

LECTURE NOTES
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SERGEY MOZGOVOY

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1. RINGS AND IDEALS

1.1. Prime and maximal ideals. In these notes a ring will always mean a commutative ring unless otherwise stated. We will allow rings A with $1_A = 0_A$. In this case $a = a \cdot 1_A = a \cdot 0_A = 0_A$ for all $a \in A$, hence $A = \{0_A\}$. This ring is called the **zero ring** and is denoted by 0 .

Definition 1.1.

- (1) An ideal $I \subset A$ is called **proper** if $I \neq A$.
- (2) A proper ideal $\mathfrak{p} \subset A$ is called **prime** if $ab \in \mathfrak{p} \implies a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}$.
- (3) A proper ideal $\mathfrak{m} \subset A$ is called **maximal** if there is no ideal I such that $\mathfrak{m} \subsetneq I \subsetneq A$.
- (4) The set of all prime ideals of A is denoted by $\text{Spec}(A)$, called the **spectrum** of A .
- (5) The set of all maximal ideals of A is denoted by $\text{Max}(A)$, called the **maximal spectrum** of A .

Example 1.2. Maximal and prime ideals in \mathbb{Z} .

Remark 1.3. An element $p \in A$ is called prime if $p \neq 0$ is not invertible and if $p \mid ab \implies p \mid a \text{ or } p \mid b$. One can see that $p \in A$ is prime $\iff (p) \subset A$ is a non-zero prime ideal.

Lemma 1.4.

- (1) An ideal \mathfrak{p} is prime $\iff A/\mathfrak{p}$ is an integral domain.
- (2) An ideal \mathfrak{m} is maximal $\iff A/\mathfrak{m}$ is a field.
- (3) A maximal ideal is prime.

Proof. (1) Let \mathfrak{p} be prime and $[a], [b] \in A/\mathfrak{p}$ be nonzero. Then $a, b \notin \mathfrak{p} \implies ab \notin \mathfrak{p}$ (as \mathfrak{p} is prime) $\implies [a] \cdot [b] = [ab] \neq 0$ in A/\mathfrak{p} . This means that A/\mathfrak{p} is an integral domain. The converse is similar.

(2) If \mathfrak{m} is maximal, then there are no ideals in A/\mathfrak{m} except 0 and A/\mathfrak{m} . If $a \in A/\mathfrak{m}$ is nonzero \implies the principal ideal (a) is nonzero $\implies (a) = A/\mathfrak{m} \implies \exists b \in A/\mathfrak{m}$ such that $ab = 1 \implies a$ is invertible. This means that A/\mathfrak{m} is a field. Conversely, if A/\mathfrak{m} is a field, then there are no ideals in A/\mathfrak{m} except 0 and A/\mathfrak{m} . This implies that there are no ideals $\mathfrak{m} \subset I \subset A$ except $I = \mathfrak{m}$ or $I = A$.

(3) Using the previous statements we obtain: if \mathfrak{m} is maximal $\implies A/\mathfrak{m}$ is a field $\implies A/\mathfrak{m}$ is an integral domain $\implies \mathfrak{m}$ is prime. Alternatively, assume that \mathfrak{m} is maximal, $ab \in \mathfrak{m}$ and $a \notin \mathfrak{m}$. Then $(a) + \mathfrak{m} = A \implies 1 - ac \in \mathfrak{m}$ for some $c \in A \implies b - abc \in \mathfrak{m} \implies b \in \mathfrak{m}$ as $ab \in \mathfrak{m}$. This proves that \mathfrak{m} is prime. \square

Exercise 1.5. Show that if R is a PID and $\mathfrak{p} \subset R$ is a non-zero prime ideal, then \mathfrak{p} is maximal.

Remark 1.6. We know that for any ideal $I \subset A$, there is a bijection between ideals $I \subset J \subset A$ and all ideals of A/I , where an ideal $J \subset A$ is mapped to the ideal J/I of A/I . This implies that there is a bijection between all prime (maximal) ideals $I \subset J \subset A$ and all prime (maximal) ideals of A/I . Indeed, an ideal $I \subset J \subset A$ is prime (maximal) $\iff A/J \simeq (A/I)/(J/I)$ is an integral domain (a field) $\iff J/I \subset A/I$ is prime (maximal).

Remark 1.7. We obtain that $\text{Max } A \subset \text{Spec } A$. The set $\text{Spec } A$ (hence also $\text{Max } A$) can be equipped with a topology as follows. For any subset (or ideal) $I \subset A$, define

$$(1) \quad Z(I) = \{\mathfrak{p} \in \text{Spec } A \mid I \subset \mathfrak{p}\}$$

and define closed sets in $\text{Spec } A$ to be subsets of the form $Z(I)$. This topology is called the **Zariski topology**.

Theorem 1.8. Any proper ideal $I \subset A$ is contained in at least one maximal ideal.

Proof. We will use **Zorn's lemma** which can be formulated as follows.

Lemma 1.9 (Zorn). Let X be a non-empty poset (partially ordered set) such that every chain $Y \subset X$ (a totally ordered subset, i.e. $x, y \in Y \implies x \leq y \text{ or } y \leq x$) has an upper bound in X (i.e. $x \in X$ such that $y \leq x \forall y \in Y$). Then X has at least one maximal element ($x \in X$ such that $x \leq y \implies x = y$).

In our case, let X be the set of proper ideals in A that contain I . We order X by inclusion. It is non-empty as $I \in X$. Given a chain of ideals (J_α) in X , consider $J = \bigcup_\alpha J_\alpha$. This is an ideal: if $a, b \in J \implies a \in J_\alpha, b \in J_\beta$ for some α, β . Then $J_\alpha \subset J_\beta$ or $J_\beta \subset J_\alpha$ by the chain assumption. Assuming that $J_\alpha \subset J_\beta$, we obtain $a, b \in J_\beta \implies a + b \in J_\beta \subset J$. Other axioms of an ideal are verified in the same way. Ideal J is proper, as otherwise $1 \in J \implies 1 \in J_\alpha$ for some $\alpha \implies J_\alpha = A$, which contradicts to the assumption that all ideals in X are proper. Therefore $J \in X$, hence J is an upper bound of the chain (J_α) . By Zorn's lemma X has a maximal element, which is the required maximal ideal containing I . \square

1.2. Contractions and extensions.

Definition 1.10. For any ideal $I \subset A$ and a subset $J \subset A$, define the product IJ to be the ideal

$$IJ = \left\{ \sum a_i b_i \mid a_i \in I, b_i \in J \forall i \right\}.$$

Definition 1.11. Let $f: A \rightarrow B$ be a ring homomorphism.

- (1) For any ideal $I \subset A$, define the **extension ideal** $I^e = Bf(I) \subset B$ (ideal generated by $f(I)$).
- (2) For any ideal $J \subset B$, define the **contraction ideal** $J^c = f^{-1}(J) \subset A$.

Example 1.12.

- (1) Consider an inclusion $\mathbb{Z} \subset \mathbb{Q}$ and an ideal $I = 2\mathbb{Z} \subset \mathbb{Z}$. Then $I^e = \mathbb{Q}$.
- (2) Consider an inclusion $\mathbb{Z} \subset \mathbb{Z}[i]$ and a prime ideal $\mathfrak{p} = 5\mathbb{Z} \subset \mathbb{Z}$. Then $\mathfrak{p}^e = 5\mathbb{Z}[i]$ is not prime as $5 = (2+i)(2-i)$ is not prime.

Therefore an extension of a prime (maximal) ideal is not necessarily prime (maximal).

Lemma 1.13. Let $f: A \rightarrow B$ be a ring homomorphism and $I \subset A$, $J \subset B$ be ideals. Then

- (1) $I \subset I^{ec}$, $J \supset J^{ce}$.
- (2) An ideal $I \subset A$ is a contraction ideal $\iff I = I^{ec}$. An ideal $J \subset B$ is an extension ideal $\iff J = J^{ce}$.
- (3) There is a bijection between the set of all contraction ideals in A and the set of all extension ideals in B .

Proof. (1) We have $f(I) \subset I^e \implies I \subset f^{-1}(f(I)) \subset f^{-1}(I^e) = I^{ec}$. Next, J^{ce} is generated by $f(f^{-1}(J)) \subset J \implies J^{ce} \subset J$.

(2) If $I \subset A$ is a contraction ideal $\implies I = J^c$ for some ideal $J \subset A \implies I^{ec} = J^{cec} \subset J^c = I \implies I^{ec} = I$. The statement for J is similar.

(3) The required mutually inverse bijections are given by $I \mapsto I^e$ and $J \mapsto J^c$. \square

An important advantage of prime ideals over maximal ideals is that prime ideals are well-behaved under contractions.

Lemma 1.14. If $f: A \rightarrow B$ is a ring homomorphism and $\mathfrak{q} \subset B$ is a prime ideal, then $\mathfrak{q}^c = f^{-1}(\mathfrak{q}) \subset A$ is also a prime ideal.

Proof. It is not difficult to see that $f^{-1}(\mathfrak{q})$ is an ideal in A . The map $f: A \rightarrow B$ induces an injective homomorphism $\bar{f}: A/f^{-1}(\mathfrak{q}) \rightarrow B/\mathfrak{q}$, $[a] \mapsto [f(a)]$. Indeed, if $\bar{f}([a]) = 0$, then $f(a) \in \mathfrak{q}$, hence $a \in f^{-1}(\mathfrak{q})$ and $[a] = 0$. As \mathfrak{q} is prime $\implies B/\mathfrak{q}$ is an integral domain $\implies A/f^{-1}(\mathfrak{q})$ is an integral domain (as a subring of an integral domain) $\implies f^{-1}(\mathfrak{q})$ is prime. \square

Example 1.15. There exist ring homomorphisms $f: A \rightarrow B$ and maximal ideals $\mathfrak{n} \subset B$ such that $f^{-1}(\mathfrak{n})$ is not necessarily maximal. For example, let $f: \mathbb{Z} \rightarrow \mathbb{Q}$ be an inclusion and $\mathfrak{n} = 0 \subset \mathbb{Q}$ be the zero ideal. Then \mathfrak{n} is a maximal ideal in \mathbb{Q} , but $f^{-1}(\mathfrak{n}) = 0$ is not a maximal ideal in \mathbb{Z} ($0 \subset (2) \subset \mathbb{Z}$).

Remark 1.16. We will see later (Hilbert Nullstellensatz) that if B is a finitely-generated algebra over a field \mathbb{k} (of the form $\mathbb{k}[x_1, \dots, x_n]/I$ for some ideal I) then B/\mathfrak{n} is a finite field extension of \mathbb{k} , for any maximal ideal $\mathfrak{n} \subset B$. If $f: A \rightarrow B$ is an algebra homomorphism and $\mathfrak{m} = f^{-1}(\mathfrak{n})$, then $\mathbb{k} \subset A/\mathfrak{m} \subset B/\mathfrak{n}$ and therefore A/\mathfrak{m} is also a finite field extension of \mathbb{k} (prove this!). This implies that \mathfrak{m} is a maximal ideal of A .

1.3. Radicals.

Definition 1.17.

- (1) An element $a \in A$ is called **nilpotent** if $a^n = 0$ for some $n > 0$.
- (2) Define the **nilradical** of A to be the set of all nilpotent elements of A

$$\mathcal{N}(A) = \{a \in A \mid a^n = 0 \text{ for some } n > 0\}.$$

- (3) For any ideal $I \subset A$, define the **radical** of I to be

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n > 0\}.$$

Note that $\mathcal{N}(A) = \sqrt{0}$, hence $\mathcal{N}(A)$ is an ideal of A by the following result.

Lemma 1.18. \sqrt{I} is an ideal of A and $\sqrt{\sqrt{I}} = \sqrt{I}$.

Proof. If $a, b \in \sqrt{I}$, then $a^m \in I$, $b^n \in I$ for some $m, n > 0$. Therefore

$$(a+b)^{m+n} = \sum_{k+l=m+n} \binom{m+n}{k} a^k b^l \in I$$

as either $k \geq m$ or $l \geq n$, hence either $a^k \in I$ or $b^l \in I$. This implies that $a+b \in \sqrt{I}$. Similarly, $-a \in \sqrt{I}$. Finally, for any $c \in A$, we have $(ca)^m = c^m a^m \in I$, hence $ca \in \sqrt{I}$. Therefore \sqrt{I} is an ideal.

If $a \in \sqrt{\sqrt{I}} \implies a^m \in \sqrt{I}$ for some $m > 0 \implies a^{mn} = (a^m)^n \in I$ for some $n > 0 \implies a \in \sqrt{I}$. \square

Lemma 1.19. $\mathcal{N}(A)$ is the intersection of all prime ideals of A .

Proof. Let $I = \mathcal{N}(A)$ and let I' be the intersection of all prime ideals of A . If $a \in I \implies a^n = 0$ for some $n > 0$. For any prime ideal $\mathfrak{p} \subset A$, we have $a^n = 0 \in \mathfrak{p} \implies a \in \mathfrak{p}$ (by the property of prime ideals). This implies that $a \in I'$ and $I \subset I'$.

Conversely, assume that $c \in I'$ and $c \notin I$. Then $c^n \notin I$ for all $n > 0$, as otherwise $c^n \in I = \mathcal{N}(A) \implies c$ is nilpotent $\implies c \in I$. Consider the set X of all ideals J such that

$$c^n \notin J \quad \forall n > 0.$$

and order it by inclusion. Then X satisfies the conditions of the Zorn lemma (see Theorem 1.8), in particular $X \neq \emptyset$ as $I \in X$. Therefore X has a maximal element, say \mathfrak{p} . We claim that \mathfrak{p} is prime. Assume that $ab \in \mathfrak{p}$ and $a \notin \mathfrak{p}$, $b \notin \mathfrak{p}$. Then $\mathfrak{p} + aA$, $\mathfrak{p} + bA$ are strictly larger than \mathfrak{p} , hence

$$c^m \in \mathfrak{p} + aA, \quad c^n \in \mathfrak{p} + bA$$

for some $m, n > 0$. Therefore $c^{m+n} \in (\mathfrak{p} + aA)(\mathfrak{p} + bA) \subset \mathfrak{p} + abA = \mathfrak{p}$, which contradicts to $\mathfrak{p} \in X$. We found a prime ideal \mathfrak{p} such that $c \notin \mathfrak{p}$, hence $c \notin I'$ (intersection of primes), a contradiction. \square

Lemma 1.20. For any ideal $I \subset A$, we have

- (1) $\sqrt{I}/I = \mathcal{N}(A/I)$.
- (2) \sqrt{I} is the intersection of all prime ideals of A that contain I .

Proof. (1) If $[a] \in \sqrt{I}/I \implies a^n \in I$ for some $n > 0 \implies [a]^n = 0$ in $A/I \implies [a] \in \mathcal{N}(A/I)$. The converse is similar.

(2) We know that $\sqrt{I}/I = \mathcal{N}(A/I)$ is the intersection of all prime ideals of A/I . These prime ideals can be identified with prime ideals of A that contain I , hence \sqrt{I} is equal to the intersection of the latter ideals. \square

Definition 1.21. Define the **Jacobson radical** $\mathcal{R}(A)$ of a ring A to be the intersection of all maximal ideals of A .

Remark 1.22. We always have $\mathcal{N}(A) \subset \mathcal{R}(A)$ as every maximal ideal is prime. Later we will show that if A is a finitely-generated algebra over a field, then every prime ideal is an intersection of maximal ideals and therefore $\mathcal{N}(A) = \mathcal{R}(A)$. Such rings are called **Jacobson rings**.

Example 1.23.

- (1) Consider the ring \mathbb{Z} and $a \in \mathcal{R}(\mathbb{Z})$. For any prime number $p \in \mathbb{Z}$, the ideal $p\mathbb{Z}$ is maximal as $\mathbb{Z}/p\mathbb{Z}$ is a field. This implies that $a \in \mathcal{R}(\mathbb{Z}) \subset p\mathbb{Z}$, hence $p \mid a$. We conclude that $p \mid a$ for all prime p , hence $a = 0$. Therefore $\mathcal{R}(A) = 0$.
- (2) Consider the ring $A = \mathbb{k}[x]/(x^2)$. Every ideal in A can be written as $I/(x^2)$ for some ideal $(x^2) \subset I \subset \mathbb{k}[x]$. We can write $I = (f)$ for some $f \in \mathbb{k}[x]$. Then $(x^2) \subset (f)$ implies $f \mid x^2 \implies f = 1, x, x^2$ up to a scalar. The corresponding ideal is maximal (or prime) only if $f = x$. We conclude that $\mathcal{N}(A) = \mathcal{R}(A) = (x)/(x^2)$.
- (3) Let $A = \mathbb{k}[[t]]$ be the ring of formal power series over a field \mathbb{k} . A power series $f = \sum_{i \geq 0} f_i t^i$ is invertible $\iff f_0 \neq 0$. Therefore $\mathfrak{m} = \{f \in A \mid f_0 = 0\}$ is a unique maximal ideal of A , hence $\mathcal{R}(A) = \mathfrak{m}$. On the other hand $\mathcal{N}(A) = 0$.

Lemma 1.24. $a \in \mathcal{R}(A) \iff 1 - ab$ is invertible for all $b \in A$.

Proof. (\implies) If $1 - ab$ is not invertible for some $b \in A \implies (1 - ab) \subset A$ is a proper ideal \implies there exists a maximal ideal $(1 - ab) \subset \mathfrak{m} \subset A$. We have $a \in \mathcal{R}(A) \subset \mathfrak{m} \implies ab \in \mathfrak{m} \implies 1 \in \mathfrak{m}$, a contradiction.

(\impliedby) We need to show that, for any maximal ideal \mathfrak{m} , we have $a \in \mathfrak{m}$. But otherwise $\mathfrak{m} + (a) = A \implies c + ab = 1$ for some $c \in \mathfrak{m}$ and $b \in A \implies 1 - ab = c$ is not invertible, a contradiction. \square

1.4. Local rings.

Definition 1.25. A ring A is called a **local ring** if it has a unique maximal ideal \mathfrak{m} . The field $\mathbb{k} = A/\mathfrak{m}$ is called the **residue field** of A . Sometimes we will write (A, \mathfrak{m}) to specify a local ring A with its maximal ideal \mathfrak{m} .

Remark 1.26. Let (A, \mathfrak{m}) be a local ring.

- (1) We have $\mathcal{R}(A) = \mathfrak{m}$.
- (2) If $I \subset A$ is a proper ideal, then $I \subset \mathfrak{m}$. Indeed, we know that I is contained in some maximal ideal of A . As there exists a unique maximal ideal \mathfrak{m} in A , we have $I \subset \mathfrak{m}$.

Example 1.27. Let $A = \mathbb{k}[[t]]$ be the ring of formal power series over a field \mathbb{k} . A power series $f = \sum_{i \geq 0} f_i t^i$ is invertible $\iff f_0 \neq 0$. Therefore $\mathfrak{m} = \{f \in A \mid f_0 = 0\}$ is a unique maximal ideal of A .

Example 1.28. Let X be a topological space and $x \in X$. Consider the set of pairs (U, f) , where $x \in U \subset X$ is open and $f: U \rightarrow \mathbb{R}$ is continuous. Define an equivalence relation on the set of such pairs as $(U, f) \sim (V, g)$ if there exists open $x \in W \subset U \cap V$ such that $f|_W = g|_W$. The corresponding equivalence classes are called **germs** and form a commutative ring A_x (with pointwise addition and multiplication), called the **ring of germs**. The set $\mathfrak{m}_x = \{[f] \in A_x \mid f(x) = 0\}$ is a maximal ideal (it is the kernel of the evaluation map $A_x \rightarrow \mathbb{R}, [f] \mapsto f(x)$). If $[f] \in A_x \setminus \mathfrak{m}_x$, then $f(x) \neq 0 \implies f$ is nonzero on some open neighborhood V of $x \implies [f]$ is invertible (the inverse is given by $g(y) = f(y)^{-1}$ for $y \in V$). If $\mathfrak{m} \neq \mathfrak{m}_x$ is another maximal ideal $\implies \exists [f] \in \mathfrak{m} \setminus \mathfrak{m}_x \implies [f]$ is invertible $\implies \mathfrak{m} = A_x$, a contradiction. We proved that \mathfrak{m}_x is the unique maximal ideal of A_x , hence A_x is local. This example explains the name “local”.

Lemma 1.29. Let A be a ring and $\mathfrak{m} \subset A$ be a proper ideal. Then f.a.e.

- (1) A is a local ring with the maximal ideal \mathfrak{m} .
- (2) \mathfrak{m} is maximal and all elements in $1 + \mathfrak{m}$ are invertible.
- (3) All elements in $A \setminus \mathfrak{m}$ are invertible.

Proof. (1) \implies (2). If $1 + a$ is not invertible for some $a \in \mathfrak{m} \implies (1 + a)$ is a proper ideal \implies it is contained in a maximal ideal $\implies (1 + a) \subset \mathfrak{m} \implies 1 = (1 + a) - a \in \mathfrak{m}$, a contradiction.

(2) \implies (3). If $a \in A \setminus \mathfrak{m} \implies (a) + \mathfrak{m} = A \implies$ there exist $b \in A$ and $c \in \mathfrak{m}$ such that $ab + c = 1 \implies ab = 1 - c$ is invertible $\implies a$ is invertible.

(3) \implies (1). Let $I \subset A$ be a proper ideal. If $I \not\subset \mathfrak{m} \implies \exists a \in I \setminus \mathfrak{m}$. By assumption a is invertible, hence $A = (a) = I$, a contradiction. We proved that every proper ideal is contained in \mathfrak{m} , hence \mathfrak{m} is the unique maximal ideal. \square

1.5. Rings of fractions. Recall that one can obtain the field of rational numbers \mathbb{Q} from the ring of integers \mathbb{Z} by formally inverting all non-zero integers. More generally, given an integral domain A , we can construct its field of fractions $Q(A)$ by formally inverting all nonzero elements of A . More precisely, we consider the set of pairs (a, s) with $a, s \in A$ and $s \neq 0$, consider an equivalence relation

$$(a, s) \sim (b, t) \iff at = bs$$

and define a ring structure on the set of equivalence classes, where we interpret an equivalence class of (a, s) as a fraction $\frac{a}{s}$. We would like to generalize this construction for an arbitrary ring A in which we invert an appropriate subset $S \subset A$.

Definition 1.30. A subset $S \subset A$ is called a **multiplicative set** (or multiplicatively closed set) if $1 \in S$ and if S is closed under multiplication (i.e. $a, b \in S \implies ab \in S$).

Remark 1.31. Equivalently, a multiplicative set is a submonoid of the multiplicative monoid $(A, *)$.

Define the **ring of fractions** of A with respect to S as follows:
Define a relation \sim on $A \times S$ by the rule

$$(a, s) \sim (b, t) \iff (at - bs)u = 0 \text{ for some } u \in S.$$

This relation is an equivalence relation:

- (1) Reflexive: $(a, s) \sim (a, s)$ is obvious.
- (2) Symmetric: $(a, s) \sim (b, t) \iff (b, t) \sim (a, s)$ is obvious.
- (3) Transitive: assume that $(a, s) \sim (b, t)$ and $(b, t) \sim (c, u)$. Then $(at - bs)v = 0$ and $(bu - ct)w = 0$ for some $v, w \in S$. Therefore

$$(atv)uw = (bsv)uw = (buw)sv = (ctw)sv$$

and $(au - cs)tvw = 0$. This implies $(a, s) \sim (c, u)$ as $tvw \in S$.

We denote by $\frac{a}{s} = a/s$ the equivalence class of (a, s) and we denote by $S^{-1}A$ the set of all equivalence classes. Define the ring structure on $S^{-1}A$ by the rule

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{ts}, \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

This ring is called the **ring of fractions** (or **localization**) of A with respect to S . There is a natural ring homomorphism

$$i: A \rightarrow S^{-1}A, \quad a \mapsto \frac{a}{1}.$$

We will usually denote $i(a)$ by a , although the map i is not necessarily injective. In particular, we denote $\frac{0}{1}$ by 0 and $\frac{1}{1}$ by 1.

Example 1.32. Let A be an integral domain and let $S = A \setminus \{0\}$. Then S is a multiplicative set (if $a, b \neq 0 \implies ab \neq 0$) and the fraction ring $S^{-1}A$ is the field of fractions of A .

Example 1.33. Assume that $0 \in S$. Then we always have $(a, s) \sim (0, 1)$. Therefore $S^{-1}A$ consists of one element $\frac{0}{1} = 0$, hence $S^{-1}A$ is the zero ring.

Remark 1.34. The map $i: A \rightarrow S^{-1}A$ is not always injective. For example, if $0 \in S$, then $S^{-1}A = 0$. For a different example, consider $A = \mathbb{Z}/6\mathbb{Z}$ and $S = \{1, 2, 4\}$. Then $(3, 1) \sim (0, 1)$ as $2(3 - 0) = 0$ in A and $2 \in S$. This implies that $i(3) = 0$.

Lemma 1.35. The map $i: A \rightarrow S^{-1}A$ is injective $\iff S$ does not contain zero divisors.

Proof. (\Leftarrow) If $i(a) = 0$ for some $a \neq 0$, then $(a, 1) \sim (0, 1) \implies ua = 0$ for some $u \in S \implies u \in S$ is a zero divisor. The converse is similar. \square

Example 1.36. Let $\mathfrak{p} \subset A$ be a prime ideal and $S = A \setminus \mathfrak{p}$. Then S is a multiplicative set. Indeed, if $a, b \in S \implies a, b \notin \mathfrak{p} \implies ab \notin \mathfrak{p} \implies ab \in S$. We denote the ring $S^{-1}A$ by $A_{\mathfrak{p}}$ in this case. The set

$$\mathfrak{m} = S^{-1}\mathfrak{p} = \left\{ \frac{a}{s} \mid a \in \mathfrak{p}, s \in S \right\} \subset A_{\mathfrak{p}}$$

is an ideal in $A_{\mathfrak{p}}$. Moreover, if $\frac{a}{s} \notin \mathfrak{m} \implies a \notin \mathfrak{p} \implies a \in S \implies \frac{a}{s}$ is invertible in $A_{\mathfrak{p}}$ with the inverse $\frac{s}{a}$. This implies that $A_{\mathfrak{p}}$ is a local ring and \mathfrak{m} is its maximal ideal. The ring $A_{\mathfrak{p}}$ is called the **localization** of A at \mathfrak{p} .

Example 1.37. Let $f \in A$ and let $S = \{f^n \mid n \geq 0\}$. Then S is a multiplicative set. We denote $S^{-1}A$ by A_f in this case. The elements of A_f are of the form $\frac{a}{f^n}$ for some $a \in A$ and $n \geq 0$. Note that if f is nilpotent, then $0 \in S$ and $A_f = 0$.

Lemma 1.38 (Universal property). *Let $f: A \rightarrow B$ be a ring homomorphism such that $f(s)$ is invertible $\forall s \in S$. Then there exists a unique ring homomorphism $\bar{f}: S^{-1}A \rightarrow B$ such that $f = \bar{f} \circ i$.*

Proof. Uniqueness: $\bar{f}(a/1) = \bar{f}i(a) = f(a) \implies \bar{f}(1/s) = f(s)^{-1} \implies \bar{f}(a/s) = f(a)/f(s)$.

Existence: let $\bar{f}(a/s) = f(a)/f(s)$. □

Remark 1.39. Given an ideal $I \subset A$, its extension I^e with respect to $i: A \rightarrow S^{-1}A$ is given by

$$I^e = S^{-1}A \cdot i(I) = \left\{ \sum \frac{a_i}{s_i} \mid a_i \in I, s_i \in S \right\} = \left\{ \frac{a}{s} \mid a \in I, s \in S \right\} = S^{-1}I.$$

Lemma 1.40. *Consider the ring homomorphism $i: A \rightarrow S^{-1}A$. Then*

- (1) *Every ideal $J \subset S^{-1}A$ is an extended ideal, hence $J = J^{ce}$.*
- (2) *If $I \subset A$ is an ideal, then $I^e = (1) \iff I \cap S \neq \emptyset$.*
- (3) *There is a bijection between the set of prime ideals in $S^{-1}A$ and the set of prime ideals in A that don't intersect S ($S^{-1}A \ni \mathfrak{q} \mapsto \mathfrak{q}^c \subset A$ and $A \ni \mathfrak{p} \mapsto \mathfrak{p}^e \subset S^{-1}A$).*

Proof. (1) Given an ideal $J \subset S^{-1}A$, we need to show that $J^{ce} = J$. If $a/s \in J \implies a \in J \implies a \in J^c \implies a/s \in S^{-1}J^c = J^{ce}$. Therefore $J \subset J^{ce}$. We always have $J^{ce} \subset J \implies J^{ce} = J$.

(2) If $I^e = (1) \implies 1 = a/s$ for some $a \in I, s \in S \implies u(a - s) = 0$ for some $u \in S \implies ua = us \in I \cap S$. Conversely, if $s \in I \cap S \implies 1 = s/s \in I^e$.

(3) If $\mathfrak{q} \subset S^{-1}A$ is prime, then \mathfrak{q}^c is prime. Moreover, $\mathfrak{q}^{ce} = \mathfrak{q} \neq (1)$ (by (1)), hence $\mathfrak{q}^c \cap S = \emptyset$ (by (2)). We need to show that if $\mathfrak{p} \subset A$ is prime and $\mathfrak{p} \cap S = \emptyset$, then \mathfrak{p}^e is prime and $\mathfrak{p}^{ec} = \mathfrak{p}$. Let \bar{S} be the image of S in A/\mathfrak{p} . It does not contain zero as $S \cap \mathfrak{p} = \emptyset$. The ring A/\mathfrak{p} is an integral domain, hence $S^{-1}A/S^{-1}\mathfrak{p} \simeq \bar{S}^{-1}(A/\mathfrak{p})$ is also an integral domain and $\mathfrak{p}^e = S^{-1}\mathfrak{p}$ is a prime ideal. We always have $\mathfrak{p} \subset \mathfrak{p}^{ec}$. If $a \in \mathfrak{p}^{ec}$, then $\frac{a}{1} = \frac{b}{s} \in \mathfrak{p}^e$ for some $b \in \mathfrak{p}$ and $s \in S \implies u(as - b) = 0$ for some $u \in S \implies aus = bu \in \mathfrak{p} \implies a \in \mathfrak{p}$ or $us \in \mathfrak{p}$. But $us \in S$ and $S \cap \mathfrak{p} = \emptyset \implies us \notin \mathfrak{p} \implies a \in \mathfrak{p}$. We proved that $\mathfrak{p}^{ec} \subset \mathfrak{p} \implies \mathfrak{p}^{ec} = \mathfrak{p}$. □

Corollary 1.41. *If $\mathfrak{p} \subset A$ is a prime ideal, then there is a 1-1 correspondence between prime ideals of $A_{\mathfrak{p}}$ and prime ideals of A contained in \mathfrak{p} .*

Remark 1.42. The above bijection does not extend to arbitrary ideals in general. For example, consider $A = \mathbb{Z}$ and $\mathfrak{p} = 2\mathbb{Z}$. Then the element 3 is invertible in $A_{\mathfrak{p}}$. Therefore $(6\mathbb{Z})^e = (2\mathbb{Z})^e$. This means that ideals $6\mathbb{Z}$ and $2\mathbb{Z}$ contained in \mathfrak{p} are mapped to the same ideal in $A_{\mathfrak{p}}$.

Remark 1.43. Let $\mathfrak{p} \subset A$ be a prime ideal and $(A_{\mathfrak{p}}, \mathfrak{m}_{\mathfrak{p}})$ be the corresponding localization. The residue field $\mathbb{k}_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ is called the **residue field at \mathfrak{p}** . It is isomorphic to the field of fractions of the integral domain A/\mathfrak{p} .

2. MODULES

2.1. **Preliminaries.** I will assume the knowledge of the following notions:

- (1) A module over a ring.
- (2) A homomorphism between two modules.
- (3) A submodule of a module and a quotient module.
- (4) The kernel and the image of a homomorphism.

Let M be a module over a ring A .

Definition 2.1.

- (1) For any ideal $I \subset A$, define the product $IM \subset M$ to be the set of all finite sums

$$IM = \left\{ \sum_{i=1}^n a_i x_i \mid a_i \in I, x_i \in M \forall 1 \leq i \leq n, n \geq 0 \right\}.$$

It is a submodule of M .

- (2) Given submodules $L, N \subset M$, define

$$(L : N) = \{a \in A \mid aN \subset L\}.$$

It is an ideal of A .

- (3) Define the **annihilator** of M to be $\text{Ann } M = \text{Ann}_A M = \{a \in A \mid aM = 0\}$.
- (4) We say that M is **faithful** if $\text{Ann } M = 0$.

Remark 2.2.

- (1) Let $I \subset A$ be an ideal and $M = A/I$. Then $\text{Ann } M = I$.
- (2) If $I \subset \text{Ann } M$ is an ideal, then $IM = 0$ and M can be considered as a module over A/I .
- (3) If $I = \text{Ann } M$, then M is faithful as a module over A/I .

Definition 2.3. Let $(M_i)_{i \in \mathcal{J}}$ be a family of submodules of M . Define the **sum** of these submodules to be the subset

$$\sum_i M_i = \left\{ \sum_i x_i \mid x_i \in M_i \forall i \in \mathcal{J}, \text{ almost all } x_i = 0 \right\} \subset M,$$

where almost all x_i (all but a finite number) are zero. It is a submodule of M and it is the minimal submodule of M which contains all the M_i .

Remark 2.4. The intersection $\bigcap_i M_i$ is a submodule of M . It is the maximal submodule of M which is contained in all the M_i .

Definition 2.5.

- (1) For any $x \in M$, define $Ax = \{ax \mid a \in A\}$. It is a submodule of M .
- (2) A module M is said to be generated by a family $(x_i)_{i \in \mathcal{J}}$ of elements in M if $M = \sum_i Ax_i$. The family $(x_i)_i$ is called the set of **generators** of M .
- (3) A module M is said to be **finitely generated** if it has a finite set of generators. This means that there exist elements $x_1, \dots, x_n \in M$ such that

$$M = Ax_1 + \dots + Ax_n = \left\{ \sum_{i=1}^n a_i x_i \mid a_i \in A \forall 1 \leq i \leq n \right\}.$$

Lemma 2.6 (Nakayama's lemma). Let M be a f.g. A -module and $I \subset \mathcal{R}(A)$ be an ideal contained in the Jacobson ideal.

- (1) If $IM = M$, then $M = 0$.
- (2) If $M = IM + N$ for some submodule N , then $M = N$.
- (3) If $f: N \rightarrow M$ is a homomorphism such that $\bar{f}: N/IN \rightarrow M/IM$ is surjective, then f is surjective.

Proof. (1) Let x_1, \dots, x_n be a minimal set of generators of M . Then $x_n \in M = IM$ implies $x_n = a_1 x_1 + \dots + a_n x_n$ with $a_i \in I$. Therefore

$$(1 - a_n)x_n = a_1 x_1 + \dots + a_{n-1} x_{n-1}.$$

We have $a_n \in \mathcal{R}(A) \implies 1 - a_n$ is invertible $\implies x_n$ is a linear combination of $x_1, \dots, x_{n-1} \implies M$ is generated by x_1, \dots, x_{n-1} , a contradiction to the minimality of n .

(2) If $M = IM + N$, then $M/N = (IM + N)/N = I(M/N) \implies M/N = 0 \implies M = N$.

(3) If \bar{f} is surjective, then $f(N) + IM = M \implies f(N) = M$. \square

Lemma 2.7. *Let (A, \mathfrak{m}) be a local ring with the residue field $\mathbb{k} = A/\mathfrak{m}$. Let M be a finitely generated A -module and x_1, \dots, x_n be elements of M that form a basis of $M/\mathfrak{m}M$ over \mathbb{k} . Then x_1, \dots, x_n generate M .*

Proof. Let $N \subset M$ be generated by x_1, \dots, x_n . Then the map $N \rightarrow M \rightarrow M/\mathfrak{m}M$ is surjective, hence $N + \mathfrak{m}M = M \implies N = M$ by Nakayama lemma as $\mathfrak{m} = \mathcal{R}(A)$. \square

2.2. Direct sums and products. Given two A -modules M, N , we define their direct sum $M \oplus N$ to be the set of all pairs (x, y) with $x \in M$ and $y \in N$ and with operations of addition and scalar multiplication

$$(x, y) + (x', y') = (x + x', y + y'), \quad a \cdot (x, y) = (ax, ay), \quad x, x' \in M, y, y' \in N, a \in A.$$

More generally

Definition 2.8. Let $(M_i)_{i \in \mathcal{J}}$ be a family of A -modules.

- (1) Define their **direct sum** $\bigoplus_{i \in \mathcal{J}} M_i = \prod_{i \in \mathcal{J}} M_i$ to be the set of families $(x_i \in M_i)_i$ with almost all $x_i = 0$ (i.e. all but a finite number).
- (2) Define their **direct product** $\prod_{i \in \mathcal{J}} M_i$ to be the set of all families $(x_i \in M_i)_i$ (we allow infinitely many x_i to be nonzero).

These sets are equipped with an A -module structure using pointwise addition and scalar multiplication. Note that if \mathcal{J} is finite, then $\bigoplus_i M_i = \prod_i M_i$.

Remark 2.9. For every $i \in \mathcal{J}$, we define a canonical inclusion

$$\alpha_i: M_i \rightarrow \bigoplus_j M_j, \quad x_i \mapsto (0, \dots, 0, x_i, 0, \dots, 0)$$

and a canonical projection

$$\pi_i: \prod_j M_j \rightarrow M_i, \quad (x_j)_j \mapsto x_i.$$

Lemma 2.10 (Universal properties). *Let M and $(M_i)_{i \in \mathcal{J}}$ be a family of A -modules.*

- (1) *Given a family of homomorphisms $(f_i: M_i \rightarrow M)_{i \in \mathcal{J}}$, there exists a unique homomorphism $\bar{f}: \bigoplus_i M_i \rightarrow M$ such that $\bar{f} \circ \alpha_i = f_i$.*
- (2) *Given a family of homomorphisms $(f_i: M \rightarrow M_i)_{i \in \mathcal{J}}$, there exists a unique homomorphism $\bar{f}: M \rightarrow \prod_i M_i$ such that $\pi_i \circ \bar{f} = f_i$.*

Proof. (1) We define $\bar{f}((x_i)_i) = \sum_i f_i(x_i) \in M$ for all $(x_i)_i \in \bigoplus_i M_i$. It's not difficult to verify that this is a well-defined homomorphism (note that almost all $x_i = 0$ and the sum on the right is finite) which satisfies the required conditions.

(2) We define $\bar{f}(x) = (f_i(x))_i \in \prod_i M_i$ for all $x \in M$. It's not difficult to verify that this is a well-defined homomorphism which satisfies the required conditions. \square

Definition 2.11.

- (1) For every $n \geq 0$, we define $A^n = \prod_{i=1}^n A = \bigoplus_{i=1}^n A$, the direct sum of n copies of A .
- (2) Given a set \mathcal{J} , define $A^{(\mathcal{J})} = \bigoplus_{i \in \mathcal{J}} A$, the direct sum of copies of A indexed by \mathcal{J} .
- (3) A module M over a ring A is called **free** if it is isomorphic to $A^{(\mathcal{J})}$ for some set \mathcal{J} . Equivalently, M has a **basis** $(x_i)_{i \in \mathcal{J}}$, meaning that $(x_i)_{i \in \mathcal{J}}$ generates M and is linearly independent: if $\sum_i a_i x_i = 0$ for some $a_i \in A$, then $a_i = 0$ for all $i \in \mathcal{J}$.

Lemma 2.12. *A module M is finitely generated $\iff M$ is isomorphic to a quotient of A^n for some $n \geq 0$.*

Proof. (\implies) Assume that M is finitely generated and let $(x_i)_{i \in I}$ be a set of its generators. Consider the map

$$f: A^n \rightarrow M, \quad (a_1, \dots, a_n) \mapsto \sum_i a_i x_i \in M.$$

This map is surjective, hence $M \simeq A^n / \text{Ker } f$.

(\impliedby) Under our assumptions there exists a surjective homomorphism $f: A^n \rightarrow M$ for some $n \geq 0$. Then the elements $x_i = f(e_i)$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in A^n$ for $1 \leq i \leq n$, generate M . \square

2.3. Hom and tensor product. Given modules M, N over A , consider the set of all A -module homomorphisms

$$\text{Hom}(M, N) = \text{Hom}_A(M, N) = \{f: M \rightarrow N \mid f \text{ is a homomorphism}\}.$$

It has an A -module structure, where the sum and the scalar product are defined by the rules

$$(f + g)(x) = f(x) + g(x), \quad (af)(x) = a \cdot f(x) \quad \forall x \in M$$

for $f, g \in \text{Hom}(M, N)$ and $a \in A$. The set of endomorphisms $\text{End}(M) = \text{Hom}(M, M)$ has a (non-commutative) ring structure, where the product is given by composition.

Remark 2.13. Homomorphisms $\alpha: M' \rightarrow M$ and $\beta: N \rightarrow N'$ induce maps

- (1) $\alpha^*: \text{Hom}(M, N) \rightarrow \text{Hom}(M', N)$, $f \mapsto f \circ \alpha$,
- (2) $\beta_*: \text{Hom}(M, N) \rightarrow \text{Hom}(M, N')$, $f \mapsto \beta \circ f$

which are again A -module homomorphisms.

We define the **tensor product** $M \otimes N = M \otimes_A N$ of two modules to be an A -module generated by elements of the form $x \otimes y$ for $x \in M$, $y \in N$ subject to the relations

- (1) $(x + x') \otimes y = x \otimes y + x' \otimes y$ for $x, x' \in M$ and $y \in N$.
- (2) $x \otimes (y + y') = x \otimes y + x \otimes y'$ for $x \in M$ and $y, y' \in N$.
- (3) $a(x \otimes y) = (ax) \otimes y = x \otimes (ay)$ for $a \in A$, $x \in M$ and $y \in N$.

Remark 2.14. A general element of $M \otimes N$ can be written in the form $\sum_i x_i \otimes y_i$, where $x_i \in M$ and $y_i \in N$, although this representation is not unique. If $(x_i)_i$ and $(y_j)_j$ are families of generators of M, N respectively, then $x_i \otimes y_j$ generate $M \otimes N$.

Proposition 2.15 (Universal property). *Let M, N be A -modules. Then*

- (1) *The map $\phi: M \times N \rightarrow M \otimes N$, $(x, y) \mapsto x \otimes y$ is A -bilinear.*
- (2) *For any A -module L and any A -bilinear map $f: M \times N \rightarrow L$, there exists a unique A -linear map $\bar{f}: M \otimes N \rightarrow L$ such that $f = \bar{f} \circ \phi$.*

Proof. (1) We have

- (1) $\phi(x + x', y) = (x + x') \otimes y = x \otimes y + x' \otimes y = \phi(x, y) + \phi(x', y)$.
- (2) $\phi(ax, y) = (ax) \otimes y = a(x \otimes y) = a\phi(x, y)$.

Other axioms of bilinearity are verified in the same way.

(2) Uniqueness follows from the fact that $\bar{f}(x \otimes y) = \bar{f}\phi(x, y) = f(x, y)$ and elements $x \otimes y$ generate $M \otimes N$. For the existence we verify that the map \bar{f} given by $\bar{f}(x \otimes y) = f(x, y)$ satisfies all the required properties. \square

Remark 2.16. Homomorphisms $\alpha: M \rightarrow M'$ and $\beta: N \rightarrow N'$ induce a homomorphism

$$\alpha \otimes \beta: M \otimes N \rightarrow M' \otimes N', \quad x \otimes y \mapsto \alpha(x) \otimes \beta(y).$$

In particular, there is a homomorphism $\alpha \otimes 1: M \otimes N \rightarrow M' \otimes N$, $x \otimes y \mapsto \alpha(x) \otimes y$.

Lemma 2.17. *We have*

- (1) $\text{Hom}(A, M) \simeq M$.
- (2) $A \otimes M \simeq M$.
- (3) $M \otimes N \simeq N \otimes M$.
- (4) $L \otimes (M \otimes N) \simeq (L \otimes M) \otimes N$.
- (5) $(L \oplus M) \otimes N \simeq (L \otimes N) \oplus (M \otimes N)$.

Lemma 2.18 (Tensor-Hom adjunction). *Given modules L, M, N , there is a natural isomorphism*

$$\text{Hom}(L \otimes M, N) \simeq \text{Hom}(L, \text{Hom}(M, N)).$$

Proof. Given $f: L \otimes M \rightarrow N$, define $f': L \rightarrow \text{Hom}(M, N)$ by the rule

$$f'(l)(m) = f(l \otimes m).$$

Conversely, given $f': L \rightarrow \text{Hom}(M, N)$, define $f'': L \times M \rightarrow N$ by the rule $f''(l, m) = f'(l)(m)$. It is easy to see that f'' is A -bilinear, hence it factors through $f: L \otimes M \rightarrow N$. \square

Definition 2.19 (Restriction and extension of scalars). Let $f: A \rightarrow B$ be a ring homomorphism.

- (1) Given a B -module N , we can equip it with an A -module structure by the rule $ax = f(a)x$ for $a \in A$, $x \in N$. It is said to be obtained from N by **restriction of scalars**. In particular, the ring B is equipped with an A -module structure. We call B an **A -algebra** in this situation.
- (2) Given an A -module M , we can equip $M_B = B \otimes_A M$ with a B -module structure by the rule $b(b' \otimes x) = (bb') \otimes x$ for $b, b' \in B$ and $x \in M$. It is said to be obtained from M by **extension of scalars**.

Example 2.20. Consider a natural projection $\pi: A = \mathbb{Z} \rightarrow B = \mathbb{Z}/2\mathbb{Z}$ and consider a \mathbb{Z} -module $M = \mathbb{Z}/3\mathbb{Z}$. Then $M_B = B \otimes_A M = (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z}) = 0$.

Lemma 2.21. We have $M_B \otimes_B N_B \simeq (M \otimes N)_B$.

Proof.

$$M_B \otimes_B (N_B) \simeq M \otimes_A B \otimes_B B \otimes_A N \simeq M \otimes_A B \otimes_A N \simeq (M \otimes N)_B$$

as $B \otimes_B B \simeq B$. □

Lemma 2.22. Given an A -module M and a B -module N , we have $\text{Hom}_B(M_B, N) \simeq \text{Hom}_A(M, N)$.

Proof. By the adjunction isomorphism

$$\text{Hom}_B(M_B, N) = \text{Hom}_B(B \otimes_A M, N) \simeq \text{Hom}_A(M, \text{Hom}_B(B, N)) \simeq \text{Hom}_A(M, N)$$

as $\text{Hom}_B(B, N) \simeq N$. □

2.4. Exact sequences.

Definition 2.23.

- (1) A sequence of modules and homomorphisms

$$\cdots \rightarrow M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \rightarrow \cdots$$

is called a (cochain) **complex** if $d_i \circ d_{i-1} = 0$ for all i . This is equivalent to the requirement $\text{Im } d_{i-1} \subset \text{Ker } d_i$.

- (2) The complex is said to be **exact** at M_i if $\text{Im } d_{i-1} = \text{Ker } d_i$. The complex (or sequence) is called exact if it is exact at every M_i .

Lemma 2.24.

- (1) $0 \rightarrow M \xrightarrow{f} N$ is exact (at M) $\iff f$ is injective.
 (2) $M \xrightarrow{g} N \rightarrow 0$ is exact (at N) $\iff g$ is surjective.
 (3) A complex $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is exact $\iff f$ is injective, g is surjective and induces an isomorphism $g: M/\text{Im } f \rightarrow N$. Such sequence is called a **short exact sequence**.

Proof. (3) If the sequence is exact, then f is injective by (1) and g is surjective by (2). We also have $\text{Im } f = \text{Ker } g$, hence $N \simeq M/\text{Ker } g = M/\text{Im } f$. Conversely, we get injectivity of f by (1) and surjectivity of g by (2). Finally, we have $\text{Im } f = \text{Ker}(M \rightarrow M/\text{Im } f) = \text{Ker}(M \rightarrow N) = \text{Ker } g$. \square

Lemma 2.25.

- (1) A complex

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$$

is exact \iff for any module N the following sequence is exact

$$0 \rightarrow \text{Hom}(N, M_1) \rightarrow \text{Hom}(N, M_2) \rightarrow \text{Hom}(N, M_3)$$

- (2) A complex

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

is exact \iff for any module N the following sequence is exact

$$0 \rightarrow \text{Hom}(M_3, N) \rightarrow \text{Hom}(M_2, N) \rightarrow \text{Hom}(M_1, N).$$

- (3) If a complex

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

is exact, then for any module N , the following complex is exact

$$M_1 \otimes N \rightarrow M_2 \otimes N \rightarrow M_3 \otimes N \rightarrow 0$$

Proof. (2) Consider an exact sequence $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \rightarrow 0$ and the corresponding sequence

$$0 \rightarrow \text{Hom}(M_3, N) \xrightarrow{g^*} \text{Hom}(M_2, N) \xrightarrow{f^*} \text{Hom}(M_1, N).$$

If $\phi \in \text{Hom}(M_3, N)$ is mapped to zero $\implies g^*(\phi) = \phi g = 0$. But g is surjective $\implies \phi = 0$. This means that $\text{Ker } g^* = 0$. If $\phi \in \text{Hom}(M_2, N)$ is mapped to zero $\implies f^*(\phi) = \phi f = 0 \implies \phi(\text{Im } f) = 0 \implies \phi: M_2 \rightarrow N$ factorizes through $\bar{\phi}: M_2/\text{Im } f \rightarrow N$. But $M_2/\text{Im } f = M_2/\text{Ker } g \simeq M_3$ and we obtain $\bar{\phi}: M_3 \rightarrow N$ satisfying $g^*(\bar{\phi}) = \phi$. This implies $\text{Ker } f^* = \text{Im } g^*$.

The converse statement is proved similarly.

- (3) According to (2), the required complex is exact \iff for any module L , the complex

$$0 \rightarrow \text{Hom}(M_3 \otimes N, L) \rightarrow \text{Hom}(M_2 \otimes N, L) \rightarrow \text{Hom}(M_1 \otimes N, L)$$

is exact. Using the isomorphism $\text{Hom}(M \otimes N, L) \simeq \text{Hom}(M, \text{Hom}(N, L))$, we can rewrite the above complex as

$$0 \rightarrow \text{Hom}(M_3, L') \rightarrow \text{Hom}(M_2, L') \rightarrow \text{Hom}(M_1, L'),$$

where $L' = \text{Hom}(N, L)$. The latter complex is exact by (2). \square

Remark 2.26. Using categories and functors we can interpret the above result by saying that

- (1) The functor $\text{Hom}(N, -): \text{Mod } A \rightarrow \text{Mod } A$ is left exact.
 (2) The contravariant functor $\text{Hom}(-, N): \text{Mod } A \rightarrow \text{Mod } A$ is left exact.

(3) The functor $- \otimes N: \text{Mod } A \rightarrow \text{Mod } A$ is right exact.

Example 2.27. Let $A = \mathbb{Z}$ and $N = \mathbb{Z}/2\mathbb{Z}$. Then $\text{Hom}(N, -)$ does not preserve exactness. For example, we can apply it to the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

and observe that $\text{Hom}(N, \mathbb{Z}) = 0$ and $\text{Hom}(N, N) = N$.

Similarly, $N \otimes -$ does not preserve exactness. We can apply $N \otimes -$ to the above sequence and obtain a sequence

$$0 \rightarrow N \xrightarrow{0} N \rightarrow N \rightarrow 0$$

which is not exact on the left.

2.5. Projective and flat modules.

Definition 2.28.

- (1) A module P is called **projective** if $\text{Hom}(P, -)$ preserves exact sequences.
- (2) A module P is called **flat** if $P \otimes -$ preserves exact sequences.

Lemma 2.29. FAE

- (1) P is projective.
- (2) $\text{Hom}(P, -)$ preserves short exact sequences.
- (3) If $f: M \rightarrow N$, $g: P \rightarrow N$ are homomorphisms and f is surjective, then there exists $h: P \rightarrow M$ such that $g = fh$.

$$\begin{array}{ccc} & & P \\ & \swarrow h & \downarrow g \\ M & \xrightarrow{f} & N \end{array}$$

- (4) Every surjective homomorphism $f: M \rightarrow N$ splits, that is, there exists $s: N \rightarrow M$ such that $fs = 1_N$.
- (5) P is a direct summand of a free module.

Proof. (1) \implies (2). is clear.

(2) \implies (1). follows from the fact that we can split every exact sequence

$$\cdots \rightarrow M_{i-1} \xrightarrow{d_{i-1}} M_i \rightarrow M_{i+1} \xrightarrow{d_i} \cdots$$

into short exact sequences

$$0 \rightarrow K_i \rightarrow M_i \rightarrow K_{i+1} \rightarrow 0,$$

where $K_i = \text{Ker } d_i$ and $K_{i+1} = \text{Ker } d_{i+1} = \text{Im } d_i$.

(2) \implies (3). Given a surjective $f: M \rightarrow N$, we consider a short exact sequence

$$0 \rightarrow \text{Ker } f \rightarrow M \xrightarrow{f} N \rightarrow 0$$

Then $f_*: \text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$ is surjective, in particular $\exists h \in \text{Hom}(P, M)$ such that $g = f_*(h) = fh$.

(3) \implies (4). Considering $g = 1_N$, we can find $h: P \rightarrow M$ such that $fh = g = 1_N$.

(4) \implies (5). Let $(x_i)_{i \in \mathbb{J}}$ be a set of generators of P . Then there exists a surjective homomorphism $f: F = A^{(\mathbb{J})} \rightarrow P$, $e_i \mapsto x_i$. By assumption, there exists $s: P \rightarrow F$ such that $fs = 1_P$. We claim that $F = \text{Im } s \oplus \text{Ker } f$. If $x \in \text{Im } s \oplus \text{Ker } f$, then $x = s(y)$ and $f(x) = 0 \implies y = fs(y) = f(x) = 0 \implies x = 0$. On the other for every $x \in F$, consider $x' = sf(x)$. Then $x' \in \text{Im } S$ and $f(x - x') = f(x) - f(x) = 0$, hence $x - x' \in \text{Ker } f$. Finally, note that $P \simeq \text{Im } s$ as $s: P \rightarrow F$ is injective. Therefore $F \simeq P \oplus \text{Ker } f$.

(5) \implies (1). Note that $\text{Hom}(A, M) \simeq M$, hence $\text{Hom}(A, -)$ preserves exactness. Similarly, $\text{Hom}(A^{(\mathbb{J})}, M) \simeq M^{(\mathbb{J})}$, hence $\text{Hom}(A^{(\mathbb{J})}, -)$ preserves exactness. This implies that for every direct summand P of $A^{(\mathbb{J})}$, $\text{Hom}(P, -)$ preserves exactness. \square

Lemma 2.30. FAE

- (1) P is flat.
- (2) $P \otimes -$ preserves short exact sequences.
- (3) If $f: M \rightarrow N$ is injective, then $f \otimes 1: P \otimes M \rightarrow P \otimes N$ is injective.

Lemma 2.31. Every projective module is flat.

Proof. If P is projective, then P is a direct summand of a free module (assume for simplicity that it is finitely generated), say A^n . We have $A^n \otimes M \simeq M^n$, hence $A^n \otimes -$ preserves exactness. This implies that P also preserves exactness. \square

2.6. Localization of modules. Let $S \subset A$ be a multiplicative system and let M be an A -module. Define an equivalence relation on $M \times S$ by the rule

$$(m, s) \sim (m', t) \iff \exists u \in S: u(tm - sm') = 0$$

We denote an equivalence class of (m, s) by $\frac{m}{s} = m/s$ and denote the set of all equivalence classes by $S^{-1}M$. It can be equipped with a structure of an $S^{-1}A$ -module in an obvious way.

Remark 2.32.

- (1) If $\mathfrak{p} \subset A$ is prime and $S = A \setminus \mathfrak{p}$, we denote $S^{-1}M$ by $M_{\mathfrak{p}}$.
- (2) If $f \in A$ and $S = \{f^n\}_{n \geq 0}$, we denote $S^{-1}M$ by M_f .

Lemma 2.33. *The map*

$$f: S^{-1}A \otimes_A M \rightarrow S^{-1}M, \quad \frac{a}{s} \otimes m \mapsto \frac{am}{s}$$

is well-defined and is an isomorphism of $S^{-1}A$ -modules.

Proof. One can see that the map

$$S^{-1}A \times M \rightarrow S^{-1}M, \quad \left(\frac{a}{s}, m\right) \mapsto \frac{am}{s}$$

is A -bilinear, hence induces the required homomorphism f . It is clear that f is surjective. Let $\sum_i \frac{a_i}{s_i} \otimes m_i \in \text{Ker } f$. We can bring it to the form $\frac{1}{s} \otimes m$. As $f(\frac{1}{s} \otimes m) = \frac{m}{s} = 0$, there exists $u \in S$ such that $um = 0$. But then

$$\frac{1}{s} \otimes m = \frac{u}{us} \otimes m = \frac{1}{us} \otimes um = 0.$$

This implies that f is injective, hence is an isomorphism. \square

Lemma 2.34. *The operation S^{-1} preserves exact sequences.*

Proof. Consider an exact sequence $L \xrightarrow{f} M \xrightarrow{g} N$ and the corresponding sequence

$$S^{-1}L \xrightarrow{f'} S^{-1}M \xrightarrow{g'} S^{-1}N$$

We have $gf = 0 \implies$ hence $g'f' = 0 \implies \text{Im}(f') \subset \text{Ker}(g')$. If $m/s \in \text{Ker}(g') \implies g(m)/s = 0$ in $S^{-1}N \implies \exists u \in S, ug(m) = 0$ in $N \implies g(um) = 0 \implies um \in \text{Ker } g = \text{Im } f \implies um = f(m')$ for some $m' \in L \implies m/s = f(m')/us = f'(m'/us) \in \text{Im}(f')$. This proves that $\text{Ker}(g') = \text{Im}(f')$. \square

Corollary 2.35. *$S^{-1}A$ is a flat A -module.*

Example 2.36. The last corollary implies that \mathbb{Q} is a flat module over \mathbb{Z} . On the other hand \mathbb{Q} is not projective: otherwise it is a direct summand of a free module, hence there is an injective map $f: \mathbb{Q} \rightarrow \mathbb{Z}^{(I)}$ for some set I . Let $f(1) = (x_i)_{i \in I} \in \mathbb{Z}^{(I)}$ and let $n = \max_i |x_i| + 1$. If $f(1/n) = (y_i)_{i \in I}$, then $(x_i)_i = nf(1/n) = (ny_i)_i$. But $|ny_i| > |x_i|$ whenever $y_i \neq 0$. A contradiction.

Lemma 2.37. *Let M be an A module. Then FAE*

- (1) $M = 0$.
- (2) $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p} \subset A$.
- (3) $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m} \subset A$.

Proof. (1) \implies (2) \implies (3) is clear.

Assume that $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m} \subset A$ and $M \neq 0$. Let $0 \neq x \in M$ and $I = \text{Ann}(A)$. Then I is a proper ideal ($1 \cdot x = x \neq 0 \implies 1 \notin I$), hence it is contained in a maximal ideal \mathfrak{m} . Since $x/1 = 0$ in $M_{\mathfrak{m}}$, there exists some $u \in A \setminus \mathfrak{m}$ such that $ux = 0$. But then $u \in \text{Ann } x = I \subset \mathfrak{m}$. A contradiction. \square

Lemma 2.38. *Let $f: M \rightarrow N$ be a module homomorphism. Then FAE*

- (1) f is injective/surjective.
- (2) $f_{\mathfrak{p}}: M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is injective/surjective for every prime ideal $\mathfrak{p} \subset A$.
- (3) $f_{\mathfrak{m}}: M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is injective/surjective for every maximal ideal $\mathfrak{m} \subset A$.

Proof. **(1) \implies (2).** As localization preserves exact sequences.

(2) \implies (3). As every maximal ideal is prime.

(3) \implies (1). Assume that $f_{\mathfrak{m}}$ is injective for every maximal ideal \mathfrak{m} . Let $L = \text{Ker } f$ and consider an exact sequence $0 \rightarrow L \rightarrow M \xrightarrow{f} N$. For every maximal ideal \mathfrak{m} , the corresponding sequence $0 \rightarrow L_{\mathfrak{m}} \rightarrow M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}}$ is exact $\implies L_{\mathfrak{m}} \simeq \text{Ker } f_{\mathfrak{m}} = 0$ as $f_{\mathfrak{m}}$ is injective. This implies that $L = 0$, hence f is injective. The proof for surjectivity is similar. \square

3. CHAIN CONDITIONS

3.1. Noetherian rings and modules.

Definition 3.1. Let A be a ring.

- (1) An A -module M is called **Noetherian** if every submodule of M is finitely generated.
- (2) The ring A is called **Noetherian** if it is Noetherian as a module over itself.

Example 3.2.

- (1) A PID A is Noetherian. Indeed, every submodule of A is an ideal, hence a principal ideal, generated by one element.
- (2) In particular, the algebra of polynomials $\mathbb{k}[x]$ over a field \mathbb{k} is Noetherian. We will see later that $\mathbb{k}[x_1, \dots, x_n]$ is also Noetherian. As well as $\mathbb{k}[x_1, \dots, x_n]/I$ for every ideal I .

Lemma 3.3. Let M be an A -module. The following conditions are equivalent

- (1) Every submodule of M is finitely generated.
- (2) Every increasing chain of submodules

$$M_1 \subset M_2 \subset \dots \subset M$$

stabilizes, that is, $M_n = M_{n+1} = \dots$ for some $n > 0$.

Proof. (1) \implies (2). Consider an increasing chain

$$M_1 \subset M_2 \subset \dots \subset M$$

and let $N = \cup_{n \geq 1} M_n \subset M$. Then N is a submodule of M and by assumption it is finitely generated. Let x_1, \dots, x_k be generators of N . Then $x_i \in M_{n_i}$ for some $n_i \geq 1$. Taking $n = \max_i n_i$, we obtain $x_i \in M_n$ for all i , hence $N \subset M_n$ and $M_n = M_{n+1} = \dots$.

(2) \implies (1). Let us show that a submodule $N \subset M$ is finitely generated. Choose $x_0 = 0 \in N$ and, assuming that elements $x_0, \dots, x_k \in N$ are constructed, let $M_k \subset N$ be the module generated by them. If $M_k = N$, then N is finitely generated and we are done. If $M_k \neq N$, we let $x_{k+1} \in N \setminus M_k$ and continue the process. In this way we obtain a chain of modules

$$M_1 \subset M_2 \subset \dots \subset N \subset M$$

with $M_k \neq M_{k+1}$ for all $k \geq 1$. A contradiction. \square

Definition 3.4. Let A be a ring.

- (1) An A -module M is called **Artin** if every decreasing chain of submodules

$$M \supset M_1 \supset M_2 \supset \dots$$

stabilizes, that is, $M_n = M_{n+1} = \dots$ for some $n > 0$.

- (2) The ring A is called **Artin** if it is Artin as a module over itself.

We will see later that every Artin ring is automatically Noetherian.

Example 3.5.

- (1) The algebra $\mathbb{k}[x_1, x_2, \dots]$ is neither Noetherian nor Artin. Indeed, it contains chains of ideals

$$(x_1) \subset (x_1, x_2) \subset \dots, \quad (x_1, x_2, \dots) \supset (x_2, \dots) \supset \dots$$

- (2) The algebra $\mathbb{k}[x]$ is not Artinian: $(x) \supset (x^2) \supset \dots$.
- (3) A vector space over a field is Noetherian \iff it is Artin \iff it is finite-dimensional.
- (4) Let $p \in \mathbb{Z}$ be a prime number and let $\mathbb{Z}_p = \{m/p^n \mid m \in \mathbb{Z}, n \geq 0\} \subset \mathbb{Q}$ be the corresponding localization. One can show that the \mathbb{Z} -module \mathbb{Z}_p/\mathbb{Z} is Artin, but not Noetherian. To see this one should verify that the only proper submodules of \mathbb{Z}_p/\mathbb{Z} are of the form $M_n = \{[m/p^n] \mid m \in \mathbb{Z}\}$ for $n \geq 0$.

Lemma 3.6. Let M be an A -module and $L \subset M$ be a submodule. Then M is Noetherian $\iff L$ and M/L are Noetherian.

Proof. First proof. Assume that M is Noetherian. Every increasing chain in L is a chain in M , hence stabilizes. Given an increasing chain $(M'_n)_n$ in M/L , we consider the chain $(\pi^{-1}(M'_n))_n$ in M , where $\pi: M \rightarrow M/L$ is the projection. Then $(\pi^{-1}(M'_n))_n$ stabilizes, hence $(M'_n)_n$ also stabilizes as $M'_n = \pi(\pi^{-1}(M'_n))$. Assume now that L and M/L are Noetherian and let $(M_n)_n$ be an increasing chain in M . Then the chain of modules

$$\pi(M_n) = (M_n + L)/L$$

stabilizes in M/L and the chain of modules $M_n \cap L$ stabilizes in L . Therefore there exists $n \geq 0$ such that $M_n + L = M_m + L$ and $M_n \cap L = M_m \cap L$ for all $m \geq n$. This implies that the inclusion $M_n \subset M_m$ is equality (hence the chain stabilizes) as otherwise $\exists x \in M_m \setminus M_n \implies x \in M_m \subset M_n + L \implies x = y + l$ for some $y \in M_n, l \in L \implies x - y = l \in M_m \cap L = M_n \cap L \implies x \in M_n$, a contradiction.

Second proof. Let M be Noetherian. If $N \subset L$ is a submodule, then $N \subset M$, hence N is finitely generated and L is Noetherian. Let $N \subset M/L$ be a submodule and let $\pi: M \rightarrow M/L$ be the projection. The module $N' = \pi^{-1}(N) \subset M$ is finitely generated, hence also $N = \pi(N')$ is finitely generated and M/L is Noetherian.

Assume that L and M/L are Noetherian and let $N \subset M$. Then $N \cap L \subset L$ is finitely generated and $N/(N \cap L) \simeq (N + L)/L \subset M/L$ is finitely generated. This implies that N is also finitely generated. \square

Corollary 3.7. *If M, N are Noetherian A -modules, then $M \oplus N$ is also Noetherian.*

Proof. Let $M' = M \oplus N$. Then $N \subset M'$ and $M'/N \simeq M$ are Noetherian. We conclude that M' is Noetherian. \square

Corollary 3.8. *If A is a Noetherian ring and M is a finitely generated A -module, then M is Noetherian.*

Proof. Let M have a generator set (x_1, \dots, x_n) . Then there is a surjection $f: A^n \rightarrow M$, $(a_i)_i \mapsto \sum_i a_i x_i$. The module A^n is Noetherian by Cor. 3.7. Therefore the module $M \simeq A^n / \text{Ker } f$ is Noetherian by Lemma 3.6. \square

Lemma 3.9. *Let A be a Noetherian ring and $S \subset A$ be multiplicative set. Then $S^{-1}A$ is Noetherian.*

Proof. Let $i: A \rightarrow S^{-1}A$, $a \mapsto a/1$ be the canonical homomorphism. We proved that for any ideal $J \subset S^{-1}A$, we have $J = S^{-1}J^c$, where $J^c = i^{-1}(J) \subset A$. By our assumption, J^c is finitely generated, say, by elements a_1, \dots, a_n . This implies that $S^{-1}J^c$ is generated by $a_1/1, \dots, a_n/1$. Therefore all ideals of $S^{-1}A$ are finitely generated. \square

Theorem 3.10 (Hilbert's basis theorem). *If A is noetherian, then $A[x]$ is noetherian.*

Proof. Let $I \subset A[x]$ be an ideal. For any $f \in A[x]$, let $\text{lc}(f)$ be its leading coefficient. The set $J = \{\text{lc}(f) \mid f \in I\}$ is an ideal in A . As A is noetherian, J is finitely generated, say by elements a_1, \dots, a_n . For every $1 \leq i \leq n$, choose $f_i \in I \subset A[x]$ such that $a_i = \text{lc}(f_i)$ and let $r_i = \deg f_i$. Let $I' = (f_1, \dots, f_n) \subset I$ and let $r = \max\{r_1, \dots, r_n\}$. For any $f \in I$, if $m = \deg f \geq r$, consider $a = \text{lc}(f) \in J$ and write $a = \sum_i b_i a_i$ for some $b_i \in A$. Then $f - \sum_i b_i f_i x^{m-r_i}$ has degree $< m$ and is still in I . Note that $\sum_i b_i f_i x^{m-r_i} \in I'$. Proceeding in this way, we obtain a decomposition $f = g + h$, where $\deg g < r$ and $h \in I'$.

Let $M \subset A[x]$ be an A -module generated by $1, x, \dots, x^{r-1}$. Then $g \in M$ and $g = f - h \in I \implies g \in M \cap I$. We proved that $I = (M \cap I) + I'$. As M is finitely generated over A , it is noetherian $\implies M \cap I$ is finitely generated over A . We also know that $I' = (f_1, \dots, f_n)$ is finitely generated over $A[x]$. This implies that $I = (M \cap I) + I'$ is finitely generated over $A[x]$. We conclude that $A[x]$ is Noetherian. \square

Definition 3.11. Let B be an A -algebra (this means that $A \subset B$ or more generally, we are given a ring homomorphism $f: A \rightarrow B$). We say that B is a **finitely generated** A -algebra if there exists a finite set of elements $b_1, \dots, b_n \in B$ such that every element in B can be written in the form $f(b_1, \dots, b_n)$ for some polynomial $f \in A[x_1, \dots, x_n]$.

Remark 3.12.

- (1) Note that there is a surjective ring homomorphism $A[x_1, \dots, x_n] \rightarrow B$, $f \mapsto f(b_1, \dots, b_n)$, hence $B \simeq A[x_1, \dots, x_n] / \text{Ker } f$. Conversely, if $B = A[x_1, \dots, x_n] / I$ for some ideal I , then B is a finitely-generated A -algebra.
- (2) Note that $\mathbb{k}[x_1, \dots, x_n]$ is a finitely-generated \mathbb{k} -algebra, but not a finitely-generated \mathbb{k} -module.

Corollary 3.13. *If A is a Noetherian ring and B is a finitely-generated A -algebra, then B is also Noetherian.*

Proof. There is a surjective ring homomorphism $A[x_1, \dots, x_n] \rightarrow B$. By the Hilbert's basis theorem, the ring $A[x_1, \dots, x_n]$ is Noetherian, therefore its quotient is also Noetherian. \square

Definition 3.14. A prime ideal $\mathfrak{p} \subset A$ is called a **minimal prime ideal** over an ideal $I \subset A$ if it is minimal among all prime ideals containing I .

Lemma 3.15 (Noether). *If A is Noetherian and $I \subset A$ is an ideal, then there are only finitely many minimal prime ideals over I (and every prime ideal over I contains one of them). In particular, \sqrt{I} is a finite intersection of prime ideals.*

Proof. Assume that the statement is wrong and let I be a maximal ideal among all ideals that do not satisfy the required condition (it exists as A is Noetherian). Then I is not prime, hence $\exists a, b \notin I$ such that $ab \in I$. Ideals (I, a) and (I, b) are strictly greater than I , hence there are finitely many minimal primes over them. We have $Z(I, a) \cup Z(I, b) \subset Z(I)$. On the other hand, if $\mathfrak{p} \supset I$ is prime, then $ab \in \mathfrak{p} \implies a \in \mathfrak{p}$ or $b \in \mathfrak{p} \implies \mathfrak{p} \in Z(I, a)$ or $\mathfrak{p} \in Z(I, b)$. Therefore $Z(I) = Z(I, a) \cup Z(I, b)$ and minimal prime ideals over I are contained in the union of (finitely many) minimal primes over (I, a) and minimal primes over (I, b) . \square

3.2. Artin rings. Our goal in this section is to get a better understanding of Artin rings and to show that they are always Noetherian.

Lemma 3.16. *In an Artin ring every prime ideal is maximal.*

Proof. Let A be an Artin ring and $\mathfrak{p} \subset A$ be a prime ideal. Then $B = A/\mathfrak{p}$ is an Artin integral domain. For every nonzero $x \in B$, the chain $(x) \supset (x^2) \supset \dots$ stabilizes $\implies (x^n) = (x^{n+1})$ for some $n \geq 1 \implies x^n = x^{n+1}y$ for some $y \in B \implies xy = 1$ and x is invertible. This implies that B is a field and $\mathfrak{p} \subset A$ is maximal. \square

Lemma 3.17. *In an Artin ring there are only finitely many maximal ideals.*

Proof. Given an infinite sequence of different maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots$, consider a decreasing chain of ideals $I_n = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n$ for $n \geq 1$. This chain stabilizes, hence $\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n \cap \mathfrak{m}_{n+1} \implies \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n \subset \mathfrak{m}_{n+1}$. This implies that $\mathfrak{m}_i \subset \mathfrak{m}_{n+1}$ for some $1 \leq i \leq n$ (otherwise $\exists a_i \in \mathfrak{m}_i \setminus \mathfrak{m}_{n+1}$, $1 \leq i \leq n \implies \prod_{i=1}^n a_i \in \cap_{i=1}^n \mathfrak{m}_i \setminus \mathfrak{m}_{n+1}$ as \mathfrak{m}_{n+1} is prime). But $\mathfrak{m}_i \subset \mathfrak{m}_{n+1}$, implies $\mathfrak{m}_i = \mathfrak{m}_{n+1}$, a contradiction. \square

Lemma 3.18. *In an Artin ring the Nilradical is equal to the Jacobson radical and is nilpotent.*

Proof. The nilradical $J = \mathcal{N}(A)$ is equal to $\mathcal{R}(A)$ as every prime ideal is maximal. By the assumption, the chain $J \supset J^2 \supset \dots$ stabilizes, hence $J^n = J^{n+1}$ for some $n \geq 0$. Assume that $J^n \neq 0$ and let $I \subset A$ be the minimal ideal such that $I \cdot J^n \neq 0$. It exists by our assumption on decreasing chains. For any $x \in I$ with $xJ^n \neq 0$, we have $I = Ax$ by minimality of I , hence I is finitely-generated. Moreover, $JI \cdot J^n = IJ^{n+1} = IJ^n \neq 0$, hence $JI = I$ by minimality of I . By Nakayama's lemma, we conclude that $I = 0$, a contradiction to $IJ^n \neq 0$. \square

Lemma 3.19. *Let $(\mathfrak{m}_1, \dots, \mathfrak{m}_n)$ be a sequence of maximal ideals in A such that $\prod_i \mathfrak{m}_i = 0$. Then A is Artin $\iff A$ is Noetherian.*

Proof. Assume that A is Artin. Consider a chain of ideals

$$A = I_0 \supset I_1 \supset \dots \supset I_n = 0,$$

where $I_i = \mathfrak{m}_1 \dots \mathfrak{m}_i$ for $0 \leq i \leq n$. As A is Artin, we conclude that I_i and I_{i-1}/I_i are Artin over A . Each factor $I_{i-1}/I_i = I_{i-1}/\mathfrak{m}_i I_{i-1}$ is a vector space over a field A/\mathfrak{m}_i . It is finite-dimensional over A/\mathfrak{m}_i as it is Artin over A and over A/\mathfrak{m}_i . But this implies that I_{i-1}/I_i is Noetherian over A/\mathfrak{m}_i , hence also over A . Assuming that we proved that I_i is Noetherian (it is automatic for $I_n = 0$), we consider an exact sequence $0 \rightarrow I_i \rightarrow I_{i-1} \rightarrow I_{i-1}/I_i \rightarrow 0$ with Noetherian modules on the sides and conclude that I_{i-1} is Noetherian. Continuing this process, we prove that A is Noetherian.

Assuming that A is Noetherian, we go through the same lines to show that A is Artin. \square

Theorem 3.20. *A ring A is Artin $\iff A$ is Noetherian and every prime ideal of A is maximal.*

Proof. Assume that A is Artin and let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be all of its maximal ideals. Then $\mathcal{R}(A) = \cap_i \mathfrak{m}_i$ and $\mathcal{R}(A)^k = 0$ for some $k \geq 0$. Therefore $\prod_i \mathfrak{m}_i^k = 0$ and we can apply the previous lemma.

Assume that A is Noetherian and its every prime ideal is maximal. Then every prime ideal is automatically a minimal prime ideal over 0, hence there are finitely many prime ideals by Lemma 3.15, say $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. The nil-radical $\mathcal{N}(A) = \cap_i \mathfrak{m}_i$ is finitely generated, hence $\mathcal{N}(A)^k = 0$ for some $k > 0$. Indeed, let $\mathcal{N}(A) = (a_1, \dots, a_l)$ with $a_i^{k_i} = 0$ for some $k_i > 0$ and let $k = \sum k_i$. For any element $\sum_i a_i b_i \in \mathcal{N}(A)$, every summand of $(\sum_i a_i b_i)^k$ is of the form $\prod_i (a_i b_i)^{t_i}$ with at least one $t_i \geq k_i$ (otherwise $k = \sum t_i < \sum k_i = k$). Therefore $\prod_i (a_i b_i)^{t_i} = 0$ and $(\sum_i a_i b_i)^k = 0$, hence $\mathcal{N}(A)^k = 0$. This implies that $\prod_i \mathfrak{m}_i^k = 0$ and we can apply the previous lemma. \square

Lemma 3.21. *Let (A, \mathfrak{m}) be a Noetherian local ring. Then A is Artin $\iff \mathfrak{m}$ is nilpotent.*

Proof. If A is Artin, then $\mathfrak{m} = \mathcal{R}(A)$ is nilpotent. Conversely, if $\mathfrak{m}^n = 0$ for some $n \geq 1$, then we can apply the previous Lemma and conclude that A is Artin. \square

Theorem 3.22. *An Artin ring is a finite product of Artin local rings.*

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be all maximal ideals of A . Then $\mathcal{R}(A) = \cap_i \mathfrak{m}_i$ and $\mathcal{R}(A)^k = 0$ for some $k \geq 1$. Therefore $\prod_i \mathfrak{m}_i^k = 0$. For any $i \neq j$, we have $\mathfrak{m}_i, \mathfrak{m}_j \subset \sqrt{\mathfrak{m}_i^k + \mathfrak{m}_j^k}$, hence $A = \mathfrak{m}_i + \mathfrak{m}_j \subset \sqrt{\mathfrak{m}_i^k + \mathfrak{m}_j^k}$. Therefore $\mathfrak{m}_i^k + \mathfrak{m}_j^k = A$ and these ideals are coprime. By the Chinese remainder theorem, there is an isomorphism $A / \cap_i \mathfrak{m}_i^k \rightarrow \prod_i A / \mathfrak{m}_i^k$, where $\cap_i \mathfrak{m}_i^k = \prod_i \mathfrak{m}_i^k = 0$. The rings A / \mathfrak{m}_i^k are Artin. They are also local, as the maximal ideal $\bar{\mathfrak{m}}_i = \mathfrak{m}_i / \mathfrak{m}_i^k$ is nilpotent (see the proof of the previous lemma). \square

Remark 3.23. Let A be an Artin, finitely generated algebra over a field \mathbb{k} . We will show that A is finite-dimensional over \mathbb{k} . Note that conversely, if A is a finite-dimensional algebra over \mathbb{k} , then A is obviously Artin. By the previous theorem we can assume that A is local, with a maximal ideal \mathfrak{m} . Then $\mathfrak{m}^n = 0$ for some $n > 0$. Every quotient $\mathfrak{m}^i / \mathfrak{m}^{i+1}$ is finitely generated over A and over the residue field A / \mathfrak{m} . Therefore $\mathfrak{m}^i / \mathfrak{m}^{i+1}$ is finite-dimensional over A / \mathfrak{m} . On the other hand A / \mathfrak{m} is finite-dimensional over \mathbb{k} (Hilbert Nullstellensatz), hence $\mathfrak{m}^i / \mathfrak{m}^{i+1}$ is finite-dimensional over \mathbb{k} . This implies that A is also finite-dimensional over \mathbb{k} .

4. ALGEBRA AND GEOMETRY

We start with a topological example that should serve us as a motivation.

Example 4.1. Let X be a compact Hausdorff topological space and let $A = C(X)$ be the algebra of continuous functions $f: X \rightarrow \mathbb{R}$. One can identify X with the set $\text{Max } A$ of maximal ideals in A

$$\phi: X \rightarrow \text{Max } A, \quad x \mapsto \mathfrak{m}_x = \{f \in A \mid f(x) = 0\}.$$

The ideal \mathfrak{m}_x is maximal as it is the kernel of the (surjective) evaluation map $e_x: A \rightarrow \mathbb{R}$, $f \mapsto f(x)$. Note that $f(x) = 0 \iff f \in \mathfrak{m}_x$. The map ϕ is injective as by Urysohn's lemma, for any $x \neq y$ in X , there exists $f \in A$ with $f(x) = 0$ and $f(y) = 1$, hence $f \in \mathfrak{m}_x \setminus \mathfrak{m}_y$ and $\mathfrak{m}_x \neq \mathfrak{m}_y$. To see that ϕ is bijective, let $I \in \text{Max } A$ and assume that $I \neq \mathfrak{m}_x$ for all $x \in X$. For every $x \in X$, let us choose $f_x \in I \setminus \mathfrak{m}_x$. Then $f_x(x) \neq 0$, hence $x \in U_x = \{y \in X \mid f_x(y) \neq 0\}$. This implies that $X = \bigcup_x U_x$ is an open cover and we can find a finite subcover $X = U_{x_1} \cup \dots \cup U_{x_n}$. The function $f = \sum f_{x_i}^2 \in I$ is nowhere zero on X , hence is invertible. Therefore $I = A$, a contradiction.

Finally, the map $\phi: X \rightarrow \text{Max } A$ is a homeomorphism if $\text{Max } A$ is equipped with the Zariski topology, where the closed sets are of the form $Z(I) = \{\mathfrak{m} \in \text{Max } A \mid \mathfrak{m} \supset I\}$ for all $I \subset A$. The map $\phi: X \rightarrow \text{Max } A$ is continuous as

$$\phi^{-1}(Z(I)) = \{x \in X \mid f(x) = 0 \forall f \in I\} = \bigcap_{f \in I} \{x \in X \mid f(x) = 0\}$$

is closed. To see that ϕ is a homeomorphism it is enough to show that $\text{Max } A$ is Hausdorff (as X is compact). Consider two ideals $\mathfrak{m}_x, \mathfrak{m}_y$ with $x \neq y$. There exist open subsets $x \in U \subset X$, $y \in V \subset X$ with $U \cap V = \emptyset$. By Urysohn's lemma $\exists f, g \in A$ such that $f(x) = 1$, $f|_{X \setminus U} = 0$ and $g(y) = 1$, $g|_{X \setminus V} = 0$. Then fg is zero on X . We have $\mathfrak{m}_x \in U' = \text{Max } A \setminus Z(f)$, $\mathfrak{m}_y \in V' = \text{Max } A \setminus Z(g)$ and $U' \cap V' = \text{Max } A \setminus (Z(f) \cup Z(g)) = \text{Max } A \setminus Z(fg) = \emptyset$.

The moral of the above example is that instead of a topological space X we can consider the algebra of functions on X and interpret the points of X as the maximal ideals of this algebra. Next, we will substitute X with an algebraic set and substitute $C(X)$ with an algebra of polynomial functions.

Definition 4.2. Let \mathbb{k} be a field and $A = \mathbb{k}[x_1, \dots, x_n]$. Given a set of polynomials $I \subset A$, we define the corresponding **algebraic subset** of \mathbb{k}^n

$$(2) \quad Z(I) = \{(a_1, \dots, a_n) \in \mathbb{k}^n \mid f(a_1, \dots, a_n) = 0 \forall f \in I\}.$$

Note that I may be replaced by the ideal it generates without changing $Z(I)$. As A is a Noetherian ring by the Hilbert's basis theorem (see Cor. 3.13), every ideal I has a finite set of generators, hence $I = (f_1, \dots, f_r)$ for some $f_i \in A$. If $I = (f_1, \dots, f_r)$, we denote $Z(I)$ by $Z(f_1, \dots, f_r)$.

Example 4.3.

- (1) Let $f(x, y) = x^2 + y^2 - 1 \in \mathbb{R}[x, y]$. Then

$$Z(f) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

is the radius one circle in \mathbb{R}^2 .

- (2) Let $f(x, y) = x^n + y^n - 1 \in \mathbb{C}[x, y]$ for $n \geq 3$. Then

$$Z(f) = \{(x, y) \in \mathbb{C}^2 \mid x^n + y^n = 1\}$$

is called the Fermat curve. Fermat's last theorem asserts that the Fermat curve has no nontrivial rational points (that is, $(x, y) \in \mathbb{Q}^2$ with $xy \neq 0$).

- (3) Given $a \in \mathbb{k}^n$, define $\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n) \subset A = \mathbb{k}[x_1, \dots, x_n]$. Then $A/\mathfrak{m}_a \simeq \mathbb{k}$, hence \mathfrak{m}_a is a maximal ideal. We have

$$Z(\mathfrak{m}_a) = \{b \in \mathbb{k}^n \mid b_i - a_i = 0 \forall i\} = \{a\}.$$

Lemma 4.4. Let $A = \mathbb{k}[x_1, \dots, x_n]$. Then

- (1) $Z(0) = \mathbb{k}^n$, $Z(A) = \emptyset$.
- (2) $Z(I) \cup Z(J) = Z(I \cap J)$ for arbitrary ideals $I, J \subset A$.
- (3) $\cap_i Z(I_i) = Z(\sum_i I_i)$ for arbitrary ideals $I_i \subset A$.
- (4) $I \subset J \implies Z(I) \supset Z(J)$.

Definition 4.5. Define the **Zariski topology** on \mathbb{k}^n with closed sets of the form $Z(I)$ for all $I \subset \mathbb{k}[x_1, \dots, x_n]$. It restricts to a topology on every algebraic subset $X \subset \mathbb{k}^n$.

Example 4.6. Consider the Zarisky topology on $\mathbb{k} = \mathbb{k}^1$, where \mathbb{k} is an algebraically closed field. Every ideal $I \subset \mathbb{k}[x]$ is principal, hence is of the form $I = (f)$ for some polynomial $f = c(x - a_1) \dots (x - a_k) \in \mathbb{k}[x]$. If $c = 0$, then $I = 0$ and $Z(I) = \mathbb{k}$. If $c \neq 0$, then $Z(I) = Z(f) = \{a_1, \dots, a_k\} \subset \mathbb{k}$ is a finite set. Hence all algebraic sets in \mathbb{k} are finite subsets of \mathbb{k} and the whole space \mathbb{k} . Therefore the open sets in the Zariski topology on \mathbb{k} are the complements of finite subsets and the empty set.

Definition 4.7. Let $X \subset \mathbb{k}^n$ be a subset.

- (1) Define the ideal of X

$$I(X) = \{f \in A \mid f(a_1, \dots, a_n) = 0 \forall (a_1, \dots, a_n) \in X\} \subset A.$$

- (2) Every polynomial $f \in \mathbb{k}[x_1, \dots, x_n]$ defines a function

$$f: \mathbb{k}^n \rightarrow \mathbb{k}, \quad (a_1, \dots, a_n) \mapsto f(a_1, \dots, a_n),$$

called a **polynomial function**. Its restriction $f: X \rightarrow \mathbb{k}$ is called a polynomial function on X .

- (3) Two polynomial functions f, g agree on X (that is, $f(x) = g(x)$ for all $x \in X$) if and only if $f - g \in I(X)$. Therefore the ring of different polynomial functions on X can be identified with

$$\mathbb{k}[X] = \mathbb{k}[x_1, \dots, x_n]/I(X)$$

called the **coordinate ring** of X .

Example 4.8. Let $a = (a_1, \dots, a_n) \in \mathbb{k}^n$. Then $\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n) \subset I(a)$ as $x_i - a_i$ vanishes at a . Note that \mathfrak{m}_a is maximal and $I(a) \neq (1)$ as $1 \notin I(a)$. Therefore $I(a) = \mathfrak{m}_a$, i.e.

- (3) $f(a) = 0 \iff f \in \mathfrak{m}_a, \quad f \in \mathbb{k}[x_1, \dots, x_n].$

We will see later that if \mathbb{k} is algebraically closed, then every maximal ideal in $\mathbb{k}[x_1, \dots, x_n]$ is of the form \mathfrak{m}_a for some $a \in \mathbb{k}^n$, hence there is a bijection between points in \mathbb{k}^n and maximal ideals in $\mathbb{k}[x_1, \dots, x_n]$.

Lemma 4.9.

- (1) $X \subset Y \implies I(X) \supset I(Y)$.
- (2) $I(\emptyset) = \mathbb{k}[x_1, \dots, x_n]$ and $I(\mathbb{k}^n) = 0$ (if \mathbb{k} is an infinite field).
- (3) $I(\cup X_i) = \cap I(X_i)$.
- (4) $I(X)$ is a radical ideal: $I(X) = \sqrt{I(X)}$.

Proof. (2) To prove that $I(\mathbb{k}^n) = 0$, we need to show that if $f \in \mathbb{k}[x_1, \dots, x_n]$ is nonzero, then $f(a) \neq 0$ for some $a \in \mathbb{k}^n$. If $n = 1$, then f can have only a finite number of roots and we are done as \mathbb{k} is infinite. For $n > 1$, consider f as a polynomial in one variable x_n over $\mathbb{k}[x_1, \dots, x_{n-1}]$

$$f = \sum_{i \geq 0} f_i x_n^i, \quad f_i \in \mathbb{k}[x_1, \dots, x_{n-1}]$$

If $f_i \neq 0$, then by induction there exists $(a_1, \dots, a_{n-1}) \in \mathbb{k}^{n-1}$ such that $f_i(a_1, \dots, a_{n-1}) \neq 0$. Then the polynomial

$$f(a_1, \dots, a_{n-1}, x_n) = \sum_{i \geq 0} f_i(a_1, \dots, a_{n-1}) x_n^i$$

is nonzero and can have only a finite number of roots. Hence we are done as \mathbb{k} is infinite.

- (4) Assume that $f \in \sqrt{I(X)}$, hence $f^k \in I(X)$ for some $k > 0$. Then $f^k(a) = 0$ for all $a \in X \implies f(a) = 0$ for all $a \in X \implies f \in I(X)$. \square

Remark 4.10. Let $\mathbb{k} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ and $f(x) = x^2 + x$. Then $f(0) = f(1) = 0 \implies f \in I(\mathbb{k}) \implies I(\mathbb{k}) \neq 0$. This is an illustration of the fact that $I(\mathbb{k}^n) \neq 0$ for finite fields \mathbb{k} and $n \geq 1$.

Lemma 4.11.

- (1) For any algebraic subset $X \subset \mathbb{k}^n$, we have $Z(I(X)) = X$.
- (2) For any ideal $I \subset \mathbb{k}[x_1, \dots, x_n]$, we have $I(Z(I)) \supset \sqrt{I}$.

Proof. (1) We always have $X \subset ZI(X)$. By assumption $X = Z(J)$ for some ideal $J \implies J \subset IZ(J) = I(X) \implies ZI(X) \subset Z(J) = X$. We conclude that $ZI(X) = X$.

(2) We know that $I \subset I(Z(I))$ and $I(Z(I))$ is a radical ideal, hence $\sqrt{I} \subset I(Z(I))$. \square

The following result is called the **Hilbert's Nullstellensatz** (zero-points-theorem in german) and will be proved later.

Theorem 4.12 (Hilbert's Nullstellensatz). *If \mathbb{k} is algebraically closed, then for any ideal $I \subset \mathbb{k}[x_1, \dots, x_n]$, we have $I(Z(I)) = \sqrt{I}$.*

Corollary 4.13. *If \mathbb{k} is algebraically closed, then there is a bijection between algebraic sets in \mathbb{k}^n and radical ideals in $\mathbb{k}[x_1, \dots, x_n]$ given by $X \mapsto I(X)$ and $I \mapsto Z(I)$.*

Let us formulate several equivalent forms of the Hilbert's Nullstellensatz (we will prove later the third statement for algebraically closed fields).

Theorem 4.14. *Given an (algebraically closed) field \mathbb{k} , the following are equivalent*

- (1) If $I \subset \mathbb{k}[x_1, \dots, x_n]$ is an ideal, then $I(Z(I)) = \sqrt{I}$.
- (2) If $I \subset \mathbb{k}[x_1, \dots, x_n]$ is an ideal and $Z(I) = \emptyset$, then $I = (1)$ (weak Nullstellensatz).
- (3) Every maximal ideal in $\mathbb{k}[x_1, \dots, x_n]$ is of the form $\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n)$ for some $a \in \mathbb{k}^n$.

Proof. Let $A = \mathbb{k}[x_1, \dots, x_n]$.

(1) \implies (2). Assume that $Z(I) = \emptyset$. Then $\sqrt{I} = I(Z(I)) = I(\emptyset) = A \implies 1 \in \sqrt{I} \implies 1^k \in I$ for some $k > 0 \implies I = A$.

(2) \implies (3). Let $I \subset A$ be a maximal ideal. Then $I \neq A \implies Z(I) \neq \emptyset$ and we can choose $a \in Z(I)$. Then $I \subset I(a) = (x_1 - a_1, \dots, x_n - a_n) \implies I = (x_1 - a_1, \dots, x_n - a_n)$ as I is maximal.

(3) \implies (2). If $I \neq A$, then there exists a maximal ideal $I \subset \mathfrak{m}$. By (3) we have $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$ for some $a \in \mathbb{k}^n$. Therefore $a \in Z(\mathfrak{m}) \subset Z(I)$ and $Z(I) \neq \emptyset$.

(2) \implies (1) (Rabinowitsch trick). We know that $\sqrt{I} \subset I(Z(I))$. Conversely, assume that $f \in I(Z(I))$. Consider the ideal

$$J = \langle I, ft - 1 \rangle \subset \mathbb{k}[x_1, \dots, x_n, t].$$

If $(a_1, \dots, a_n, c) \in Z(J)$, then $(a_1, \dots, a_n) \in Z(I) \implies f(a_1, \dots, a_n) = 0 \implies ft - 1$ does not vanish at this point. Therefore $Z(J) = \emptyset$. By (2) we have $J = (1)$ and we can write

$$1 = (ft - 1)g_0 + \sum_i f_i g_i$$

for some $g_i \in \mathbb{k}[x_1, \dots, x_n, t]$ and $f_i \in I$. After substitution $t = 1/f$, we get

$$1 = \sum_i f_i g_i(x_1, \dots, x_n, 1/f)$$

and after multiplication with a sufficiently high power of f , we get $f^N = \sum_i f_i h_i \in I$ for some $N \gg 0$ and $h_i = f^N g_i(x_1, \dots, x_n, 1/f) \in \mathbb{k}[x_1, \dots, x_n]$. Thus $f^N \in I \implies f \in \sqrt{I}$. \square

Corollary 4.15. *Let \mathbb{k} be an algebraically closed field and $X \subset \mathbb{k}^n$ be an algebraic set. Then*

- (1) *There is a bijection between \mathbb{k}^n and the set of maximal ideals of $\mathbb{k}[x_1, \dots, x_n]$.*
- (2) *There is a bijection between X and the set of maximal ideals of $\mathbb{k}[X] = \mathbb{k}[x_1, \dots, x_n]/I(X)$.*

Proof. (1) Follows from the previous theorem and Hilbert's Nullstellensatz. (2) If $J = I(X)$, then $X = ZI(X) = Z(J)$. We have $a \in X \iff f(a) = 0 \forall f \in J \iff f \in \mathfrak{m}_a \forall f \in J \iff J \subset \mathfrak{m}_a$. Therefore $a \in X$ corresponds to the maximal ideal $\mathfrak{m}_a/J \subset \mathbb{k}[x_1, \dots, x_n]/J$. \square

5. INTEGRAL DEPENDENCE

Definition 5.1. Let B be a ring and $A \subset B$ be a subring.

- (1) An element $x \in B$ is called **integral** over A if it is a root of a monic polynomial with coefficients in A , that is, there exist $a_0, \dots, a_{n-1} \in A$ such that

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0.$$

- (2) The ring B is said to be **integral** over A if all elements of B are integral over A .
 (3) The ring B is said to be **finite** over A if B is finitely generated as an A -module.
 (4) The ring B is said to be of **finite type** over A if B is a finitely generated A -algebra.

Example 5.2. The algebra of polynomials $\mathbb{k}[x]$ over a field \mathbb{k} is finitely-generated over \mathbb{k} as an algebra, but it is not finite as it is not finitely-generated over \mathbb{k} as a module. On the hand, the algebra $B = \mathbb{k}[x]/(x^n - 1)$ is finite over \mathbb{k} . It is generated (as a module) by the elements $1, x, \dots, x^{n-1}$. The element $x \in B$ is integral over \mathbb{k} as it is a root of the polynomial $x^n - 1$.

Remark 5.3. If $A \subset B$ are fields and B is integral over A , then B is called an algebraic field extension of A . If B is finite over A , then B is called a finite field extension of A .

Remark 5.4. Let $f: A \rightarrow B$ be a ring homomorphism, so that B is an A -algebra. Then f is called integral (and B is called an integral A -algebra) if B is integral over $f(A)$. Similarly, f is called finite (and B is called a finite A -algebra) if B is finite over $f(A)$.

Remark 5.5. If $A \subset B \subset C$ are finite inclusions, then C is finite over A . Indeed, let C have generators c_1, \dots, c_n over B and B have generators b_1, \dots, b_m over A . Then $(c_i b_j)$ are generators of C over A (as a module).

Exercise 5.6. Let $A \subset B$ be rings and B be integral over A .

- (1) If $J \subset B$ is an ideal, then B/J is integral over $A/(A \cap J)$.
 (2) If $S \subset A$ is a multiplicative set, then $S^{-1}B$ is integral over $S^{-1}A$.

Lemma 5.7. Let $A \subset B$ be rings and $x \in B$. The following are equivalent:

- (1) x is integral over A .
 (2) $A[x]$ is finite over A .
 (3) $A[x]$ is contained in a subring $C \subset B$, finite over A .
 (4) There exists a faithful $A[x]$ -module, finitely generated over A (M is faithful if $\text{Ann } M = 0$).

Proof. (1) \implies (2). We have

$$x^n = -(a_1 x^{n-1} + \dots + a_n)$$

for some $a_1, \dots, a_n \in A$. Hence

$$x^{n+k} = -(a_1 x^{n+k-1} + \dots + a_n x^k), \quad k \geq 0$$

and, by induction, we can express x^{n+k} as a linear combination of $1, x, \dots, x^{n-1}$ with coefficients in A . This means that $A[x]$ is generated by $1, x, \dots, x^{n-1}$ as an A -module.

(2) \implies (3). Take $C = A[x]$.

(3) \implies (4). Take $M = C$. It is faithful over $A[x]$, as $fC = 0 \implies f = f \cdot 1 = 0$ for any $f \in A[x]$. It is finite over A by assumption.

(4) \implies (1). Let u_1, \dots, u_n generate module M over A and let $xu_i = \sum_j b_{ij}u_j$ for some $b_{ij} \in A$. Then

$$(xI_n - B)u = 0, \quad u = (u_1, \dots, u_n)^t,$$

where $B = (b_{ij}) \in M_{n \times n}(A)$ and I_n is an identity matrix. Multiplying the last equation by the adjoint of the matrix $(xI_n - B)$, we obtain $\det(xI_n - B)u = 0$, where $\det(xI_n - B) \in A[x]$. Therefore $\det(xI_n - B)u_i = 0$ for all $i \implies \det(xI_n - B)M = 0 \implies \det(xI_n - B) = 0$ as M is a faithful module. But

$$\det(xI_n - B) = x^n + c_1 x^{n-1} + \dots + c_n$$

for some $c_i \in A$. Therefore x is integral over A . □

Corollary 5.8. Let $x_1, \dots, x_n \in B$ be integral over A . Then $A[x_1, \dots, x_n] \subset B$ is finite over A .

Proof. By induction, the ring $A' = A[x_1, \dots, x_{n-1}]$ is finite over A . The element x_n is integral over A' , hence $A[x_1, \dots, x_n] = A'[x_n]$ is finite over A' by 5.7.2. This implies that $A[x_1, \dots, x_n]$ is finite over A by Ref. 5.5. \square

Corollary 5.9. *An algebra B is finite over $A \iff$ it is finitely-generated and integral over A .*

Proof. If B is finite, then it automatically finitely-generated. For every $x \in B$, $A[x] \subset B$ and B is finite over A , hence x is integral over A .

On the other hand, let B be integral over A and let $x_1, \dots, x_n \in B$ be generators of B over A . Then $B = A[x_1, \dots, x_n]$ is finite over A by the previous result. \square

Lemma 5.10. *Let $A \subset B \subset C$ be rings such that B is integral over A and C is integral over B . Then C is integral over A .*

Proof. Given $x \in C$, there exist $b_1, \dots, b_n \in B$ such that

$$x^n + b_1 x^{n-1} + \dots + b_n = 0.$$

Then $B' = A[b_1, \dots, b_n]$ is finite over A by Cor. 5.8 and $B'[x]$ is finite over B' . This implies that $B'[x]$ is finite over A , hence x is integral over A . \square

Lemma 5.11. *The set C of elements in B that are integral over A is a subring of B containing A .*

Proof. Let $x, y \in C$ be integral over A . Then $A[x, y]$ is finite over A by Cor. 5.8. Therefore $x \pm y$ and xy are integral over A by 5.7.3. \square

Definition 5.12. Let $A \subset B$ be rings.

- (1) The ring \bar{A} of all elements in B integral over A is called the **integral closure** of A in B .
- (2) $A \subset B$ is called **integrally closed** in B if the integral closure of A coincides with A .

Example 5.13. Let $A = \mathbb{Z}$ and $B = \mathbb{Q}$. If $x = m/n \in B$ (with coprime m, n) is integral over \mathbb{Z} , then $x^r + a_{r-1}x^{r-1} + \dots + a_0 = 0$ for some $a_i \in \mathbb{Z}$, hence $m^r + a_{r-1}m^{r-1}n + \dots + a_0n^r = 0$. This implies $n \mid m^r$. As m, n are coprime, we conclude that $n = \pm 1$, hence $x \in \mathbb{Z}$. This implies that \mathbb{Z} is integrally closed in \mathbb{Q} .

Remark 5.14. A ring A is called **integrally closed** (without a reference to any larger ring) if A is integrally closed in the ring of fractions $Q(A) = S^{-1}A$, where $S \subset A$ is the set of non-zero-divisors of A . The above example can be generalized to show that every UFD is integrally closed.

Lemma 5.15. *Let $A \subset B$ be rings and let \bar{A} be the integral closure of A in B . Then \bar{A} is integrally closed in B .*

Proof. Let $x \in B$ be integral over \bar{A} . Then $A \subset \bar{A} \subset \bar{A}[x]$ are integral inclusions, hence $\bar{A}[x]$ is integral over A . In particular x is integral over A , hence $x \in \bar{A}$. \square

Lemma 5.16. *Let $A \subset B$ be rings and let \bar{A} be the integral closure of A in B . If $S \subset A$ is a multiplicative set, then $S^{-1}\bar{A}$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Proof. By Ex. 5.6, $S^{-1}\bar{A}$ is integral over $S^{-1}A$. Assume that $b/s \in S^{-1}B$ is integral over $S^{-1}A$. Then we can write

$$(b/s)^n + \sum_{i=0}^{n-1} \frac{a_i}{s_i} (b/s)^i = 0$$

for some $a_i/s_i \in S^{-1}A$. Multiplying this equation with $(st)^n$, where $t = s_0 \dots s_{n-1}$, we obtain integral dependence of bt over A . This implies $bt \in \bar{A}$, hence $b/s = bt/st \in S^{-1}\bar{A}$. \square

5.1. Going-up theorem.

Proposition 5.17. *Let $A \subset B$ be integral domains such that B is integral over A . Then B is a field if and only if A is a field.*

Proof. Let A be a field. Every $0 \neq x \in B$ is integral over A , hence

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

for some $a_i \in A$. Assume that n is minimal possible. Then $a_0 \neq 0$ as otherwise $x^{n-1} + \cdots + a_1 = 0$ (B is an integral domain) and n would be not minimal possible. This implies

$$x^{-1} = -a_0^{-1}(x^{n-1} + \cdots + a_1) \in B.$$

Hence B is a field.

Assume that B is a field. Given $0 \neq x \in A$, the element $x^{-1} \in B$ is integral over A , hence

$$x^{-n} + a_{n-1}x^{-n+1} + \cdots + a_0 = 0$$

for some $a_i \in A$. Therefore

$$x^{-1} = -(a_{n-1} + \cdots + a_0x^{n-1}) \in A.$$

Hence A is a field. □

Lemma 5.18. *Let $f: A \rightarrow B$ be an integral ring homomorphism and $\mathfrak{q} \subset B$ be a prime ideal. Then \mathfrak{q} is maximal $\iff \mathfrak{p} = f^{-1}(\mathfrak{q})$ is maximal.*

Proof. The rings $A/\mathfrak{p} \subset B/\mathfrak{q}$ are integral domains and B/\mathfrak{q} is integral over A/\mathfrak{p} . By the previous result B/\mathfrak{q} is a field $\iff A/\mathfrak{p}$ is a field. □

Corollary 5.19. *Let $f: A \rightarrow B$ be an integral ring homomorphism and $\mathfrak{q} \subset \mathfrak{q}' \subset B$ be prime ideals such that $f^{-1}(\mathfrak{q}) = f^{-1}(\mathfrak{q}')$. Then $\mathfrak{q} = \mathfrak{q}'$.*

Proof. Taking $\mathfrak{p} = f^{-1}(\mathfrak{q})$ and localizations $A_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$, we can assume that \mathfrak{p} is maximal. But then \mathfrak{q} and \mathfrak{q}' are maximal by the previous result, hence $\mathfrak{q} = \mathfrak{q}'$. □

Theorem 5.20. *Let $f: A \rightarrow B$ be an integral ring homomorphism. Then, for every prime ideal $\mathfrak{p} \subset A$, there exists a prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Equivalently, the following map is surjective*

$$f^*: \text{Spec } B \rightarrow \text{Spec } A, \quad \mathfrak{q} \mapsto f^{-1}(\mathfrak{q}).$$

Proof. Consider $S = A \setminus \mathfrak{p}$, $A_{\mathfrak{p}} = S^{-1}A$, $B_{\mathfrak{p}} = S^{-1}B$ and a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ i \downarrow & & \downarrow j \\ A_{\mathfrak{p}} & \xrightarrow{f_{\mathfrak{p}}} & B_{\mathfrak{p}} \end{array}$$

The induced map $f_{\mathfrak{p}}: A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$ is integral. If $\mathfrak{n} \subset B_{\mathfrak{p}}$ is maximal, then $\mathfrak{m} = f_{\mathfrak{p}}^{-1}(\mathfrak{n}) \subset A_{\mathfrak{p}}$ is also maximal by Lemma 5.18. Hence $\mathfrak{m} = \mathfrak{p}_{\mathfrak{p}}$, the unique maximal ideal of $A_{\mathfrak{p}}$. We have $\mathfrak{p} = i^{-1}(\mathfrak{p}_{\mathfrak{p}}) = i^{-1}(f_{\mathfrak{p}}^{-1}(\mathfrak{n})) = f^{-1}(j^{-1}(\mathfrak{n}))$ and define $\mathfrak{q} = j^{-1}(\mathfrak{n}) \subset B$. □

Corollary 5.21 (Going-up). *If $f: A \rightarrow B$ is integral, then for any chain of prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n \subset A$, there exists a chain of prime ideals $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_n \subset B$ with $\mathfrak{p}_i = f^{-1}(\mathfrak{q}_i)$.*

Exercise 5.22. If $f: A \rightarrow B$ is finite, then for any $\mathfrak{p} \in \text{Spec } A$, the set $\{\mathfrak{q} \in \text{Spec } B \mid f^{-1}(\mathfrak{q}) = \mathfrak{p}\}$ is finite.

Hint: Substitute A by A/\mathfrak{p} and B by $B/\mathfrak{p}B$, then invert nonzero elements of A .

5.2. Proof of the Nullstellensatz.

Theorem 5.23 (Noether's normalization theorem). *Let \mathbb{k} be a field and B be a finitely-generated \mathbb{k} -algebra. Then there exists a subalgebra $A \subset B$ isomorphic to a polynomial algebra $\mathbb{k}[y_1, \dots, y_r]$ and such that B is finite over A .*

Proof. We will assume that \mathbb{k} is infinite. Let x_1, \dots, x_n be generators of B over \mathbb{k} . If they are algebraically independent, then we can take $A = B$. Otherwise, there exists a polynomial $f \neq 0$ over \mathbb{k} in n variables such that $f(x_1, \dots, x_n) = 0$. Let $f = \sum_{k=0}^N f_k$ be the sum of its homogeneous parts with $f_N \neq 0$. Then $f_N(x_1, \dots, x_{n-1}, 1) \neq 0$, hence there exist $a_1, \dots, a_{n-1} \in \mathbb{k}$ such that $f_N(a_1, \dots, a_{n-1}, 1) \neq 0$ (as \mathbb{k} is infinite, see Lemma 4.9). Using the substitution $x'_i = x_i - a_i x_n$ for $1 \leq i \leq n-1$, we obtain

$$\begin{aligned} 0 = f(x_1, \dots, x_n) &= \sum_{k=0}^N f_k(x'_1 + a_1 x_n, \dots, x'_{n-1} + a_{n-1} x_n, x_n) \\ &= f_N(a_1, \dots, a_{n-1}, 1) x_n^N + \sum_{k=0}^{N-1} g_k(x'_1, \dots, x'_{n-1}) x_n^k \end{aligned}$$

for some polynomials g_k in $n-1$ variables. Dividing this equation by $f_N(a_1, \dots, a_{n-1}, 1)$ we can see that x_n is integral over $A' = \mathbb{k}[x'_1, \dots, x'_{n-1}] \subset B$, hence $B = A'[x_n]$ is finite over A' . By induction on n , there exists a polynomial subalgebra $A = \mathbb{k}[y_1, \dots, y_r] \subset A'$ such that A' is finite over A . But then B is also finite over A . \square

Theorem 5.24 (General Nullstellensatz). *Let \mathbb{k} be a field. Then*

- (1) *If A is a finitely-generated \mathbb{k} -algebra and is a field, then A is a finite field extension of \mathbb{k} .*
- (2) *If $\mathfrak{m} \subset \mathbb{k}[x_1, \dots, x_n]$ is a maximal ideal, then $\mathbb{k}[x_1, \dots, x_n]/\mathfrak{m}$ is a finite field extension of \mathbb{k} .*

Proof. (1) By the Noether normalization theorem, there exists a polynomial subalgebra $B = \mathbb{k}[y_1, \dots, y_r] \subset A$ such that A is integral over B . Then B is a field by Prop. 5.17, hence $B = \mathbb{k}$. This implies that A is integral over \mathbb{k} . As it is also finitely-generated over \mathbb{k} and all generators are algebraic (integral) over \mathbb{k} , we obtain that A is a finite field extension of \mathbb{k} .

(2) The algebra $A = \mathbb{k}[x_1, \dots, x_n]/\mathfrak{m}$ is finitely generated and is a field. By (1) it is a finite field extension of \mathbb{k} . \square

Theorem 5.25 (Nullstellensatz). *Let \mathbb{k} be an algebraically closed field. Then every maximal ideal in $\mathbb{k}[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in \mathbb{k}$.*

Proof. By the previous theorem, if $\mathfrak{m} \subset \mathbb{k}[x_1, \dots, x_n]$ is a maximal ideal, then the field $L = \mathbb{k}[x_1, \dots, x_n]/\mathfrak{m}$ is a finite field extension of \mathbb{k} . Hence every element $a \in L$ is algebraic over \mathbb{k} . If \mathbb{k} is algebraically closed, then the minimal polynomial of a has degree one and $a \in \mathbb{k}$. We conclude that $L = \mathbb{k}$. Consider the projection map

$$\pi: \mathbb{k}[x_1, \dots, x_n] \rightarrow \mathbb{k}[x_1, \dots, x_n]/\mathfrak{m} = L = \mathbb{k}$$

and let $a_i = \pi(x_i) \in \mathbb{k}$ for $1 \leq i \leq n$. Then $x_i - a_i \in \ker \pi = \mathfrak{m}$, hence $(x_1 - a_1, \dots, x_n - a_n) \subset \mathfrak{m}$. But $(x_1 - a_1, \dots, x_n - a_n)$ is a maximal ideal, hence $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$. \square

Theorem 5.26. *Let A be a finitely-generated \mathbb{k} -algebra. Then*

- (1) $N(A) = R(A)$.
- (2) *For any ideal $I \subset A$, the intersection of the maximal ideals that contain I is equal to \sqrt{I} .*
- (3) *For any prime ideal $\mathfrak{p} \subset A$, the intersection of the maximal ideals containing \mathfrak{p} is equal to \mathfrak{p} .*

Proof. (3) We can substitute A by A/\mathfrak{p} and assume that $\mathfrak{p} = 0$ and A is an integral domain. Then we need to show that $R(A) = 0$. Given $0 \neq f \in A$, consider the ring $A[f^{-1}]$, a maximal ideal $\mathfrak{m} \subset A[f^{-1}]$ and $\mathfrak{q} = \mathfrak{m} \cap A$. Then $A[f^{-1}]/\mathfrak{m}$ is a finite field extension of \mathbb{k} (as $A[f^{-1}]$ is a finitely-generated \mathbb{k} -algebra). Moreover, $\mathbb{k} \subset A/\mathfrak{q} \subset A[f^{-1}]/\mathfrak{m}$, hence A/\mathfrak{q} is a field. This implies that $\mathfrak{q} \subset A$ is a maximal ideal. We have $f \notin \mathfrak{q}$ as otherwise $f \in \mathfrak{q} \subset \mathfrak{m}$, hence $\mathfrak{m} = A[f^{-1}]$ would be not maximal. We conclude that $f \notin R(A)$, hence $R(A) = 0$.

(3) \implies (1). We know that $N(A)$ is the intersection of all prime ideals of A . Every maximal ideal is prime, hence $N(A) \subset R(A)$. By our assumption, every prime ideal is an intersection of maximal ideals, hence $N(A) = R(A)$.

(1) \implies (2). Let J be the intersection of all maximal ideals containing I . Then $J/I = R(A/I) = N(A/I) = \sqrt{I}/I$, hence $J = \sqrt{I}$.

□

6. DEDEKIND DOMAINS

6.1. Valuation rings.

Definition 6.1. Let A be an integral domain and K be its field of fractions. Then A is called a **valuation ring** of K if for every $0 \neq x \in K$, either $x \in A$ or $x^{-1} \in A$.

Exercise 6.2. Let A be a valuation ring with the field of fractions K . Show that

- (1) The set $\Gamma = K^\times / A^\times$ with the relation $x \geq y$ if $x/y \in A$ is a totally ordered set.
- (2) The map $v: K^\times \rightarrow \Gamma$, $x \mapsto [x]$, satisfies
 - (a) $v(xy) = v(x) + v(y)$ (using additive notation for the multiplication in Γ).
 - (b) $v(x + y) \geq \min \{v(x), v(y)\}$.

Lemma 6.3. If A is a valuation ring of a field K , then

- (1) A is a local ring.
- (2) A is integrally closed in K .

Proof. (1) Let $\mathfrak{m} \subset A$ be the set of non-invertible elements of A . Then \mathfrak{m} consists of $x \in K$ such that $x = 0$ or $x^{-1} \in A$. Let $a \in A$ and $x \in \mathfrak{m}$. If $ax \notin \mathfrak{m}$, then $(ax)^{-1} \in A$, hence $a(ax)^{-1} = x^{-1} \in A$, contradicting to $x \in \mathfrak{m}$. Let $0 \neq x, y \in \mathfrak{m}$. Then either xy^{-1} or $x^{-1}y$ is in A . Assuming the former, we get $x + y = (1 + xy^{-1})y \in A\mathfrak{m} \subset \mathfrak{m}$. Therefore \mathfrak{m} is an ideal. As all proper ideals consist of non-invertible elements, we deduce that all such ideals are contained in \mathfrak{m} .

(2) Assume that $x \in K \setminus A$ is integral over A . Then we have

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

for some $a_i \in A$. As $x \notin A$, we obtain $x^{-1} \in A$, hence $x = -(a_{n-1} + \cdots + a_0x^{1-n}) \in A$. A contradiction. \square

6.2. Discrete valuation rings.

Definition 6.4. A **discrete valuation** on a field K is a surjective map $v: K^* \rightarrow \mathbb{Z}$ such that

- (1) $v(xy) = v(x) + v(y)$ (that is, v is a group homomorphism).
- (2) $v(x + y) \geq \min \{v(x), v(y)\}$.

We define $v(0) = +\infty$ for convenience.

Remark 6.5. An **absolute value** on a field K is a map $|\cdot|: K \rightarrow \mathbb{R}$ such that

- (1) $|x| \geq 0$ and $|x| = 0 \iff x = 0$.
- (2) $|xy| = |x| \cdot |y|$.
- (3) $|x + y| \leq |x| + |y|$.

It is called non-archimedean if a stronger condition is satisfied:

- (3') $|x + y| \leq \max \{|x|, |y|\}$.

Given a discrete valuation v and a constant $0 < c < 1$, we can define a non-archimedean absolute value $|x| = c^{v(x)}$.

Lemma 6.6. Let v be a discrete valuation on a field K , then

$$A = \{x \in K \mid v(x) \geq 0\}$$

is a valuation ring (hence local), called the **discrete valuation ring (DVR)** of v . Its maximal ideal is

$$\mathfrak{m} = \{x \in K \mid v(x) \geq 1\}.$$

Proof. It is clear that A is a ring. We have $v(1) = 0$. If $0 \neq x \in K$ is not in A , then $v(x) < 0$, hence $v(x^{-1}) = v(1) - v(x) = -v(x) > 0$ and $x^{-1} \in A$. This implies that A is a valuation ring.

It is clear that \mathfrak{m} is an ideal. An element $x \in A$ is invertible $\iff v(x) \geq 0$ and $v(x^{-1}) \geq 0 \iff v(x) = 0$. This means that \mathfrak{m} consists of all non-invertible elements, hence is the unique maximal ideal. \square

Example 6.7. Let $K = \mathbb{Q}$ and p be prime. Every non-zero $x \in \mathbb{Q}$ can be written in the form $p^k \frac{m}{n}$, where m, n are coprime with p and $k \in \mathbb{Z}$. We define $v_p(x) = k$. The valuation ring of v_p consists of fractions $p^k \frac{m}{n}$ with $k \geq 0$. This is the local ring $Z_{(p)} = S^{-1}Z$, where $S = \mathbb{Z} \setminus (p)$.

Example 6.8. Let $K = \mathbb{k}((x))$ be the field of Laurent power series $f = \sum_{i=-N}^{\infty} a_i x^i$ over a field \mathbb{k} . We define valuation $v(i) = \min \{i \mid a_i \neq 0\}$. Its local ring is $\mathbb{k}[[x]]$, the ring of power series.

An element $t \in A$ with $v(t) = 1$ is called a **uniformizer**.

Lemma 6.9. *If $t \in A$ is a uniformizer, then every element $x \in K$ can be expressed uniquely in the form ut^k , where $u \in A$ is a unit and $k \in \mathbb{Z}$.*

Proof. Let $k = v(x)$ and $u = xt^{-k}$. Then $v(u) = 0$, hence $u \in A$ is invertible. Note that if $x = ut^k$, where u is invertible, then $v(x) = k$, hence k is uniquely determined. Therefore $u = xt^{-k}$ is also uniquely determined. \square

Lemma 6.10. *Every non-zero ideal $I \subset A$ is of the form $\mathfrak{m}^n = (t^n)$ for some $n \geq 0$.*

Proof. Let $n = \min \{v(a) \mid a \in I\}$ and let $a \in I$ satisfy $v(a) = n$. It is clear, that $(a) \subset I$. Conversely, if $b \in I$, then $v(ba^{-1}) = v(b) - v(a) \geq 0$, hence $ba^{-1} \in A$ and $b = (ba^{-1})a \in (a)$. Therefore $I = (a)$. We can write $a = ut^n$ for some invertible $u \in A$. Then $I = (a) = (t^n)$. In particular, $\mathfrak{m} = (t)$, hence $I = (t^n) = \mathfrak{m}^n$. \square

The above result implies that A is a Noetherian local domain of dimension one (every nonzero prime ideal is maximal).

Theorem 6.11. *Let (A, \mathfrak{m}) be a Noetherian local domain with the residue field $\mathbb{k} = A/\mathfrak{m}$. Then FAE*

- (1) A is a DVR
- (2) A is integrally closed and every nonzero prime ideal is maximal.
- (3) \mathfrak{m} is principal.
- (4) $\dim_{\mathbb{k}} \mathfrak{m}/\mathfrak{m}^2 = 1$.
- (5) Every nonzero ideal is a power of \mathfrak{m} .
- (6) There exists $t \in A$ such that every nonzero ideal is of the form (t^n) for some $n \geq 0$
- (7) A is a PID.

Proof. (1) \implies (2). A is a valuation ring, hence integrally closed by Lemma 6.3. If $\mathfrak{p} \subset A$ is a non-zero prime ideal, then $\mathfrak{p} = \mathfrak{m}^n$ for some $n \geq 0 \implies \mathfrak{m} \subset \mathfrak{p} \implies \mathfrak{p} = \mathfrak{m}$.

(2) \implies (3). Let $0 \neq a \in \mathfrak{m}$. Then $\sqrt{(a)}$ is an intersection of prime ideals, hence $\sqrt{(a)} = \mathfrak{m}$. This implies that $\mathfrak{m}^n \subset (a)$ for some $n \geq 1$ and we can assume that $\mathfrak{m}^{n-1} \not\subset (a)$. Let $b \in \mathfrak{m}^{n-1} \setminus (a)$ and $x = \frac{b}{a} \notin A$. If $x\mathfrak{m} \subset \mathfrak{m}$, then \mathfrak{m} is a faithful $A[x]$ -module, finitely generated over A . Therefore x is integral over A , hence $x \in A$, a contradiction. On the other hand $x\mathfrak{m} \subset \frac{\mathfrak{m}^{n-1}}{a}\mathfrak{m} \subset \frac{1}{a}(a) = A$. We conclude that $x\mathfrak{m} = A$, hence $\mathfrak{m} = x^{-1}A$ is principal.

(3) \implies (4). If \mathfrak{m} is principal, then $\dim \mathfrak{m}/\mathfrak{m}^2 \leq 1$. On the other hand, if $\mathfrak{m}^2 = \mathfrak{m}$, then $\mathfrak{m} = 0$ by Nakayama lemma. Therefore $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$.

(4) \implies (3). If $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$, then $Ax + \mathfrak{m}^2 = \mathfrak{m}$ for some $x \in \mathfrak{m}$. By Nakayama lemma, this implies that $Ax = \mathfrak{m}$, hence \mathfrak{m} is principal.

(3) \implies (5). Let $\mathfrak{m} = (t)$. We claim that $\cap_n \mathfrak{m}^n = 0$. If $a \in \cap_n \mathfrak{m}^n$, then we can write $a = b_n t^n$ for some $b_n \in A$. This implies $b_n t^n = b_{n+1} t^{n+1} \implies b_n = b_{n+1} t \implies (b_n) \subset (b_{n+1})$. This chain of ideals stabilizes, hence $b_{n+1} = ub_n$ for some invertible u . Therefore $b_n = utb_n$ and $b_n = 0$ as otherwise $ut = 1$ and $\mathfrak{m} = (t) = A$. We conclude that $a = 0$.

Let $0 \neq I \subset A$ be a proper ideal. Then $I \subset \mathfrak{m}$ and $I \not\subset \cap_n \mathfrak{m}^n$, hence there exists $n \geq 0$ such that $I \subset \mathfrak{m}^n$, but $I \not\subset \mathfrak{m}^{n+1}$. If $a \in I \setminus \mathfrak{m}^{n+1} \implies a \in \mathfrak{m}^n = (t^n) \implies a = ut^n$ for some $u \notin \mathfrak{m}$ (otherwise $a \in \mathfrak{m}^{n+1}$). This implies that u is invertible, hence $\mathfrak{m}^n = (t^n) = (a) \subset I \subset \mathfrak{m}^n$ and $I = \mathfrak{m}^n$.

(5) \implies (6). We have $\mathfrak{m} \neq \mathfrak{m}^2$ by Nakayama's lemma. Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then $(t) = \mathfrak{m}^n$ for some $n \geq 1$. If $n \geq 2$, then $(t) = \mathfrak{m}^n \subset \mathfrak{m}^2$, hence $t \in \mathfrak{m}^2$, a contradiction. Therefore $n = 1$ and $(t) = \mathfrak{m}$. By assumption, every nonzero ideal is of the form $\mathfrak{m}^n = (t^n)$ for some $n \geq 0$.

(6) \implies (1). We have $\mathfrak{m} = (t)$. If $(t^n) = (t^{n+1}) = \mathfrak{m}(t^n)$, then $(t^n) = 0$ by Nakayama lemma, a contradiction. For every $a \neq 0$, we have $(a) = (t^k)$ for exactly one $k \geq 0$. We define $v(a) = k$ and extend v to K^* by defining $v(a/b) = v(a) - v(b)$. One can check that v is a discrete valuation and A is its valuation ring.

(6) \implies (7). Obvious. (7) \implies (3). Obvious. \square

Example 6.12. Let $A = \mathbb{k}[x]_{(x)}$ (localization of the ring $\mathbb{k}[x]$ at the prime ideal $\mathfrak{p} = (x)$). This is a Noetherian local domain with the maximal ideal $\mathfrak{m} = (x)$ which is principal. Therefore A is a DVR. Its field of fractions is $\mathbb{k}(x)$, the field of rational functions over \mathbb{k} . The valuation is given by the formula $v(x^k f/g) = k$ for the polynomials $f, g \in \mathbb{k}[x]$ with non-trivial constant coefficient.

6.3. Dedekind domains.

Definition 6.13. Let A be an integral domain.

- (1) A is said to have dimension one if every nonzero prime ideal of A is maximal. In particular, DVR have dimension one.
- (2) A is called **integrally closed** if it is integrally closed in its field of fractions.
- (3) An integral domain A is called a **Dedekind domain** if A is Noetherian, integrally closed and has dimension one.

Example 6.14.

- (1) Any DVR is a Dedekind domain.
- (2) \mathbb{Z} is a Dedekind domain. We proved earlier that \mathbb{Z} is integrally closed. Every nonzero prime ideal of the form (p) for some prime number $p \in \mathbb{Z}$. But $\mathbb{Z}/(p)$ is a field, hence (p) is a maximal ideal.
- (3) More generally, every PID is a Dedekind domain.
- (4) $\mathbb{Z}[-\sqrt{5}]$ is a Dedekind domain (although it is not a PID).
- (5) Let K be a finite field extension of \mathbb{Q} , called a **number field**. The integral closure A of \mathbb{Z} in K is called the **ring of integers** of K . One can prove that A is a Dedekind domain. Moreover, A is a free module of finite rank over \mathbb{Z} .

Lemma 6.15. Let A be an integral domain. Then f.a.e.

- (1) A is integrally closed.
- (2) $A_{\mathfrak{p}}$ is integrally closed for every prime ideal \mathfrak{p} .
- (3) $A_{\mathfrak{m}}$ is integrally closed for every maximal ideal \mathfrak{m} .

Proof. Let K be the field of fractions of A and C be the integral closure of A in K . Then $C_{\mathfrak{p}}$ is the integral closure of $A_{\mathfrak{p}}$ in K by 5.16. A is integrally closed $\iff A = C \iff C/A = 0 \iff C_{\mathfrak{p}}/A_{\mathfrak{p}} = (C/A)_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \iff A_{\mathfrak{p}} = C_{\mathfrak{p}} \iff A_{\mathfrak{p}}$ is integrally closed for all prime ideals \mathfrak{p} . The same proof works for maximal ideals. \square

Theorem 6.16. Let A be a Noetherian domain of dimension one. Then f.a.e.

- (1) A is a Dedekind domain.
- (2) $A_{\mathfrak{p}}$ is a DVR for all prime ideals $\mathfrak{p} \neq 0$.

Proof. (1) \implies (2). By assumption $A_{\mathfrak{p}}$ has dimension one. We proved in Lemma 5.16 that if $A \subset B$ is integrally closed and $S \subset A$ is a multiplicative system, then $S^{-1}A$ is integrally closed in $S^{-1}B$. Taking $B = K$ and $S = A \setminus \mathfrak{p}$, we obtain that $A_{\mathfrak{p}}$ is integrally closed. This implies that $A_{\mathfrak{p}}$ is a DVR. (2) \implies (1). As $A_{\mathfrak{p}}$ is a DVR, it is integrally closed. This implies that A is also integrally closed. If $0 \neq \mathfrak{p} \subset \mathfrak{q}$ are prime ideals, then $0 \neq \mathfrak{p}_{\mathfrak{q}} \subset A_{\mathfrak{q}}$ is a non-maximal prime ideal. This contradicts to the assumption that A is a DVR. \square

6.3.1. *Fractional ideals.* Let A be an integral domain and K be its field of fractions.

Definition 6.17.

- (1) An A -submodule $I \subset K$ is called a **fractional ideal** of A if $aI \subset A$ for some $0 \neq a \in A$.
- (2) For any A -submodule $I \subset K$, define $I^{-1} = (A : I) = \{a \in K \mid aI \subset A\}$.
- (3) An A -submodule $I \subset K$ is called **invertible** if there exists an A -submodule $J \subset K$ such that $IJ = A$. We have then $J = I^{-1}$ as $J \subset I^{-1} = I^{-1}IJ \subset AJ = J$.

Remark 6.18. We have $I^{-1}I \subset A$. If $I \subset A$ is an ideal, then $A \subset I^{-1}$.

Remark 6.19. If $I \subset K$ is invertible, then I is a fractional ideal. Indeed, if $I^{-1}I = A$, then $1 = \sum_{i=1}^n x_i y_i$ for some $x_i \in I^{-1}$ and $y_i \in I$. For any $x \in I$, we have $x = \sum_i (xx_i)y_i$ with $xx_i \in A$. This implies that I is generated by y_1, \dots, y_n , hence is a fractional ideal. The same argument implies that I^{-1} is a fractional ideal.

Remark 6.20. If I, J are fractional ideals, then $aI \subset A$ and $bJ \subset A$ for some nonzero $a, b \in A$. Therefore $ab(IJ) \subset A$ with $ab \neq 0$, hence IJ is also a fractional ideal. This implies that fractional ideals form a commutative monoid (with an identity given by A).

Exercise 6.21. Show that if $KI = K$, then I^{-1} is isomorphic to $\text{Hom}_A(I, A)$.

Lemma 6.22. *Let A be a Noetherian integral domain and $I \subset K$ be an A -submodule. Then I is a fractional ideal $\iff I$ is finitely generated over A .*

Proof. If I is a fraction ideal, then $aI \subset A$ for some $a \in A$, hence $I \subset \frac{1}{a}A$. The A -module $\frac{1}{a}A \subset K$ is Noetherian, hence I is finitely generated. Let I be finitely generated, say by elements a_i/b_i for $1 \leq i \leq n$. Taking $b = \prod b_i$, we obtain $bI \subset (a_1, \dots, a_n) \subset A$, hence I is a fractional ideal. \square

Lemma 6.23. *Let A be a Noetherian ring and $I \subset A$ be an ideal. Then I contains a product of prime ideals $\mathfrak{p}_1 \dots \mathfrak{p}_n$ such that $I \subset \mathfrak{p}_i$ for all i .*

Proof. Assume the contrary and let I be a maximal ideal that does not satisfy the required property (it exists as A is Noetherian). Then I is not prime, hence $\exists a, b \in A$ such that $ab \in I$, $a \notin I$, $b \notin I$. By maximality of I , ideals $I + aA$, $I + bA$ contain products of non-zero prime ideals (all of primes contain I). Then the product of all these prime ideals is contained in $(I + aA)(I + bA) \subset I + abA = I$, a contradiction. \square

Lemma 6.24. *Let A be a Dedekind domain. For any prime $0 \neq \mathfrak{p} \subset A$, we have $\mathfrak{p}^{-1} \neq A$ and $\mathfrak{p}^{-1}\mathfrak{p} = A$.*

Proof. We claim that $\mathfrak{p}^{-1} \neq A$. Choose $0 \neq a \in \mathfrak{p}$ and choose the smallest $n > 0$ such that aA contains a product of non-zero primes $\mathfrak{p}_1 \dots \mathfrak{p}_n$ (there is such n by the previous result). Then $\mathfrak{p}_1 \dots \mathfrak{p}_n \subset aA \subset \mathfrak{p}$ and \mathfrak{p} contains one of the factors, say \mathfrak{p}_1 . The prime ideal \mathfrak{p}_1 is maximal, hence $\mathfrak{p}_1 = \mathfrak{p}$. By minimality of n , we have $\mathfrak{p}_2 \dots \mathfrak{p}_n \not\subset aA$ and we can choose $b \in \mathfrak{p}_2 \dots \mathfrak{p}_n \setminus aA$. Then $b\mathfrak{p} \subset \mathfrak{p}_1 \dots \mathfrak{p}_n \subset aA$, hence $b/a \in \mathfrak{p}^{-1}$. On the other hand $b/a \notin A$ as $b \notin aA$. We conclude that $\mathfrak{p}^{-1} \neq A$.

To show that $\mathfrak{p}^{-1}\mathfrak{p} = A$, we note that $\mathfrak{p} \subset \mathfrak{p}^{-1}\mathfrak{p} \subset A$ and \mathfrak{p} is maximal. Assume that $\mathfrak{p}^{-1}\mathfrak{p} = \mathfrak{p}$. Then, for any $x \in \mathfrak{p}^{-1}$, we have $x\mathfrak{p} \subset \mathfrak{p}$. This implies that x is integral over A (see Lemma 5.7), hence $x \in A$. We conclude that $\mathfrak{p}^{-1} = A$, a contradiction. \square

Theorem 6.25. *Let A be a Dedekind domain. Then every ideal $I \subset A$ can be uniquely (up to a permutation of factors) written as a product of prime ideals $I = \mathfrak{p}_1 \dots \mathfrak{p}_n$.*

Proof. Let $I \neq 0$ be a maximal ideal that can not be written as a product of prime ideals. There exists a maximal ideal \mathfrak{p} that contains I . Then $I_1 = \mathfrak{p}^{-1}I \subset \mathfrak{p}^{-1}\mathfrak{p} = A$ is an ideal in A and $\mathfrak{p}I_1 = (\mathfrak{p}\mathfrak{p}^{-1})I = I$. If I_1 can be written as a product of prime ideals, then we are done. Otherwise, from $I \subset \mathfrak{p}^{-1}I = I_1$ and maximality of I , we conclude that $I = I_1 = \mathfrak{p}^{-1}I$. Then for any $x \in \mathfrak{p}^{-1}$, $xI \subset I$, hence x is integral over A and $x \in A$. This implies $\mathfrak{p}^{-1} = A$, a contradiction.

Assume that $I = \mathfrak{p}_1 \dots \mathfrak{p}_m = \mathfrak{q}_1 \dots \mathfrak{q}_n$. Then $\prod \mathfrak{q}_j \subset \mathfrak{p}_1$, hence $\mathfrak{q}_j \subset \mathfrak{p}_1$ for some j , say $j = 1$. By maximality of \mathfrak{q}_1 , we conclude that $\mathfrak{p}_1 = \mathfrak{q}_1$. Multiplying both sides with \mathfrak{p}_1^{-1} , we obtain $\mathfrak{p}_2 \dots \mathfrak{p}_m = \mathfrak{q}_2 \dots \mathfrak{q}_n$ and conclude by induction that $m = n$ and $\mathfrak{p}_i = \mathfrak{q}_i$ up to a permutation. \square

Corollary 6.26. *For any non-zero fractional ideal I , we have $I^{-1}I = A$.*

Proof. We have $aI \subset A$ for some $0 \neq a \in A$. Then $(aI)^{-1} = a^{-1}I^{-1}$ and $II^{-1} = (aI)(aI)^{-1}$. Substituting I by aI , we can assume that $I \subset A$. We can write $I = \mathfrak{p}_1 \dots \mathfrak{p}_n$, where \mathfrak{p}_i are prime. Then $I^{-1} = \mathfrak{p}_1^{-1} \dots \mathfrak{p}_n^{-1}$, hence $II^{-1} = A$. \square

6.4. AKLB setup.

Theorem 6.27. *Let A be an integrally closed domain and K be its field of fractions with $\text{char } K = 0$. Let L/K be a finite field extension and B be the integral closure of A in L . Then*

- (1) $L = KB$ is the field of fractions of B .
- (2) There exists a basis v_1, \dots, v_n of L over K such that $B \subset \bigoplus_i Av_i$.
- (3) If A is a Dedekind domain, then B is a Dedekind domain, finite over A .

Proof. (1) If $x \in L$, then x is algebraic over K , hence satisfies an equation of the form

$$a_m x^m + a_{m-1} x^{m-1} + \dots + a_0 = 0, \quad a_i \in A, a_m \neq 0.$$

Multiplying this equation by a_m^{m-1} , we see that $a_m x$ is integral over A , hence $a_m x \in B$. Applying this procedure to a basis of L over K , we find a new basis u_1, \dots, u_n with $u_i \in B$, hence $L = KB$. The field of fractions of B contains K and B , hence is equal to L .

(2) For any $x \in L$, consider the multiplication operator $x: L \rightarrow L$, $y \mapsto xy$, which is K -linear (we obtain an embedding $L \subset \text{End}_K(L)$). Define a K -bilinear form $(x, y) = \text{Tr}_{L/K}(xy)$ on L . This bilinear form is non-degenerate as $(x, x^{-1}) = \text{Tr}_{L/K}(\text{Id}) = n \neq 0$ for any $0 \neq x \in L$. Consider the basis v_1, \dots, v_n of L over K dual to u_1, \dots, u_n , that is, satisfying $(u_i, v_j) = \delta_{ij}$.

For any $x \in B$, consider the characteristic polynomial $\chi_x(t) = \det_{L/K}(t \cdot \text{Id} - x) \in K[t]$. Then x is a root of this polynomial. On the other hand x is a root of some monic polynomial $f \in A[t]$, hence $\chi_x(t)$ is a factor of some power of f . All roots of f (in some finite field extension of K) are integral over A , hence all the coefficients of $\chi_x(t)$ (and in particular $\text{Tr}_{L/K}(x)$) are integral over A , hence $\text{Tr}_{L/K}(x) \in A$. We have $u_i \in B$, hence $xu_i \in B$ and $\text{Tr}_{L/K}(xu_i) \in A$. This implies that $x = \sum_i (x, u_i) v_i \in \sum_i Av_i$, hence $B \subset \bigoplus_i Av_i$.

(3) As A is Noetherian, $B \subset \bigoplus_i Av_i$ is Noetherian as an A -module (in particular, finite over A), hence also Noetherian as a B -module. B is integrally closed by its definition. Let $0 \neq \mathfrak{q} \subset B$ be a prime ideal. Given $0 \neq x \in \mathfrak{q}$, we have

$$x^m + a_{m-1} x^{m-1} + \dots + a_0 = 0$$

for some $a_i \in A$ with $a_0 \neq 0$. This implies that $a_0 \in Bx \cap A \subset \mathfrak{q} \cap A$, so the prime ideal $\mathfrak{q} \cap A$ is nonzero, hence maximal as A is a Dedekind domain. We conclude that \mathfrak{q} is also maximal by Lemma 5.18. \square

Corollary 6.28. *Let K be a finite field extension of \mathbb{Q} and A be the integral closure of \mathbb{Z} in K . Then A is a Dedekind domain and is a free \mathbb{Z} -module of rank $[K:\mathbb{Q}]$. The corresponding basis of A over \mathbb{Z} is called an **integral basis**.*

Proof. As \mathbb{Z} is a Dedekind domain, we conclude that A is a Dedekind domain, finite over \mathbb{Z} . This implies that A is a free \mathbb{Z} -module of finite rank. If $A = \bigoplus_{i=1}^n \mathbb{Z}v_i$, then v_i are linearly independent over \mathbb{Q} (otherwise there would be a linear dependence over \mathbb{Z}). The elements v_1, \dots, v_n generate K over \mathbb{Q} as $K = \mathbb{Q}A$, hence they form a basis of K over \mathbb{Q} . \square

Example 6.29. Let B be the integral closure of \mathbb{Z} in $L = \mathbb{Q}[\sqrt{m}]$, where $m \in \mathbb{Z}$ is square-free. An element $a + b\sqrt{m} \in L$ has a minimal polynomial $p(x) = x^2 - 2ax + (a^2 - mb^2)$. If this element is integral over \mathbb{Z} , then $p(x)$ is a factor of some polynomial in $\mathbb{Z}[x]$, hence $p \in \mathbb{Z}[x]$ by Gauss lemma. This implies that $a + b\sqrt{m} \in B \iff 2a \in \mathbb{Z}$ and $a^2 - mb^2 \in \mathbb{Z}$. In particular, $4mb^2 \in \mathbb{Z}$, hence $2b \in \mathbb{Z}$. If $a = \frac{1}{2}k$ and $b = \frac{1}{2}l$ for $k, l \in \mathbb{Z}$, then one requires $4 \mid (k^2 - ml^2)$. For example

- (1) If $m = 2$, then we obtain $2 \mid k$ and $2 \mid l$, hence $B = \mathbb{Z}[\sqrt{2}]$.
- (2) If $m = 5$, then we obtain $4 \mid (k^2 - l^2) \iff k \equiv l \pmod{2}$. Therefore $B = \mathbb{Z}[\frac{1}{2}(1 + \sqrt{5})]$.
- (3) If $m = -5$, then we obtain $4 \mid (k^2 + l^2) \iff 2 \mid k, 2 \mid l$. Therefore $B = \mathbb{Z}[\sqrt{-5}]$.
- (4) Generally, if $m \not\equiv 1 \pmod{4}$, then $B = \mathbb{Z}[\sqrt{m}]$. If $m \equiv 1 \pmod{4}$, then $B = \mathbb{Z}[\frac{1}{2}(1 + \sqrt{m})]$.

7. DIMENSION

7.1. Krull dimension.

Remark 7.1. Consider a vector space \mathbb{k}^n over a field \mathbb{k} . A possible interpretation of its dimension n is that there is a maximal strictly increasing chain of vector spaces

$$0 \subset \mathbb{k} \subset \mathbb{k}^2 \subset \dots \subset \mathbb{k}^n$$

of length n , where

$$\mathbb{k}^m = \{(x_1, \dots, x_m, 0, \dots, 0) \mid x_i \in \mathbb{k} \ \forall i \leq m\} = Z(x_{m+1}, \dots, x_n) \subset \mathbb{k}^n.$$

On the level of ideals in $\mathbb{k}[x_1, \dots, x_n]$ we have a chain of length n

$$(x_1, \dots, x_n) \supset (x_2, \dots, x_n) \supset \dots \supset (x_n) \supset 0.$$

Note that all these ideals are prime as $\mathbb{k}[x_1, \dots, x_n]/(x_{m+1}, \dots, x_n) \simeq \mathbb{k}[x_1, \dots, x_m]$ is an integral domain. We can use this interpretation to define the dimension of $\mathbb{k}[x_1, \dots, x_n]$ or any other ring to be the maximal length of a chain of prime ideals. We don't use arbitrary ideals here as, for example, there is an infinite chain of ideals in $\mathbb{k}[x]$ (only one of them is prime)

$$(x) \supset (x^2) \supset (x^3) \supset \dots$$

Definition 7.2. Let A be a ring.

- (1) A finite sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n \subset A$$

of prime ideals is called a **prime chain** of length n .

- (2) For any prime ideal $\mathfrak{p} \subset A$, the supremum of the lengths of the prime chains with $\mathfrak{p}_n = \mathfrak{p}$ is called the **height** of \mathfrak{p} , denoted by $\text{ht } \mathfrak{p}$.
(3) Define the **Krull dimension** of A to be the supremum of the lengths of all prime chains of A , denoted by $\dim(A)$. Equivalently,

$$\dim(A) = \sup \{ \text{ht } \mathfrak{p} \mid \mathfrak{p} \in \text{Spec } A \}.$$

Example 7.3.

- (1) Let \mathbb{k} be a field. Then $\dim(\mathbb{k}) = 0$ (Krull dimension).
(2) Let A be a principal ideal domain. Then $\dim A = 1$.
(3) Let A be a Dedekind domain, then $\dim A = 1$.
(4) Let $A = \mathbb{k}[x_1, \dots, x_n]$, where \mathbb{k} is a field. Then we have a prime chain

$$0 \subset (x_1) \subset \dots \subset (x_1, \dots, x_n),$$

hence $\dim(A) \geq n$. We will show later that $\dim(A) = n$.

Lemma 7.4. For any prime ideal $\mathfrak{p} \subset A$, we have

- (1) $\text{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}})$.
(2) $\dim(A) \geq \text{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p})$.

Lemma 7.5. Let A be a Noetherian ring. Then $\dim(A) = 0 \iff A$ is Artinian.

Proof. We have $\dim(A) = 0 \iff$ every prime ideal of A is maximal. This is equivalent to A being Artinian by Theorem 3.20. \square

Let (A, \mathfrak{m}) be a Noetherian local ring. Then $\dim(A) = \text{ht}(\mathfrak{m})$. We will say that an ideal $I \subset A$ is **\mathfrak{m} -primary** (or is an **ideal of definition**) if $\mathfrak{m}^n \subset I \subset \mathfrak{m}$ for some $n > 0$. Define $\delta(A)$ to be the minimal number of generators of all \mathfrak{m} -primary ideals of A . We will prove later

Theorem 7.6. We have $\dim(A) = \delta(A)$.

Theorem 7.7. Let $f: A \rightarrow B$ be a finite (or integral) ring homomorphism. Then $\dim A = \dim B$.

Proof. Any prime chain $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_n$ in A can be lifted to a prime chain $\mathfrak{q}_0 \subset \dots \subset \mathfrak{q}_n$ in B by Cor. 5.21, hence $\dim A \leq \dim B$. Conversely, given a prime chain in B , its preimage in A consists of distinct prime ideals by Cor. 5.19, hence $\dim A \geq \dim B$. \square

7.2. Hilbert-Poincaré series.

Definition 7.8.

- (1) A **graded ring** A is a ring equipped with a decomposition $A = \bigoplus_{n \geq 0} A_n$, where $A_n \subset A$ are subgroups and $A_m A_n \subset A_{m+n}$.
- (2) A **graded A -module** M is an A -module equipped with a decomposition $M = \bigoplus_{n \geq 0} M_n$, where $M_n \subset M$ are subgroups and $A_m M_n \subset M_{m+n}$.
- (3) An element $x \in M$ is called **homogeneous** of degree $n \geq 0$ if $x \in M_n$.

Example 7.9. (1) Every ring can be considered as a graded ring concentrated in degree zero.

(2) Let A be a ring and $I \subset A$ be an ideal. Then $A^* = \bigoplus_{n \geq 0} I^n / I^{n+1}$ is a graded ring, with $A_0^* = A/I$, $A_1^* = I/I^2$. Multiplication $(I^m / I^{m+1}) \times (I^n / I^{n+1}) \rightarrow I^{m+n} / I^{m+n+1}$ is induced by the multiplication in A . Similarly, if M is an A -module, then $M^* = \bigoplus_{n \geq 0} I^n M / I^{n+1} M$ is a graded A^* -module.

Lemma 7.10. *Let A be a graded ring. Then FAE*

- (1) A is Noetherian.
- (2) A_0 is Noetherian and A is finitely generated as an A_0 -algebra.

Proof. (1) \implies (2). If A is Noetherian, then $A_0 = A/I$, where $I = \bigoplus_{n \geq 1} A_n$, is also Noetherian. The ideal I is finitely generated over A , say by homogeneous elements x_1, \dots, x_r of degrees $d_1, \dots, d_r > 0$. Let us show that these elements generate A as an algebra over A_0 . For any $x \in A_n$, we can write $x = \sum a_i x_i$ for some $a_i \in A_{n-d_i}$. By induction, A_{n-d_i} is contained in the algebra generated by x_1, \dots, x_r over A_0 , therefore x is also contained in this algebra.

(2) \implies (1). This follows from the Hilbert's basis theorem. \square

Lemma 7.11. *Let A be a Noetherian graded ring and M be a finitely-generated graded A -module. Then every M_n is a finitely-generated A_0 -module.*

Proof. We can assume that $A = A_0[x_1, \dots, x_r]$, where x_i has degree $d_i > 0$. Let M be generated by homogeneous elements m_1, \dots, m_s of degree k_1, \dots, k_s . Every element in M_n can be written in the form $\sum_{i=1}^s a_i m_i$, where $a_i \in A_{n-k_i}$. Therefore M_n is generated over A_0 by the elements $f(x_1, \dots, x_r) m_i$, where f is a monomial in x_1, \dots, x_r of total degree $n - k_i$ (there are finitely many such monomials for every $1 \leq i \leq s$). \square

Later we will assume that A_0 is Artinian. Let us discuss this situation in more detail.

Theorem 7.12. *Let A be an Artinian ring and M be a finitely-generated A -module. Then*

- (1) M is both Artinian and Noetherian.
- (2) M has a **composition series**, that is, a chain of submodules

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M$$

such that $M_i / M_{i-1} \neq 0$ is simple for all $1 \leq i \leq n$. We say that M has **finite length** in this case.

- (3) The length n is independent of the composition series. We call it the **length** of M and denote by $\ell(M)$.

Proof. (1) A is Artinian, hence also Noetherian. We can represent M as a quotient of A^r for some $r > 0$. Therefore M is also Artinian and Noetherian.

(2) As M is Artinian, we can choose a minimal nonzero submodule $M_1 \subset M$. Then M_1 / M_0 is simple. Similarly, we choose a minimal submodule $M_1 \subsetneq M_2 \subset M$ and show that M_2 / M_1 is simple. Continuing this process, we obtain a chain of submodules $0 = M_0 \subset M_1 \subset M_2 \subset \dots = M$ such that $M_i / M_{i-1} \neq 0$ are simple. As M is Noetherian, this chain should stabilize, hence $M_n = M$ for some $n \geq 0$.

(3) This follows from the Jordan-Hölder theorem. \square

Example 7.13. If A is a field and M is a finitely-generated A -module, then M is finite-dimensional and $\ell(M) = \dim M$.

Lemma 7.14. *If*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is an exact sequence, then $\ell(M) = \ell(L) + \ell(N)$.

From now on we assume that A is a Noetherian graded ring, M is a f.g. graded A -module and A_0 is Artinian (for example a field). Then every M_n is f.g. over A_0 , hence has a finite length. Then we define the **Hilbert-Poincaré series** of M

$$P(M, t) = \sum_{n \geq 0} \ell(M_n) t^n \in \mathbb{Z}[[t]].$$

Example 7.15. Let $A = \mathbb{k}[x]$ be graded ring with $\deg x = d > 0$. Then

$$P(A, t) = \sum_{k \geq 0} t^{kd} = \frac{1}{1 - t^d}$$

More generally, let $A = \mathbb{k}[x_1, \dots, x_r]$ be a graded ring with $\deg x_i = d_i > 0$. Then

$$P(A, t) = \sum_{k_1, \dots, k_r \geq 0} t^{k_1 d_1 + \dots + k_r d_r} = \prod_{i=1}^r \frac{1}{1 - t^{d_i}}.$$

Theorem 7.16 (Hilbert-Serre). *Assume that A is generated over A_0 by homogeneous elements x_1, \dots, x_r of degrees $d_1, \dots, d_r > 0$. Then $P(M, t)$ can be written in the form $f(t)/\prod_i (1 - t^{d_i})$ for some polynomial $f \in \mathbb{Z}[t]$.*

Proof. We can assume that $A = A_0[x_1, \dots, x_r]$, where x_i has degree d_i . If $r = 0$, then M is f.g. over $A = A_0$, hence $M_n = 0$ for $n \gg 0$. Therefore $P(M, t)$ is a polynomial.

For any $d \in \mathbb{Z}$, we define the shifted graded A -module $M(d)$ by the rule $M(d)_n = M_{d+n}$. It satisfies $P(M(d), t) = t^{-d} P(M, t)$. There is an exact sequence of graded A -modules

$$0 \rightarrow K(-d_r) \rightarrow M(-d_r) \xrightarrow{x_r} M \rightarrow L \rightarrow 0$$

As the length ℓ is additive (with respect to exact sequences), we obtain

$$P(K(-d_r), t) - P(M(-d_r), t) + P(M, t) - P(L, t) = 0,$$

hence

$$(1 - t^{d_r}) P(M, t) = P(L, t) - t^{d_r} P(K, t).$$

Multiplication by x_r is trivial on K and L , hence we can consider them as modules over the graded ring $A_0[x_1, \dots, x_{r-1}]$ and apply induction on r . \square

Let $d(M)$ denote the order of the pole of $P(M, t)$ at $t = 1$. Then $d(M) \leq r$ by the previous theorem.

Lemma 7.17. *Assume that A is generated over A_0 by homogeneous elements x_1, \dots, x_r of degree 1. Then $\ell(M_n)$ is a polynomial in n (with rational coefficients) of degree $d(M) - 1$ for $n \gg 0$. It is called the **Hilbert polynomial** of M .*

Proof. By the previous theorem, we can write $P(M, t) = f(t)/(1 - t)^d$, where $d = d(M)$ and $f(t) \in \mathbb{Z}[t]$ with $f(1) \neq 0$. We have Taylor series

$$(1 - t)^{-d} = \sum_{k \geq 0} \frac{d(d+1) \dots (d+k-1)}{k!} t^k = \sum_{k \geq 0} \binom{d+k-1}{d-1} t^k$$

If $f(t) = \sum_{i=0}^N f_i t^i$, then

$$P(M, t) = \sum_{i=0}^N f_i t^i (1 - t)^{-d} = \sum_{k \geq 0} \sum_{i=0}^N f_i \binom{d+k-1}{d-1} t^{k+i}$$

hence

$$\ell(M_n) = \sum_{i=0}^N f_i \binom{d+n-i-1}{d-1}, \quad \text{for } n \gg 0.$$

This is a polynomial in n with the leading term $\sum_{i=0}^N f_i \frac{n^{d-1}}{(d-1)!} = f(1) \frac{n^{d-1}}{(d-1)!} \neq 0$. \square

7.3. Dimension theorem. Let (A, \mathfrak{m}) be a Noetherian local ring, $I \subset A$ be an \mathfrak{m} -primary ideal ($\mathfrak{m}^n \subset I \subset \mathfrak{m}$ for some $n > 0$) and let M be a finitely-generated A -module. Then A/I is Artinian, being a quotient of A/\mathfrak{m}^n . Consider the graded ring and the graded module

$$A^* = \bigoplus_{n \geq 0} I^n / I^{n+1}, \quad M^* = \bigoplus_{n \geq 0} I^n M / I^{n+1} M.$$

Then M^* is finitely-generated over A^* . If I is generated by the elements x_1, \dots, x_r over A , then A^* is generated as an algebra over A/I by the corresponding elements in I/I^2 . By the previous results, the function

$$\chi_I(M, n) = \ell(M/I^n M) = \sum_{k=0}^{n-1} \ell(I^k M / I^{k+1} M) = \sum_{k=0}^{n-1} \ell(M_k^*)$$

is a polynomial in n for $n \gg 0$, called a **Hilbert-Samuel function** (polynomial). We denote $\chi_I(A, n)$ by $\chi_I(n)$. We know that $\chi_I(M, n+1) - \chi_I(M, n) = \ell(M_n^*)$ is a polynomial of degree $d(M^*) - 1$, hence $\chi_I(M, n)$ is a polynomial of degree $d(M^*)$. Note that if I has r generators over A , then A^* has r generators over A/I , hence $d(M^*) \leq r$.

Lemma 7.18. *If $\mathfrak{m}^r \subset I \subset \mathfrak{m}$, then $\deg \chi_I(M, n) = \deg \chi_{\mathfrak{m}}(M, n)$. We denote it by $d(M)$.*

Proof. We have $\mathfrak{m}^{rn} M \subset I^n M \subset \mathfrak{m}^n M$, hence $\chi_{\mathfrak{m}}(M, rn) \geq \chi_I(M, n) \geq \chi_{\mathfrak{m}}(M, n)$. Therefore these polynomials have equal degrees. \square

Lemma 7.19. *Consider a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$. Then*

- (1) $d(M) = \max\{d(L), d(N)\}$.
- (2) $\chi_I(M, n) - \chi_I(L, n) - \chi_I(N, n)$ is a polynomial of degree $< d(L)$.

Proof. We have $\ell(N/I^n N) = \ell(M/L + I^n M) \leq \ell(M/I^n M)$, hence $d(N) \leq d(M)$. Moreover,

$$(4) \quad \chi(M, n) - \chi(N, n) = \ell(M/I^n M) - \ell(M/L + I^n M) = \ell(L + I^n M / I^n M) = \ell(L/L \cap I^n M).$$

One can show that $I(L \cap I^k M) = L \cap I^{k+1} M$ for $k \gg 0$ (Artin-Rees lemma). Therefore

$$I^n L \subset L \cap I^n M = I^{n-k}(L \cap I^k M) \subset I^{n-k} L$$

for all $n > k$, hence

$$(5) \quad \ell(L/I^n L) \geq \ell(L/L \cap I^n M) \geq \ell(L/I^{n-k} L).$$

We obtain from (4) and (5) that $\chi(M, n) - \chi(N, n)$ and $\chi(L, n)$ have the same degree and the same leading coefficient. This implies the second statement. If $d(N) < d(M)$, we obtain $d(M) = d(L) = \max\{d(L), d(N)\}$. If $d(N) = d(M)$, we obtain $d(L) \leq d(M)$, hence $d(M) = \max\{d(L), d(N)\}$. \square

Corollary 7.20. *Let $x \in A$ be a non-zero-divisor. Then $d(A/xA) < d(A)$.*

Proof. Consider an exact sequence $0 \rightarrow A \xrightarrow{x} A \rightarrow A/xA \rightarrow 0$. By the previous result $\chi(A/xA, n)$ is a polynomial of degree $< d(A)$. \square

Theorem 7.21 (Dimension theorem). *For any Noetherian local ring (A, \mathfrak{m}) , the following integers are equal*

- (1) *The Krull dimension $\dim(A)$.*
- (2) *The degree of the polynomial $\ell(A/\mathfrak{m}^n)$ for $n \gg 0$, denoted by $d(A)$.*
- (3) *The minimal number of generators of an \mathfrak{m} -primary ideal of A , denoted by $\delta(A)$.*

Proof. We will prove inequalities

$$\delta(A) \leq \dim(A) \leq d(A) \leq \delta(A).$$

(1) $d(A) \leq \delta(A)$. Let I be an \mathfrak{m} -primary ideal with $r = \delta(A)$ generators over A . Then the algebra A^* is generated over A/I by r elements. Therefore $d(A) = d(A^*) \leq r$ by the previous results.

(2) $\dim(A) \leq d(A)$. If $d(A) = 0$, then $\ell(A/\mathfrak{m}^n)$ is constant for $n \gg 0$, hence $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for $n \gg 0$. By Nakayama lemma, $\mathfrak{m}^n = 0$. Therefore A is Artinian, hence $\dim A = 0$. Let $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_k$, be a prime chain. Choose $x \in \mathfrak{p}_1 \setminus \mathfrak{p}$. Then $\dim(A/Ax + \mathfrak{p}) \geq k - 1$. The element $[x] \in A/\mathfrak{p}$ is a non-zero-divisor, hence $d(A/Ax + \mathfrak{p}) < d(A/\mathfrak{p}) \leq d(A)$. Therefore, by induction on $d(A)$,

$$k - 1 \leq \dim(A/Ax + \mathfrak{p}) \leq d(A/Ax + \mathfrak{p}) < d(A),$$

hence $k \leq d(A)$ and $\dim(A) \leq d(A)$.

(3) $\delta(A) \leq \dim(A)$. If $\dim(A) = 0$, then A is Artinian, hence $\mathfrak{m}^n = 0$ for some $n > 0$. Taking $I = \mathfrak{m}^n = 0$, we obtain $\delta(A) = 0$.

Let $r = \dim(A) > 0$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be minimal prime ideals over 0 such that $\dim(A/\mathfrak{p}_i) = r$. If $\mathfrak{m} \subset \bigcup_i \mathfrak{p}_i$, then $\mathfrak{m} \subset \mathfrak{p}_i$ for some i (prime avoidance), hence $\mathfrak{m} = \mathfrak{p}_i$ is a minimal prime ideal and $\dim A = 0$, a contradiction. Consider any $x \in \mathfrak{m} \setminus \bigcup_i \mathfrak{p}_i$. Then $\dim(A/xA) \leq r - 1$, hence by induction there exists an ideal $\mathfrak{m}^n + xA \subset I \subset \mathfrak{m}$ such that I/xA has $r - 1$ generators over A/xA . Then I has r generators over A , hence $\delta(A) \leq r$. \square

Lemma 7.22 (Prime avoidance). *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r \subset A$ be prime ideals and $I \subset A$ be an ideal such that $I \subset \bigcup_i \mathfrak{p}_i$. Then $I \subset \mathfrak{p}_i$ for some i .*

Proof. Let $I \subset \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r$. We may assume that \mathfrak{p}_i are not contained in each other. Assume that $I \not\subset \mathfrak{p}_i$ for all i . Then $I\mathfrak{p}_1 \dots \mathfrak{p}_{r-1} \not\subset \mathfrak{p}_r$ and $I \not\subset \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_{r-1}$ (by induction). Consider $a \in I\mathfrak{p}_1 \dots \mathfrak{p}_{r-1} \setminus \mathfrak{p}_r$ and $b \in S = I \setminus (\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_{r-1}) \subset \mathfrak{p}_r$. Then $a + b \in I$ and $a + b \notin \mathfrak{p}_i$ for $1 \leq i < r$. Therefore $a + b \in S \subset \mathfrak{p}_r$, hence $a = (a + b) - b \in \mathfrak{p}_r$, a contradiction. \square

Corollary 7.23. *We have $\dim A \leq \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$. In particular, $\dim A$ is finite.*

Proof. Let $x_1, \dots, x_r \in \mathfrak{m}$ be elements such that their images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. By Nakayama lemma, these elements generate \mathfrak{m} . Therefore $\dim(A) = \delta(A) \leq r = \dim \mathfrak{m}/\mathfrak{m}^2$. \square

Example 7.24. Let A be a DVR. Then $\dim(A) = 1$ and $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$, hence $\dim A = \dim \mathfrak{m}/\mathfrak{m}^2$ in this case.

Theorem 7.25 (Krull's principal ideal theorem). *Let A be a Noetherian ring and $x \in A$ be a non-zero-divisor and not a unit. Then every minimal prime ideal \mathfrak{p} over (x) has height 1.*

Proof. Taking $A_{\mathfrak{p}}$, we can assume that A is local and \mathfrak{p} is its maximal ideal, hence $\text{ht } \mathfrak{p} = \dim(A)$. Then \mathfrak{p} is the only prime ideal containing (x) , hence $\sqrt{(x)} = \mathfrak{p}$ and $\mathfrak{p}^n \subset (x) \subset \mathfrak{p}$ for some $n > 0$. Therefore $\dim(A) = \delta(A) \leq 1$. If $\dim(A) = 0$, then A is Artinian, hence $\mathfrak{p}^n = 0$ for some $n > 0$. But this would imply that x is a zero-divisor. \square

Lemma 7.26. *Let (A, \mathfrak{m}) be a Noetherian local ring and $x \in \mathfrak{m}$ be a non-zero divisor. Then $\dim A/(x) = \dim A - 1$.*

Proof. Let $(x) \subset \mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_k = \mathfrak{m}$ be a prime chain with $k = \dim A/(x)$. Then $\dim A \geq k + 1$ by the previous result, hence $\dim A/(x) \leq \dim A - 1$. On the other hand, let $\mathfrak{m}^n + xA \subset I \subset \mathfrak{m}$ be an ideal such that I/xA has $r = \delta(A/xA)$ generators. Then I has $r + 1$ generators, hence $\delta(A) \leq r + 1 = \delta(A/xA) + 1$. We conclude that $\dim A/xA = \dim A - 1$. \square

Lemma 7.27. *Let A be Noetherian, $a_1, \dots, a_k \in A$ and \mathfrak{p} be a minimal prime ideal over (a_1, \dots, a_k) . Then $\text{ht } \mathfrak{p} \leq k$.*

Proof. Taking $A_{\mathfrak{p}}$ we can assume that A is local and \mathfrak{p} is its maximal ideal. As before, \mathfrak{p} is a minimal prime over $I = (a_1, \dots, a_k)$, hence is the only prime containing I . Therefore $\sqrt{I} = \mathfrak{p}$ and $\mathfrak{p}^n \subset I \subset \mathfrak{p}$ for some $n > 0$. We conclude that $\text{ht } \mathfrak{p} = \dim A = \delta(A) \leq k$. \square

Corollary 7.28. *If \mathbb{k} is algebraically closed, then $\dim \mathbb{k}[x_1, \dots, x_n] = n$.*

Proof. We have seen that $A = \mathbb{k}[x_1, \dots, x_n]$ has dimension $\dim A \geq n$. Every maximal ideal of A is of the form $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in \mathbb{k}$. Therefore $\text{ht } \mathfrak{m} \leq n$, hence $\dim A \leq n$. We conclude that $\dim A = n$. \square

7.4. Transcendence degree.

Definition 7.29. Let L/K be a field extension.

- (1) We say that elements $a_1, \dots, a_n \in L$ are **algebraically independent** over K if we have $f(a_1, \dots, a_n) \neq 0$ for any polynomial $0 \neq f \in K[x_1, \dots, x_n]$.
- (2) Define the **transcendence degree** $\text{trdeg}(L/K)$ of L over K to be the maximal number of algebraically independent elements of L over K .
- (3) A collection (a_1, \dots, a_n) of elements in L is called a **transcendence base** of L/K if they are algebraically independent over K and L is algebraic over $K(a_1, \dots, a_n)$.

Example 7.30. We will see that $\text{trdeg}(K(x_1, \dots, x_n)/K) = n$.

Lemma 7.31. Let $a_1, \dots, a_n \in L$ be algebraically independent over K . Then $b \in L$ is algebraic over $K(a_1, \dots, a_n) \iff a_1, \dots, a_n, b$ are algebraically dependent over K .

Proof. Assume that b is algebraic over $K(a_1, \dots, a_n)$. Multiplying the corresponding polynomial by the common denominator, we obtain $0 \neq f \in K[a_1, \dots, a_n][x]$ such that $f(b) = 0$. It can be interpreted as a polynomial $f \in K[x_1, \dots, x_n, x]$ such that $f(a_1, \dots, a_n, b) = 0$. The converse is similar. \square

Lemma 7.32. Let $a_1, \dots, a_n \in L$ be algebraically independent over K and $b \in L$ be algebraic over $K(a_1, \dots, a_n)$, but not algebraic over $K(a_2, \dots, a_n)$. Then a_1 is algebraic over $K(b, a_2, \dots, a_n)$.

Proof. By the previous lemma, b, a_2, \dots, a_n are algebraically independent over K , while b, a_1, \dots, a_n are algebraically dependent. Applying the lemma again, we obtain that a_1 is algebraic over $K(b, a_2, \dots, a_n)$. \square

Theorem 7.33. If $a_1, \dots, a_n \in L$ is a transcendence base over K , then $\text{trdeg}(L/K) = n$.

Proof. We only have to show that $\text{trdeg}(L/K) \leq n$ and we can assume that n is minimal with the property that $a_1, \dots, a_n \in L$ is a transcendence base over K . Assume that $b_1, \dots, b_m \in L$ are algebraically independent over K . We will prove by induction on k that $S_k = \{b_1, \dots, b_k, a_{k+1}, \dots, a_n\}$ is a transcendence base (after reordering a_i). This is true for $k = 0$. Assume that it is true for k . Then b_{k+1} is algebraic over $K(S_k)$. This implies that $b_1, \dots, b_{k+1}, a_{k+1}, \dots, a_n$ are algebraically dependent over K . Then there exists $i \geq k+1$ such that $S = \{b_1, \dots, b_{k+1}, a_{k+1}, \dots, a_{i-1}\}$ is algebraically independent, while $S \cup \{a_i\}$ is algebraically dependent. Then a_i is algebraic over $K(S_{k+1})$, where $S_{k+1} = S_k \cup \{b_{k+1}\} \setminus \{a_i\}$, hence L is also algebraic over $K(S_{k+1})$. The set S_{k+1} is algebraically independent by minimality of n , hence is a transcendence base. If $m > n$, then $S_n = \{b_1, \dots, b_n\}$ is a transcendence base, hence b_{n+1} is algebraic over $K(b_1, \dots, b_n)$, a contradiction. This implies that $\text{trdeg}(L/K) \leq n$. \square

Lemma 7.34. Let $K \subset L \subset M$ be field extensions. Then

$$\text{trdeg}(M/K) = \text{trdeg}(L/K) + \text{trdeg}(M/L).$$

Proof. Let $S = \{a_1, \dots, a_m\}$ be a transcendence base of L/K and $T = \{b_1, \dots, b_n\}$ be a transcendence base of M/L . Then $S \cup T$ is algebraically independent over K . Moreover, L is algebraic over $K(S)$, hence $L(T)$ is algebraic over $K(S \cup T)$. As M is algebraic over $L(T)$, we obtain that M is algebraic over $K(S \cup T)$. \square

Let now A be an integral domain, finitely generated over a field K and let L be the fraction field of A . It is a finitely generated field extension of K , hence $\text{trdeg}(L/K) < \infty$.

Theorem 7.35. We have $\dim A = \text{trdeg}(L/K)$.

Proof. By Noether normalization theorem, we can embed $B = K[x_1, \dots, x_n] \subset A$ so that A is finite over B . Then $\dim A = \dim B = n$ by Theorem 7.7 and Cor. 7.28.

Let $L' = K(x_1, \dots, x_n)$ be the field of fractions of B . Then L is a finite field extension of L' , hence $\text{trdeg}(L/K) = \text{trdeg}(L'/K) = n$. \square

Remark 7.36. One can show that for any maximal ideal $\mathfrak{m} \subset A$, we have $\text{ht } \mathfrak{m} = \text{trdeg}(L/K)$.

APPENDIX A. CATEGORIES AND FUNCTORS

Definition A.1. A **category** \mathcal{A} consists of the following data

- (1) A family $\text{Ob } \mathcal{A}$, whose elements are called objects of \mathcal{A} .
- (2) For all objects X, Y of \mathcal{A} , a set $\text{Hom}(X, Y) = \text{Hom}_{\mathcal{A}}(X, Y)$, whose elements are called morphisms from X to Y .
- (3) For all objects X, Y, Z of \mathcal{A} , a map

$$\text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z), \quad (f, g) \mapsto g \circ f,$$

called the composition map.

This data should satisfy

- (1) $\forall X \in \text{Ob } \mathcal{A}, \exists 1_X \in \text{Hom}(X, X)$ s.t. $1_Y \circ f = f \circ 1_X = f$ for any $f \in \text{Hom}(X, Y)$.
- (2) The composition of morphisms is associative.

Remark A.2.

- (1) The element 1_X is unique for every $X \in \text{Ob } \mathcal{A}$.
- (2) We write $f: X \rightarrow Y$ for $f \in \text{Hom}(X, Y)$.
- (3) A morphism $f: X \rightarrow Y$ is called an isomorphism if $\exists g: Y \rightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$.

Example A.3.

- (1) The category $\text{Mod } A$ of modules over a ring A and homomorphisms between them.
- (2) The category Sets of sets and all maps between them.
- (3) The category Top of topological spaces and continuous maps between them.
- (4) The category Com of commutative rings and ring homomorphisms.
- (5) The category Grp of groups and group homomorphisms.
- (6) The category Ab of abelian groups and group homomorphisms. It can be identified with $\text{Mod } \mathbb{Z}$.

Definition A.4. Let \mathcal{A} and \mathcal{B} be two categories. A **(covariant) functor** F from \mathcal{A} to \mathcal{B} consists of the following data

- (1) A map $F: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\mathcal{B})$.
- (2) A map $F: \text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{B}}(FX, FY)$ for all objects $X, Y \in \text{Ob}(\mathcal{A})$.

This data should satisfy

- (1) $F(1_X) = 1_{FX}$ for all $X \in \text{Ob}(\mathcal{A})$.
- (2) $F(g \circ f) = F(g) \circ F(f)$.

Definition A.5.

- (1) Given a category \mathcal{A} , we define the **opposite category** \mathcal{A}^{op} using the data

$$\text{Ob}(\mathcal{A}^{\text{op}}) = \text{Ob}(\mathcal{A}), \quad \text{Hom}_{\mathcal{A}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{A}}(Y, X).$$

- (2) A functor from \mathcal{A}^{op} to \mathcal{B} is called a **contravariant functor** from \mathcal{A} to \mathcal{B} .

Example A.6.

- (1) Given a category \mathcal{A} and an object X , there is a (covariant) functor

$$\text{Hom}(X, -): \mathcal{A} \rightarrow \text{Sets}, \quad Y \mapsto \text{Hom}(X, Y).$$

There is also a contravariant functor

$$\text{Hom}(-, X): \mathcal{A} \rightarrow \text{Sets}, \quad Y \mapsto \text{Hom}(Y, X).$$

- (2) For the category $\text{Mod } A$ and an A -module M , we have similar functors $\text{Hom}(M, -)$ and $\text{Hom}(-, M)$ from $\text{Mod } A$ to $\text{Mod } A$.
- (3) For any A -module N , there is a functor

$$- \otimes N: \text{Mod } A \rightarrow \text{Mod } A, \quad M \mapsto M \otimes N.$$

Definition A.7. Let F, G be two functors from \mathcal{A} to \mathcal{B} . A **morphism** (or natural transformation) ϕ from F to G consists of the data

- (1) Morphism $\phi_X: FX \rightarrow GX$ for every object $X \in \text{Ob}(\mathcal{B})$

such that for every $f \in \text{Hom}_A(X, Y)$ the following diagram commutes

$$\begin{array}{ccc} FX & \xrightarrow{F(f)} & FY \\ \phi_X \downarrow & & \downarrow \phi_Y \\ GX & \xrightarrow{G(f)} & GY \end{array}$$

Definition A.8. Let $f: A \rightarrow B$ be a ring homomorphism.

- (1) Given a B -module M , we can consider it as an A -module by setting $ax = f(a)x$ for $a \in A$, $x \in M$. In this way we obtain a functor $\text{Mod } B \rightarrow \text{Mod } A$, called a **restriction of scalars**.
- (2) Given an A -module M , we consider a B -module

$$M_B = B \otimes_A M, \quad b(b' \otimes x) = bb' \otimes x, \quad b, b' \in B, x \in M.$$

In this way we obtain a functor $B \otimes_A -: \text{Mod } A \rightarrow \text{Mod } B$, called an **extension of scalars**.

Definition A.9. Two functors $F: \mathcal{A} \rightarrow \mathcal{B}$, $G: \mathcal{B} \rightarrow \mathcal{A}$ are called **adjoint** if there exist natural bijections

$$\text{Hom}_{\mathcal{B}}(F(X), Y) \simeq \text{Hom}_{\mathcal{A}}(X, G(Y)) \quad \forall X \in \text{Ob}(\mathcal{A}), Y \in \text{Ob}(\mathcal{B}).$$

In this case F is called a **left adjoint** functor to G and G is called a **right adjoint** functor to F .

Example A.10. There is a Tensor-Hom adjunction (see Lemma 2.18)

$$\text{Hom}(L \otimes M, N) \simeq \text{Hom}(L, \text{Hom}(M, N)).$$

for A -modules L, M, N . It implies that the functors

$$\begin{aligned} F: \text{Mod } A &\rightarrow \text{Mod } A, & L &\mapsto L \otimes M, \\ G: \text{Mod } A &\rightarrow \text{Mod } A, & N &\mapsto \text{Hom}(M, N) \end{aligned}$$

are adjoint.

APPENDIX B. LIMITS

Recall that a **poset** (a **partially ordered set**) \mathcal{J} is a set equipped with a relation \leq (a subset $R \subset \mathcal{J} \times \mathcal{J}$, where $(x, y) \in R$ is denoted by $x \leq y$) satisfying

- (1) $x \leq x$ for $x \in \mathcal{J}$.
- (2) $x \leq y$ and $y \leq x \implies x = y$ for $x, y \in \mathcal{J}$.
- (3) $x \leq y$ and $y \leq z \implies x \leq z$ for $x, y, z \in \mathcal{J}$.

A poset \mathcal{J} is called a **chain** (or a **totally ordered set**) if $x \leq y$ or $y \leq x$ for all $x, y \in \mathcal{J}$.

Example B.1.

- (1) The sets $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ with the usual order are chains.
- (2) The set \mathbb{N}^2 with the order $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$ is a poset. Note that $(1, 0)$ and $(0, 1)$ are incomparable, hence the order is partial.
- (3) A set \mathcal{J} with the only relations $x \leq x$ for $x \in \mathcal{J}$ is a poset.

Definition B.2. Let \mathcal{J} be a poset.

- (1) Define an \mathcal{J} -**diagram** of modules to be a family $(M_i)_{i \in \mathcal{J}}$ of A -modules together with homomorphisms $\phi_{ij}: M_i \rightarrow M_j$ for $i \leq j$ such that $\phi_{ii} = \text{Id}$ for $i \in \mathcal{J}$ and $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ for all $i \leq j \leq k$.
- (2) Given an \mathcal{J} -diagram $(M_i)_{i \in \mathcal{J}}$ and modules M, N , define a morphism $f: (M_i)_i \rightarrow M$ to be a collection of homomorphisms $(f_i: M_i \rightarrow M)_i$ such that $f_j \circ \phi_{ij} = f_i$ for all $i \leq j$. Define a morphism $g: N \rightarrow (M_i)_i$ to be a collection of homomorphisms $(g_i: N \rightarrow M_i)_i$ such that $\phi_{ij} \circ g_i = g_j$ for all $i \leq j$.

$$\begin{array}{ccc} M_i & & M_i \\ \downarrow \phi_{ij} & \searrow f_i & \nearrow g_i \\ & M & N \\ & \nearrow f_j & \searrow g_j \\ M_j & & M_j \end{array}$$

- (3) Given an \mathcal{J} -diagram of modules $(M_i)_{i \in \mathcal{J}}$, define its **direct limit** (or colimit) $\varinjlim_{i \in \mathcal{J}} M_i$ to be an A -module M with a morphism $\psi: (M_i)_i \rightarrow M$ such that, for any module M' with a morphism $f: (M_i)_i \rightarrow M'$ there exists a unique morphism $\bar{f}: M \rightarrow M'$ such that $\bar{f} \circ \psi_i = f_i \forall i \in \mathcal{J}$.

$$\begin{array}{ccccc} & & f_i & & \\ & \searrow & \curvearrowright & \searrow & \\ M_i & \xrightarrow{\psi_i} & M & \xrightarrow{\bar{f}} & M' \end{array}$$

- (4) Given an \mathcal{J} -diagram of modules $(M_i)_{i \in \mathcal{J}}$, define its **inverse limit** $\varprojlim_{i \in \mathcal{J}} M_i$ to be an A -module N with a morphism $\psi: N \rightarrow (M_i)_i$ such that, for any module N' with a morphism $f: N' \rightarrow (M_i)_i$ there exists a unique morphism $\bar{f}: N' \rightarrow N$ such that $\psi_i \circ \bar{f} = f_i \forall i \in \mathcal{J}$.

$$\begin{array}{ccccc} & & f_i & & \\ & \searrow & \curvearrowright & \searrow & \\ N' & \xrightarrow{\bar{f}} & N & \xrightarrow{\psi_i} & M_i \end{array}$$

Example B.3. Let $(M_i)_{i \in \mathcal{J}}$ be a family of modules. Equip \mathcal{J} with a partial order $i \leq j$ for $i \in \mathcal{J}$. Then $\varinjlim_i M_i = \bigoplus_i M_i$ and $\varprojlim_i M_i = \prod_i M_i$.

Remark B.4. Note that direct and inverse limits are unique up to an isomorphism.

Theorem B.5. Any \mathcal{J} -diagram of modules has a direct limit and an inverse limit.

Theorem B.6. Given a poset \mathcal{J} and an \mathcal{J} -diagram of A -modules $(M_i)_{i \in \mathcal{J}}$, we have

$$\varprojlim_i M_i = \left\{ x \in \prod_i M_i \mid x_j = \phi_{ij}(x_i) \ \forall i \leq j \right\}.$$

Proof. It is easy to see that the above subset $M \subset \prod_i M_i$ is an A -submodule. We define $\psi_i: M \hookrightarrow \prod_i M_i \xrightarrow{\pi_i} M_i$, where the first map is an embedding and the second map is the canonical projection. If $f: M' \rightarrow (M_i)_i$ is a morphism, then we obtain a canonical morphism $\bar{f}: M' \rightarrow \prod_i M_i$ by the universal property of products. Given $y \in M'$ and $x = \bar{f}(y) \in \prod_i M_i$, we have $x_i = f_i(y)$ and $\phi_{ij}(x_i) = \phi_{ij}f_i(y) = f_j(y) = x_j$ for all $i \leq j$. This implies that $\bar{f}(y) = x \in M$ and we obtain a homomorphism $\bar{f}: M' \rightarrow M$ as required in the definition of an inverse limit. \square

Theorem B.7. *Let \mathcal{I} be a **filtered poset** (i.e. $\forall i, j \in \mathcal{I} \exists k \in \mathcal{I}$ with $i \leq k, j \leq k$) and $(M_i)_{i \in \mathcal{I}}$ be an \mathcal{I} -diagram of A -modules. Then $\varinjlim_i M_i = \bigcup_i M_i / \sim$, where for $x_i \in M_i, x_j \in M_j$*

$$x_i \sim x_j \iff \exists k \geq i, j \text{ with } \phi_{ik}(x_i) = \phi_{jk}(x_j).$$

APPENDIX C. PRIMARY DECOMPOSITION

Throughout this section we will assume that A is a Noetherian ring. This implies that every set of ideals of A has a maximal element.

Definition C.1. Let M be an A -module.

- (1) A prime ideal $\mathfrak{p} \subset A$ is called **associated** to M if $\mathfrak{p} = \text{Ann } x$ for some $x \in M$. Equivalently, M contains a submodule isomorphic to A/\mathfrak{p} (consider $A/\mathfrak{p} \rightarrow M$, $[a] \mapsto ax$). The set of all primes associated to M is denoted by $\text{Ass}(M) = \text{Ass}_A(M)$.
- (2) M is called **co-primary** if it has only one associated prime.
- (3) A submodule $N \subset M$ (or an ideal $I \subset A$) is called **primary** if M/N (respectively A/I) is co-primary. Submodule N is called **\mathfrak{p} -primary** if $\text{Ass}(M/N) = \{\mathfrak{p}\}$.

Example C.2. For any prime ideal $\mathfrak{p} \subset A$, we have $\text{Ass}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$. Indeed, for every $0 \neq [x] \in A/\mathfrak{p}$, we have $\text{Ann}_A[x] = \{a \in A \mid ax \in \mathfrak{p}\} = \mathfrak{p}$: if $a \in \mathfrak{p}$, then $ax \in \mathfrak{p}$ and conversely, if $ax \in \mathfrak{p}$, then $a \in \mathfrak{p}$ as $x \notin \mathfrak{p}$ and \mathfrak{p} is prime.

Remark C.3. We will see later that an ideal $I \subset A$ is primary \iff

$$(6) \quad ab \in I \implies a^n \in I \text{ for some } n \geq 1 \text{ or } b \in I$$

Moreover, the associated prime to A/I is equal to \sqrt{I} . Equivalently, every zero divisor in A/I is nilpotent.

Let us show currently that condition (6) implies that $\mathfrak{p} = \sqrt{I}$ is prime and I is \mathfrak{p} -primary. If $ab \in \sqrt{I} \implies (ab)^m \in I$ for some $m > 0 \implies a^{mn} \in I$ for some $n > 0$ or $b^m \in I \implies a \in \sqrt{I}$ or $b \in \sqrt{I}$. If $\mathfrak{q} = \text{Ann}_A[b]$ is prime for some $0 \neq [b] \in A/I$, then $I \subset \mathfrak{q} \subset \sqrt{I} = \mathfrak{p}$. The first inclusion implies $\mathfrak{p} = \sqrt{I} \subset \mathfrak{q}$, hence $\mathfrak{q} = \mathfrak{p}$. Therefore $\text{Ass } A/I = \{\mathfrak{p}\}$ (assuming that it is non-empty, which we will prove shortly) and I is \mathfrak{p} -primary.

Condition (6) means that a primary ideal is an analogue of an ideal $(p^n) \subset \mathbb{Z}$ for a prime number $p \in \mathbb{Z}$. If $ab \in (p^n)$, then $p^n \mid ab$. Therefore either $p \mid a$ and then $p^n \mid a^n$ or $p \nmid a$ and then $p^n \mid b$. This means that either $a^n \in (p^n)$ or $b \in (p^n)$. The primary decomposition that we will discuss later is an analogue of the fact that every nonzero integer $m \in \mathbb{Z}$ can be written in the form $m = \prod_i p_i^{n_i}$ with distinct p_i . In terms of ideals this means $(m) = \prod_i (p_i^{n_i}) = \bigcap_i (p_i^{n_i})$.

Example C.4.

- (1) A prime ideal is primary.
- (2) An ideal $(n) \subset \mathbb{Z}$ is primary $\iff n = 0$ or n is a prime power.
- (3) If $\mathfrak{m} = \sqrt{I}$ is a maximal ideal, then I is primary and \mathfrak{m} is the associated prime. The maximal ideal $\mathfrak{m}/I = \sqrt{I}/I = \mathcal{N}(A/I)$ is contained in every prime ideal of A/I , hence \mathfrak{m}/I is the only prime ideal. Every zero divisor of A/I is contained in $\mathfrak{m}/I = \mathcal{N}(A/I)$, hence is nilpotent.
- (4) Let $A = \mathbb{k}[x, y]$ and $I = (x, y^2)$. Then $\mathfrak{m} = \sqrt{I} = (x, y)$ is a maximal ideal, hence I is primary. Note that I is not a power of \mathfrak{m} .
- (5) Let $A = \mathbb{k}[x, y, z]/(xy - z^2)$ and $\mathfrak{p} = (x, z)_A$. Then $A/\mathfrak{p} \simeq \mathbb{k}[y]$, hence \mathfrak{p} is prime. The ideal $I = \mathfrak{p}^2 = (x^2, xz, xy)_A = x(x, y, z)_A$ is not primary. The element $x + I \in A/I$ has annihilator $(x, y, z)_A$ which is maximal and different from $\mathfrak{p} = \sqrt{I}$. Another way to see that \mathfrak{p}^2 is not primary is to consider $xy = z^2 \in \mathfrak{p}^2$ and note that $x \notin \mathfrak{p}^2$ while $y \notin \sqrt{\mathfrak{p}^2} = \mathfrak{p}$.

Lemma C.5. Let M be a nonzero A -module.

- (1) If \mathfrak{p} is a maximal element in the set of ideals $\{\text{Ann } x \mid 0 \neq x \in M\}$, then $\mathfrak{p} \in \text{Ass}(M)$. In particular, $\text{Ass } M$ is nonempty.
- (2) For every $0 \neq x \in M$, the ideal $\text{Ann } x$ is contained in some prime associated to M .

Proof. (1) We need to show that \mathfrak{p} is prime. Let $\mathfrak{p} = \text{Ann}(x)$, $ab \in \mathfrak{p}$ and $b \notin \mathfrak{p}$. Then $bx \neq 0$ and $abx = 0$. As $\mathfrak{p} = \text{Ann}(x) \subset \text{Ann}(bx)$ is maximal, we obtain $\mathfrak{p} = \text{Ann}(bx)$, hence $a \in \text{Ann}(bx) = \mathfrak{p}$.

(2) Follows from (1). □

Lemma C.6. Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of A -modules. Then

$$\text{Ass } L \subset \text{Ass } M \subset \text{Ass } L \cup \text{Ass } N.$$

Proof. If $\mathfrak{p} \in \text{Ass } L$, then A/\mathfrak{p} is isomorphic to a submodule of L , hence to a submodule of M . Therefore $\mathfrak{p} \in \text{Ass } M$.

Let $\mathfrak{p} \subset M$ and $M' \subset M$ be isomorphic to A/\mathfrak{p} . If $L \cap M' = 0$, then $M' \rightarrow N$ is injective, hence $\mathfrak{p} \in \text{Ass } N$. If $L \cap M' \neq 0$, let $0 \neq x \in L \cap M'$. We have seen that $\text{Ann } x = \mathfrak{p}$ for all $0 \neq x \in M' \simeq A/\mathfrak{p}$, hence $\mathfrak{p} \in \text{Ass } L$. \square

Lemma C.7. *If $N, N' \subset M$ are \mathfrak{p} -primary, then $N \cap N'$ is also \mathfrak{p} -primary.*

Proof. There is an injection $M/N \cap N' \rightarrow M/N_1 \oplus M/N_2$, hence $\emptyset \neq \text{Ass}(M/N_1 \cap N_2) \subset \text{Ass}(M/N_1) \cup \text{Ass}(M/N_2) = \{\mathfrak{p}\}$. \square

Definition C.8. Let $N \subset M$ be a submodule. A **primary decomposition** of N is an expression of N as a finite intersection

$$N = \bigcap_{i=1}^n Q_i$$

of primary submodules $Q_i \subset M$. Such a decomposition is called **irredundant** (or minimal) if no Q_i can be omitted and the associated primes of M/Q_i are all distinct.

Theorem C.9 (Lasker-Noether). *Let M be a finitely-generated module over a Noetherian ring A . Then every (proper) submodule $N \subset M$ has an irredundant primary decomposition.*

Proof. We say that $N \subset M$ is irreducible in M if whenever $N = N_1 \cap N_2$, we have $N = N_1$ or $N = N_2$. Let $N \subset M$ be a maximal submodule that is not a finite intersection of irreducible submodules. Then N is not irreducible, hence $N = N_1 \cap N_2$, where N_1, N_2 are strictly greater than N , hence can be written as finite intersections of irreducibles. This implies that N is also a finite intersection of irreducibles. We claim that if $N \subset M$ is irreducible, then N is primary. Taking M/N , we can assume that $N = 0$. Assume that $\mathfrak{p}, \mathfrak{q} \in \text{Ass}(M)$ and $\mathfrak{p} \neq \mathfrak{q}$. Then M contains submodules isomorphic to A/\mathfrak{p} and A/\mathfrak{q} . We have seen that $\text{Ann}_A[x] = \mathfrak{p}$ for every $0 \neq [x] \in A/\mathfrak{p}$ and similarly for \mathfrak{q} . Therefore the intersection of these submodules is equal to zero, hence $N = 0$ is not irreducible, a contradiction.

If any Q_i can be removed from the intersection, we omit it. If Q_i, Q_j have the same associated prime, we consider $Q_i \cap Q_j$ which is again primary. \square

Lemma C.10. *Let $N \subset M$ be a submodule and $N = \bigcap_{i=1}^n Q_i$ be an irredundant primary decomposition with $\text{Ass } M/Q_i = \{\mathfrak{p}_i\}$. Then $\text{Ass } M/N = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.*

Proof. There is an injection $M/N \rightarrow \bigoplus_i M/Q_i$, hence $\text{Ass } M/N \subset \bigcup_i \text{Ass } M/Q_i = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

Let $N' = Q_2 \cap \dots \cap Q_n$. Then $N = Q_1 \cap N'$ and $N'/N = N'/Q_1 \cap N' \simeq (Q_1 + N')/Q_1 \subset M/Q_1$, hence $\text{Ass } N'/N \subset \text{Ass } M/Q_1 = \{\mathfrak{p}_1\} \implies \mathfrak{p}_1 \in \text{Ass } N'/N \subset \text{Ass } M/N$. Similarly $\mathfrak{p}_i \in \text{Ass } M/N$ for all i . \square

Lemma C.11. *If M is a finitely generated A -module, then there exists a chain of submodules*

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M$$

such that $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i \subset A$ and $1 \leq i \leq n$.

Proof. If $M \neq 0$, choose $\mathfrak{p}_1 \in \text{Ass } M$ and $M_1 \subset M$ isomorphic to A/\mathfrak{p}_1 . Then apply the same procedure to M/M_1 to find M_2 , and so on. As M is finitely-generated and A is Noetherian, we obtain that M is also Noetherian, hence an increasing chain of submodules stabilizes and our process stops after a finite number of steps. \square

We have seen already that a Noetherian module has a primary decomposition, hence has a finite number of associated ideals. Here is a direct proof of this fact.

Lemma C.12. *Let M be a finitely-generated A -module. Then $\text{Ass } M$ is a finite set.*

Proof. Consider the chain from Lemma C.11. Then $\text{Ass } M \subset \bigcup_i \text{Ass } M_i/M_{i-1}$. Moreover, we have $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$, hence $\text{Ass } M_i/M_{i-1} = \{\mathfrak{p}_i\}$. \square

Let $S \subset A$ be a multiplicative set and $i: A \rightarrow S^{-1}A$, $a \mapsto a/1$ be the natural ring homomorphism. We proved in Lemma 1.40 that there is a bijection

$$i^*: \text{Spec}(S^{-1}A) \xrightarrow{\sim} \{\mathfrak{p} \in \text{Spec } A \mid \mathfrak{p} \cap S = \emptyset\}, \quad \mathfrak{q} \mapsto i^{-1}(\mathfrak{q})$$

with an inverse given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

Lemma C.13. *Given an A -module M , the map $i^*: \text{Spec } S^{-1}A \rightarrow \text{Spec } A$ induces a bijection*

$$\text{Ass}_{S^{-1}A} S^{-1}M \simeq \{\mathfrak{p} \in \text{Ass}_A M \mid \mathfrak{p} \cap S = \emptyset\}.$$

Proof. Let $\mathfrak{q} = \text{Ann}_{S^{-1}A}(m/s)$ be prime, for some $m/s \in S^{-1}M$, and let $\mathfrak{p} = i^{-1}(\mathfrak{q}) = (a_1, \dots, a_n)$ (\mathfrak{p} is finitely-generated as A is Noetherian). Then $a_i m/s = 0$ in $S^{-1}M$, hence $a_i t_i m = 0$ for some $t_i \in S$. With $t = \prod_i t_i$, we have $a_i t m = 0 \implies a_i \in \text{Ann}(tm) \implies \mathfrak{p} \subset \text{Ann}(tm)$. Conversely, if $a \in \text{Ann}(tm)$, then $ats \cdot m/s = 0$ in $S^{-1}M \implies ats/1 \in \mathfrak{q} \implies a/1 \in \mathfrak{q}$ and $a \in \mathfrak{p}$.

On the other hand, let $\mathfrak{p} = \text{Ann}_A(m)$ be prime, for some $m \in M$, and let $\mathfrak{p} \cap S = \emptyset$. We claim that $\mathfrak{q} = S^{-1}\mathfrak{p} \subset S^{-1}A$ is equal to $\text{Ann}_{S^{-1}A}(m/1)$. For any $a/s \in S^{-1}\mathfrak{p}$, we have $a/s \cdot m/1 = am/s = 0$, hence $a/s \in \text{Ann}_{S^{-1}A}(m/1)$. Conversely, if $a/s \in \text{Ann}_{S^{-1}A}(m/1)$, then $am/s = 0 \implies atm = 0$ for some $t \in S \implies at \in \text{Ann}_A(m) = \mathfrak{p} \implies a \in \mathfrak{p}$ as $t \in S$ and $S \cap \mathfrak{p} = \emptyset$. \square

Remark C.14. For any prime ideal $\mathfrak{p} \subset A$, consider the multiplicative set $S = A \setminus \mathfrak{p}$. Then we obtain a bijection between $\text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$ and $\{\mathfrak{p}' \in \text{Ass}_A M \mid \mathfrak{p}' \subset \mathfrak{p}\}$.

Theorem C.15. *We have*

$$\text{Ass}(M) \subset \text{Supp}(M) = \{\mathfrak{p} \in \text{Spec } A \mid M_{\mathfrak{p}} \neq 0\}$$

and every minimal element of $\text{Supp}(M)$ is in $\text{Ass}(M)$.

Proof. If $\mathfrak{p} \in \text{Ass } M$, then there exists an exact sequence $0 \rightarrow A/\mathfrak{p} \rightarrow M$, hence $0 \rightarrow (A/\mathfrak{p})_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}}$. The residue field $(A/\mathfrak{p})_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is nonzero, hence $M_{\mathfrak{p}} \neq 0$.

Let $\mathfrak{p} \in \text{Supp } M$ be a minimal element. Localizing with respect to $S = A \setminus \mathfrak{p}$, we obtain that $\mathfrak{p} \in \text{Ass}_A M \iff \mathfrak{p}A_{\mathfrak{p}} \in \text{Ass}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$ (by the previous lemma). After localization we can assume that $A = A_{\mathfrak{p}}$ and $M = M_{\mathfrak{p}}$. Then all prime ideals are contained in \mathfrak{p} and by the minimality of \mathfrak{p} , $\text{Supp } M = \{\mathfrak{p}\}$. As $\text{Ass } M \subset \text{Supp } M$ is non-empty, we conclude that $\mathfrak{p} \in \text{Ass } M$. \square

Lemma C.16. *Let M be a finitely-generated non-zero A -module. Then*

- (1) M is co-primary $\iff \text{Ann } x \subset \sqrt{\text{Ann } M}$ for $0 \neq x \in M$. The associated prime of M is $\sqrt{\text{Ann } M}$.
- (2) Ideal $I \subset A$ is primary $\iff ab \in I$ implies $a \in \sqrt{I}$ or $b \in I$, for $a, b \in A$. The associated prime of A/I is \sqrt{I} .

Proof. (1) Let $\text{Ass } M = \{\mathfrak{p}\}$. By Lemma C.5, $\text{Ann } x$ is contained in some prime associated to M , hence $\text{Ann } x \subset \mathfrak{p}$. As M is finitely-generated, we have $\text{Supp } M = Z(\text{Ann } M)$ (prove this!). By Theorem C.15, \mathfrak{p} is the unique minimal prime ideal over $\text{Ann } M$ (see Lemma 3.15), hence $\sqrt{\text{Ann } M} = \mathfrak{p}$ and $\text{Ann } x \subset \mathfrak{p} = \sqrt{\text{Ann } M}$.

Conversely, let $I = \sqrt{\text{Ann } M}$. For every $\mathfrak{p} \in \text{Ass } M$, we have $\mathfrak{p} = \text{Ann } x \subset \sqrt{\text{Ann } M} = I$, for some $0 \neq x \in M$, by our assumption. On the other hand $\text{Ann } M \subset \text{Ann } x = \mathfrak{p}$, hence $I = \sqrt{\text{Ann } M} \subset \mathfrak{p}$. We conclude that $I = \mathfrak{p}$, hence I is prime and $\text{Ass } M = \{I\}$.

(2) Consider $M = A/I$. Then $\text{Ann } M = I$ and we apply the previous statement. \square

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