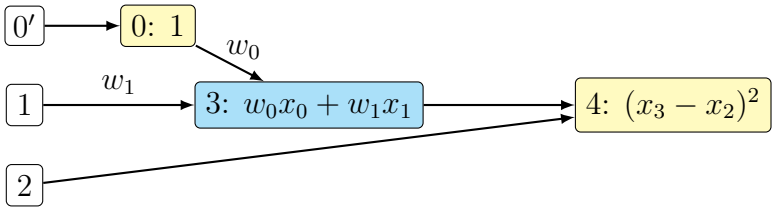
Remark B.5. Usually $S_i = \mathbb{R}^{d_i}$, $W_i = \mathbb{R}^{e_i}$ and the map $f_j\colon W_j \times S_j^- \to S_j$ is bilinear if $W_j \ne 0$ (we call such $j \in V$ *parametric*). If $W_j = 0$, then the operation $f_j\colon S_j^- = \prod_{a\colon i\to j} S_i \to S_j$

is called an *activation function* and can be non-linear. For all $w \in W = \mathbb{R}^k$, $k = \sum_{i \in V} e_i$, we have

$$F_w \colon S_{\mathbf{in}} = \mathbb{R}^m \to S_{\mathbf{out}} = \mathbb{R}^n, \qquad m = \sum\nolimits_{i \in \mathbf{in}(G)} d_i, \qquad n = \sum\nolimits_{i \in \mathbf{out}(G)} d_i.$$

Remark B.6. Every vertex $i \in V$ with $S_i = \mathbb{R}^d$, $d > 0$, can be substituted by vertices $i_1, \dots, i_d$ with $S_{i_k} = \mathbb{R}$. Therefore we can assume that $S_i = \mathbb{R}$ (or $S_i = 0$) for all $i \in V$. For parametric $j \in V$, we can take $W_j = \prod_{a \colon i \to j} W_a$, where $W_a = \mathbb{R}$, with the bilinear map $f_j \colon W_j \times S_j^- \to S_j$, $(w_a, x_i)_{a \colon i \to j} \mapsto x_j = \sum_{a \colon i \to j} w_a x_i$.

Example B.7. Let us construct a neural network that maps a point $(x, y) \in \mathbb{R}^2$ to $f(x, y) = (y - w_1 x - w_0)^2$, where $w_0, w_1 \in \mathbb{R}$ are weights. Minimization of f corresponds to finding the best fitting line of the form $y = w_1 x + w_0$. Consider the directed graph (boxes represent vertices, with activation functions written inside)



with state spaces $S_{0'} = 0$ and $S_i = \mathbb{R}$ for all other vertices, the weight space $W_3 = W_{03} \times W_{13} = \mathbb{R}^2$, and activation functions $f_0 \colon S_{0'} \to S_0$, $0 \mapsto 1$, and $f_4 \colon S_3 \times S_2 = \mathbb{R}^2 \to S_4 = \mathbb{R}$, $(x_3, x_2) \mapsto (x_3 - x_2)^2$. For $w = (w_0, w_1) = W_3 = \mathbb{R}^2$ and $(x, y) \in S_{\mathbf{in}} = S_1 \times S_2 = \mathbb{R}^2$, we have $x_0 = f_0(0) = 1$, $x_1 = x$, $x_2 = y$, $x_3 = w_0 x_0 + w_1 x_1 = w_0 + w_1 x$ and

$$F_w(x, y) = f_4(x_3, x_2) = f_4(w_0 + w_1 x, y) = (w_0 + w_1 x - y)^2.$$

Example B.8. In the example with cats and dogs we have $x \in \mathbb{R}^m = \mathbb{R}^{100 \times 100}$ and $y \in \{0, 1\} \subset \mathbb{R}$. Assume that we have an appropriate NN with the transfer function $F_w \colon \mathbb{R}^m \to \mathbb{R}$ for $w \in \mathbb{R}^k$. Using the activation function $\sigma(t) = \frac{1}{1+e^{-t}}$ (sigmoid), we can always assume that $F_w(x) \in [0, 1]$. The loss function $f_w(x, y) = (F_w(x) - y)^2$ can be also encoded by a NN (cf. Example B.7) and our training consists of minimization of this function (see below). After training (optimization), we obtain a vector $w \in \mathbb{R}^k$ and the transfer function $F_w \colon \mathbb{R}^m \to [0, 1]$. Any new picture of a cat or a dog is encoded by $x \in \mathbb{R}^m$. In order to classify the picture, we compute $F_w(x) \in [0, 1]$ and we say that x encodes a cat if $F_w(x) < 0.5$ and a dog if $F_w(x) \geq 0.5$.

B.1. **Back propagation.** Back propagation is a method of training a neural network using gradient descent. Consider a NN with the transfer function $F_w\colon \mathbb{R}^m \to \mathbb{R}^n$ for $w \in \mathbb{R}^k$. The loss function

$$L\colon \mathbb{R}^k \times \mathbb{R}^{m+n} \to \mathbb{R}, \qquad (w,z) \mapsto \|F_w(x) - y\|^2, \qquad w \in \mathbb{R}^k,\ z = (x,y) \in \mathbb{R}^{m+n},$$

can be itself encoded by a NN and has to be minimized (we need to find $w \in \mathbb{R}^k$ that minimizes $L(w,-)$ on the training data). For $z = (x,y) \in \mathbb{R}^{m+n}$ in the training data, we apply gradient descent to the function

$$f\colon \mathbb{R}^k \to \mathbb{R}, \qquad w \mapsto L(w,z),$$

hence we define the new weight

$$w' = w - \varepsilon \nabla f(w),$$

where $\nabla f(w) = \left(\frac{\partial f}{\partial w_a}(w)\right)_a \in \mathbb{R}^k$. We repeat this step for every $z_i \in \mathbb{R}^{m+n}$, $1 \le i \le N$, in the training data (or apply stochastic gradient descent).

To apply the above approach we need an effective way to compute the derivatives $\frac{\partial f}{\partial w_a}$. We assume for simplicity that $S_i = \mathbb{R}$ or 0 for $i \in V$ and $W_a = \mathbb{R}$ or 0 for $a \in E$ (see Remark B.6). Note that $f(w) = x_k$ for the unique outgoing vertex $k \in V$ (with $E(k,V) = \varnothing$). Therefore

$$\frac{\partial f}{\partial w_a} = \frac{\partial x_k}{\partial w_a} = \frac{\partial x_k}{\partial x_j}\frac{\partial x_j}{\partial w_a} = \frac{\partial x_k}{\partial x_j} x_i, \qquad a\colon i \to j.$$

where we used the fact that $x_j = \sum_{a\colon i\to j} w_a x_i$. We compute the derivatives $\frac{\partial x_k}{\partial x_i}$ inductively, going from k to its predecessors (therefore the method is called *back propagation*). Assume that we want to determine $\frac{\partial x_k}{\partial x_i}$ and we know already $\frac{\partial x_k}{\partial x_j}$ for all arrows $a\colon i \to j$. By the chain rule

$$\frac{\partial x_k}{\partial x_i} = \sum_{a\colon i\to j} \frac{\partial x_k}{\partial x_j}\frac{\partial x_j}{\partial x_i}.$$

The first factor in every summand is known by assumption and the second factor is

$$\frac{\partial x_j}{\partial x_i} = \begin{cases} w_a & j \text{ is parametric (see Remark B.5)},\\ \frac{\partial f_j}{\partial x_i} & \text{otherwise.}\end{cases}$$

Example B.9. Consider Example B.7. For $w_0\colon 0 \to 3$, we have $\frac{\partial x_4}{\partial w_0} = \frac{\partial x_4}{\partial x_3}x_0 = \frac{\partial f_4}{\partial x_3} = 2(x_3 - x_2) = 2(w_0 + w_1 x - y)$. For $w_1\colon 1 \to 3$, we have $\frac{\partial x_4}{\partial w_1} = \frac{\partial x_4}{\partial x_3}x_1 = \frac{\partial f_4}{\partial x_3}x_1 = 2(w_0 + w_1 x - y)x$. These are exactly the derivatives of $f_w(x,y) = (w_0 + w_1 x - y)^2$ with respect to w_0 and w_1.

Appendix C. Portfolio optimization

Assume that there are potential investments $j \in [n]$ with the return R_j (a random variable) over a given time period. We determine a portfolio by specifying the ratio $x_j \geq 0$ of the investment j so that $\sum_{j=1}^{n} x_j = 1$. The return of the portfolio is $R = \sum_j x_j R_j$. The expected return or reward of the portfolio is

$$\mathbb{E}R = \sum_j x_j \mathbb{E}R_j,$$

where $\mathbb{E}R_j$ is the expected value (to be estimated later). We define the risk of the portfolio to be the mean absolute deviation

$$\mathbb{E}\,|R - \mathbb{E}R|$$

Fixing a risk tolerance factor $q \geq 0$ ($q = 0$ corresponds to a portfolio with minimal risk), we want to minimize

$$\mathbb{E}\,|R - \mathbb{E}R| - q\mathbb{E}R.$$

To estimate $\mathbb{E}R_j$, let us assume that we have historic data, meaning the returns $R_j(k)$ on investment j over time periods $k \in [N]$. Then we estimate

$$r_j = \mathbb{E}R_j = \frac{1}{N}\sum_{k=1}^{N} R_j(k), \qquad \mathbb{E}R = \sum_j x_j r_j.$$

We estimate the risk

$$\mathbb{E}\,|R - \mathbb{E}R| = \frac{1}{N}\sum_{k=1}^{N} \left|\sum_j x_j(R_j(k) - r_j)\right|$$

Therefore our optimization problem is

$$\begin{aligned} \text{minimize} \quad & \frac{1}{N}\sum_{k\in[N]} \left|\sum_j x_j(R_j(k) - r_j)\right| - q\sum_j x_j r_j \\ \text{subject to} \quad & \sum_j x_j = 1 \\ & x_j \geq 0 \quad \forall j \in [n] \end{aligned}$$

The objective function is not linear in x_j as there are absolute values in the sum. To resolve this issue we use the following trick. We replace each absolute value with a new variable y_k and we impose new inequality constraints to ensure that y_k is equal to the corresponding absolute value. We obtain the linear program

$$\begin{aligned} \text{minimize} \quad & \frac{1}{N}\sum_{k\in[N]} y_k - q\sum_j x_j r_j \\ \text{subject to} \quad & -y_k \leq \sum_j x_j(R_j(k) - r_j) \leq y_k \quad \forall k \in [N] \\ & \sum_j x_j = 1,\ x_j \geq 0 \quad \forall j \in [n] \end{aligned}$$

Every portfolio profile ($x \in \mathbb{R}^n_+$ with $\sum_j x_j = 1$) can be mapped to the point $(\mathsf{risk}, \mathsf{reward}) \in \mathbb{R}^2$. The left boundary of the image consists of points such that we can not decrease risk having the same reward. This boundary is called the efficient frontier and one should only invest in portfolios that lie on it. Solving the above optimization problem for a fixed q, we obtain a portfolio such that its image in $\mathbb{R}^2$ lies on the efficient frontier. Note that the polyhedron of our linear program is the same for all parameters q and the optimal value is attained at a vertex of the polyhedron. Therefore we can consider all vertices of the

polyhedron, take their images in $\mathbb{R}^2$ and construct the efficient frontier as the boundary of the convex hull of these points.

C.0.1. *Markowitz model.* In the Markowitz model one defines the risk to be the variance of the return

$$\mathrm{Var}(R) = \mathbb{E}(R - \mathbb{E}R)^2$$

and one wants to minimize (for a fixed $q \geq 0$)

$$\mathrm{Var}(R) - q\mathbb{E}R.$$

Let $\tilde{R}_j = R_j - \mathbb{E}R_j$. We have

$$\mathrm{Var}(R) = \sum_{i,j} x_i x_j \mathbb{E}(\tilde{R}_i \tilde{R}_j) = \sum_{i,j} x_i x_j c_{ij} = x^\mathsf{T} C x$$

where C is the covariance matrix with entries $c_{ij} = \mathbb{E}(\tilde{R}_i \tilde{R}_j)$. Therefore we have an optimization problem (with $r = (\mathbb{E}R_j)_j \in \mathbb{R}^n$)

$$\begin{aligned} \text{minimize} \quad & x^\mathsf{T} C x - q r^\mathsf{T} x \\ \text{subject to} \quad & \sum_j x_j = 1 \\ & x_j \geq 0 \quad \forall j \in [n] \end{aligned}$$

This is a quadratic programming problem. As before, one can construct the efficient frontier in $\mathbb{R}^2$ by solving the optimization problem for different q. This time the efficient frontier will be a parabola.