LECTURES ON HOMOLOGY THEORY
DRAFT VERSION

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Contents

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1. Simplicial homology

1.1. **Motivation.** Algebraic topology relates problems of topology and algebra. At the first level which is the content of this course one reduces topological problems to algebra and in particular to linear algebra. At the next level one reduces algebraic problems to topology.

The main approach is the following. Given a topological space X, one constructs an algebraic object $H(X)$. This can be a group (like a fundamental group), a vector space, an algebra, etc. This construction is usually functorial, meaning that given a continuous map $f\colon X \to Y$, there is a homomorphism $f_*\colon H(X) \to H(Y)$. In this way we obtain a functor (see Appendix B) from the category of topological spaces to the category of groups, vector spaces, algebras, etc. The objective is to relate topological properties of X to algebraic properties of $H(X)$. We will study just one construction of this type, called the homology theory.

As applications of the theory that we will develop, we will later prove the following statements:

(1) $\mathbb{R}^n$ are not homeomorphic to each other for different n 3.35.
(2) A continuous map $f\colon D^n \to D^n$ has a fixed point (Brouwer theorem) 3.37.
(3) One can not comb a hedgehog smoothly, meaning that there is no continuous non-vanishing tangent vector field on S^2 4.4.
(4) If $f\colon S^1 \to \mathbb{R}^2$ is injective and continuous, then $\mathbb{R}^2\backslash f(S^1)$ consists of exactly two connected components (Jordan curve theorem) 4.7.
(5) One can not embed S^n in $\mathbb{R}^n$ 4.10. Generally, if $f\colon M \to N$ is an embedding of topological n-manifolds, with compact M and connected N, then f is a homeomorphism 4.13.
(6) The field $\mathbb{C}$ is algebraically closed (Fundamental theorem of algebra) 4.17.
(7) If $f\colon S^n \to \mathbb{R}^n$ is continuous, then there exists $x \in S^n$ with $f(x) = f(-x)$ (Borsuk-Ulam theorem) 4.18.
(8) If F_1, F_2, F_3 is a closed covering of S^2, then at least one F_i contains antipodal points ($x, -x \in F_i$). Generally, if $F_1, \dots, F_{n+1}$ is a closed covering of S^n, then at least one F_i contains antipodal points (Borsuk-Ulam theorem) 4.18.

Example 1.1. Let us consider a finite connected graph on a plane. It consists of several points (called vertices) connected by line segments (called edges), without intersections. Connected components of the complement are called faces (we consider also the unbounded component). We assume that all bounded faces are homeomorphic to an open disc. Let v, e, f be the numbers of vertices, edges and faces respectively. Then *Euler's formula* states that

$$v - e + f = 2.$$

For example, consider a graph consisting of one point. Then $v = 1, e = 0, f = 1$ and the formula is satisfied. For a triangle on a plane, we have $v = e = 3, f = 2$ and the formula is again satisfied.

The above decomposition of $\mathbb{R}^2$ can be interpreted as a decomposition (also called a triangulation if all faces are triangles) of the two-dimensional sphere S^2 – the sphere is obtained from $\mathbb{R}^2$ by adding one point at infinity and we consider the unbounded component as containing this additional point. The left hand side of Euler's formula can be associated with any triangulation of S^2. According to the formula, this number is independent of the triangulation, hence it is an invariant of S^2, called the *Euler characteristic* of S^2. Homology theory in a nutshell is a generalization of Euler's formula to other topological spaces.

1.2. **Triangulated spaces.** By a space we will always mean a topological space. By a map between spaces we will always mean a continuous map, unless otherwise stated. We will study spaces that can be obtained by gluing together points, segments, triangles and higher-dimensional building blocks, called simplices. The structure that one obtains is called a triangulated space. Its combinatorial counterpart is called a Δ-set or a semi-simplicial set. Having this combinatorial structure, one can apply linearization to it and get an algebraic structure (an abelian group or a vector space), called the homology of the original space.

Definition 1.2. The *standard n-simplex* (simplex of dimension n) is

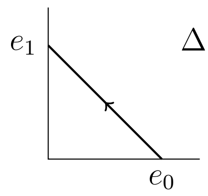
$$\Delta^n = \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, t_i \geq 0,\, \sum\nolimits_i t_i = 1\right\}.$$

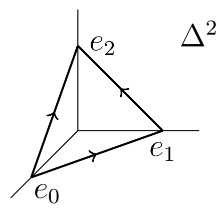
Remark 1.3. For a subset $X \subset \mathbb{R}^d$, define the convex hull

$$\operatorname{conv}(X) = \left\{\sum\nolimits_i t_i x_i \,\middle|\, x_i \in X,\, t_i \geq 0,\, \sum\nolimits_i t_i = 1\right\}.$$

Then $\Delta^n = \operatorname{conv}\{e_0, \dots, e_n\}$, where $e_0, \dots, e_n$ is the standard basis of $\mathbb{R}^{n+1}$.

Example 1.4. $\Delta^0 = \{1\} \subset \mathbb{R}$ is a point, Δ^1 is a line segment (edge, interval), Δ^2 is a triangle, Δ^3 is a tetrahedron.





Remark 1.5. Let us define an *n-dimensional disc*

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\},$$

an *n-dimensional sphere*

$$S^n = \left\{x \in \mathbb{R}^{n+1} \,\middle|\, \|x\| = 1\right\},$$

and an *n-dimensional cube*

$$I^n = \underbrace{I \times \ldots \times I}_{n \text{ times}}, \qquad I = [0, 1].$$

Then

$$\Delta^n \simeq D^n \simeq I^n$$

and $\partial\Delta^n \simeq \partial I^n \simeq \partial D^n = S^{n-1}$. We can obtain S^n by gluing two hemispheres $D^n_\pm$ (both homeomorphic to D^n) along their boundary S^{n-1}.

Definition 1.6. For $n \geq 0$, let $[n] = \{0, \dots, n\}$, considered as an ordered set.

(1) The simplex Δ^n has vertices $e_i \in \mathbb{R}^{n+1}$ for $i \in [n]$ (where $e_i = (t_0, \dots, t_n)$ satisfies $t_i = 1$). Every $t \in \Delta^n$ can be uniquely written as $t = \sum_{i=0}^n t_i e_i$, where $t_i \geq 0$ and $\sum_i t_i = 1$.

(2) For every nonempty subset $I \subset [n]$, define the *face* of Δ^n

$$\Delta^I = \operatorname{conv}\{e_i \,|\, i \in I\} = \{t \in \Delta^n \,|\, t_i = 0 \text{ for } i \notin I\}.$$

It is a simplex of dimension $m = \#I - 1$. The subset $I \subset [n]$ can be identified with the (strictly) increasing map $f\colon [m] \to [n]$ such that $f([m]) = I$.

(3) For $f\colon [m] \to [n]$, consider the continuous map

$$f_*\colon \Delta^m \to \Delta^n, \qquad \sum\nolimits_i s_i e_i \mapsto \sum\nolimits_i s_i e_{f(i)}.$$

Equivalently, $(s_0, \dots, s_m) \mapsto (t_0, \dots, t_n)$, where $t_j = \sum_{f(i)=j} s_i$. If $f\colon [m] \to [n]$ is increasing, then f_* is injective and $f_*(\Delta^m) = \Delta^I$ for $I = f([m]) \subset [n]$.

(4) A *facet* of Δ^n is a face of dimension $n-1$. There are $n+1$ facets in Δ^n, corresponding to (strictly) increasing maps $\delta_i\colon [n-1] \to [n]$ that miss $i \in [n]$, called *coface* maps.
(5) The boundary $\partial\Delta^n$ of Δ^n is the union of all facets (or all faces of dimension $< n$). It consists of $t \in \Delta^n$ with at least one $t_i = 0$.
(6) The *open simplex* $\mathring{\Delta}^n$ is the interior of Δ^n

$$\mathring{\Delta}^n = \{t \in \Delta^n \mid t_i > 0 \ \forall i\} = \Delta^n \backslash \partial\Delta^n.$$

Note that $\partial\Delta^0 = \varnothing$ and $\mathring{\Delta}^0 = \Delta^0$.

Example 1.7. Consider the faces of Δ^2 and observe how they correspond to subsets of [2] or to (strictly) increasing maps $f\colon [m] \to [2]$. For $f\colon [0] \to [2]$, consider the map $f_*\colon \Delta^0 = \{1\} \to \Delta^2$ and the corresponding vertex of Δ^2.

Definition 1.8. A *triangulation* K of a space X is a collection of maps (called simplices)

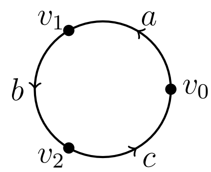
$$(\phi_\sigma = \sigma\colon \Delta^n \to X)_{\sigma \in K},$$

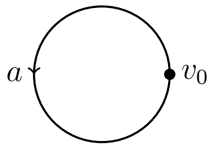
where n depends on σ and is called its dimension, such that
(1) The restriction $\sigma|_{\mathring{\Delta}^n}$ is injective and X is the disjoint union of *cells* $e_\sigma = \sigma(\mathring{\Delta}^n)$.
(2) The restriction of σ to a face of Δ^n is again a simplex $\tau\colon \Delta^m \to X$ from K.
(3) A subset $A \subset X$ is open $\iff$ $\sigma^{-1}(A)$ is open in Δ^n for each $\sigma \in K$.

The pair (X, K), consisting of a space X and a triangulation K, is called a *triangulated space* (or a Δ-complex).

Remark 1.9. A map $\sigma\colon \Delta^n \to X$ from a triangulation is not necessarily injective even though its restriction to the open simplex $\mathring{\Delta}^n$ is injective. Nevertheless, we will often identify σ with its image $\sigma(\Delta^n) \subset X$ (especially on the drawings of triangulations).

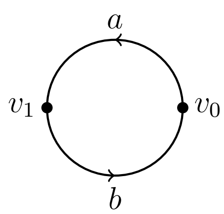
Example 1.10. We have the following triangulations of the circle S^1.

(1) $S^1 \simeq \partial\Delta^2$ inherits its triangulation with 3 vertices and 3 edges: 

(2) S^1 can be obtained by gluing 1 vertex and 1 edge: 

We have $K = \{v_0, a\}$, where v_0 is a 0-simplex and a is a 1-simplex. We represent S^1 as $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and consider the maps

$$\phi_{v_0}\colon \Delta^0 = \{1\} \to S^1, \quad 1 \mapsto 1, \qquad \phi_a\colon \Delta^1 \simeq [0,1] \to S^1, \quad t \mapsto e^{2\pi i t}.$$

(3) S^1 can be obtained by gluing 2 vertices and 2 edges: 

Example 1.11. We have the following triangulations of the sphere S^2.
(1) S^2 is homeomorphic to $\partial\Delta^3$ (the boundary of a tetrahedron). The corresponding triangulation has 4 0-simplices, 6 1-simplices and 4 2-simplices. Note that $4 - 6 + 4 = 2$ in accordance with Euler's formula.

(2) S^2 can be obtained by gluing two hemispheres or two triangles along the boundary. The corresponding triangulation has 3 vertices, 3 1-simplices and 2 2-simplices. Note that $3-3+2=2$ in accordance with Euler's formula.

Let K_n be the set of all n-dimensional simplices of a triangulation. For a (strictly) increasing map $f\colon [m]\to[n]$ and a simplex $\sigma\in K_n$, the composition $\Delta^m \xrightarrow{f_*} \Delta^n \xrightarrow{\sigma} X$ is equal to $\tau\colon \Delta^m\to X$ for a unique simplex $\tau\in K_m$ which we denote by $f^*(\sigma)\in K_m$:

$$\begin{tikzcd} \Delta^m \arrow[r, "f_*"] \arrow[rr, bend left, "\tau"] & \Delta^n \arrow[r, "\sigma"] & X \end{tikzcd}$$

In this way we obtain a map $f^*\colon K_n\to K_m$, $f^*(\sigma)=\sigma f_*$, for every increasing map $f\colon[m]\to[n]$. Given two increasing maps $[\ell]\xrightarrow{g}[m]\xrightarrow{f}[n]$, we consider $f^*\colon K_n\to K_m$, $g^*\colon K_m\to K_l$ and $(fg)^*\colon K_n\to K_l$. We have

$$(fg)^*(\sigma)=\sigma(fg)_*=(\sigma f_*)g_*=g^*(f^*\sigma),$$

where we used the fact that $(fg)_*=f_*g_*$. We conclude that $(fg)^*=g^*f^*$. This leads us to the definition of a Δ-set.

Definition 1.12. A *Δ-set* (or a *semi-simplicial set*) K is a collection of sets $(K_n)_{n\ge0}$ together with maps $f^*\colon K_n\to K_m$, for every (strictly) increasing map $f\colon[m]\to[n]$, such that

$$\mathbb{1}^*=\mathbb{1},\qquad (fg)^*=g^*f^*,\qquad [\ell]\xrightarrow{g}[m]\xrightarrow{f}[n].$$

Elements of K_n are called *n-dimensional simplices* of K. Elements of $V=K_0$ are called vertices of K. We denote $\bigsqcup_{n\ge0}K_n$ by K.

Example 1.13. The faces of Δ^n are parametrized by non-empty subsets $I\subset[n]$ or by increasing maps $\sigma\colon[m]\to[n]$, for $0\le m\le n$. They form a triangulation of Δ^n. The corresponding Δ-set is denoted by $\Delta[n]$. The set $\Delta[n]_m$ of m-simplices consists of increasing maps $\sigma\colon[m]\to[n]$. If $f\colon[k]\to[m]$ is an increasing map, then f^* is given by

$$f^*\colon\Delta[n]_m\to\Delta[n]_k,\qquad \Delta[n]_m\ni\sigma\mapsto\sigma\circ f\in\Delta[n]_k.$$

Example 1.14. Given a Δ-set K, let ∂K be the Δ-set obtained from K by removing all maximal simplices (simplices $\sigma\in K_m$ that are not of the form $f^*(\tau)$ for some increasing $f\colon[m]\to[n]$, $m<n$). For example, $\partial\Delta[n]$ is obtained from $\Delta[n]$ by removing the unique n-simplex $\mathrm{id}\colon[n]\to[n]$. The simplices of $\partial\Delta[n]$ correspond to all proper non-empty subsets $I\subset[n]$. Note that $\partial\Delta[n]$ is a triangulation of the boundary $\partial\Delta^n\simeq S^{n-1}$.

Remark 1.15. Let K be a Δ-set. For every $0\le i\le n$, we consider the maps

$$u_i\colon[0]\to[n],\qquad 0\mapsto i,\qquad v_i=u_i^*\colon K_n\to K_0=V.$$

We call $v_i(\sigma)$ the i-th vertex of $\sigma\in K_n$. In particular, for an edge $a\in K_1$, we consider its vertices $x_0=v_0(a)$, $x_1=v_1(a)$ and interpret a as an arrow $x_0 \xrightarrow{a} x_1$

If the map

$$v\colon K_n\to V^{n+1},\qquad \sigma\mapsto(v_0(\sigma),\dots,v_n(\sigma)),$$

is injective, then every n-simplex is uniquely determined by the sequence of its vertices. In this case we will denote a simplex $\sigma\in K_n$ by $[x_0,\dots,x_n]$ or just $x_0\dots x_n$, where $x_i=v_i(\sigma)$.

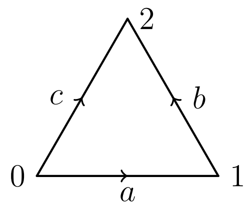
Example 1.16.

(1) Triangulation of an interval. We have $K_0=\{v_0,v_1\}$ and $K_1=\{a=[v_0,v_1]\}$.

$$v_0 \xrightarrow{a} v_1$$

(2) Triangulations of S^1:
  (1) $K_0 = \{v_0\}$, $K_1 = \{a = [v_0, v_0]\}$ (one vertex and one loop).
  (2) $K_0 = \{v_0, v_1\}$, $K_1 = \{a = [v_0, v_1], b = [v_1, v_0]\}$.
  (3) $K = \partial\Delta[2]$ with $K_0 = \{0, 1, 2\}$ and $K_1 = \{[0, 1], [1, 2], [0, 2]\}$.

(3) Consider the canonical triangulation K of the standard simplex Δ^2:



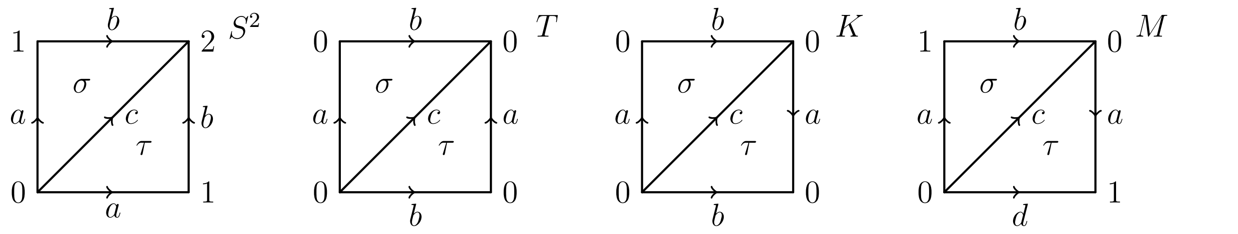
Every face of Δ^2 is a simplex of the triangulation. Therefore we have

$$K_0 \colon 0, 1, 2, \qquad K_1 \colon a = [0, 1],\ b = [1, 2],\ c = [0, 2], \qquad K_2 \colon \sigma = [0, 1, 2].$$

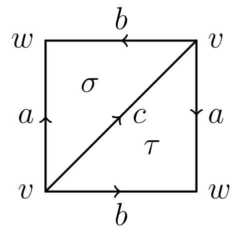
Note that the simplices of K correspond to all nonempty subsets of $[2] = \{0, 1, 2\}$.

(4) Since, $S^{n-1} \simeq \partial\Delta^n$, it has a triangulation parametrized by the Δ-set $\partial\Delta[n]$.

Example 1.17. Triangulations of S^2, Torus, Klein bottle, Möbius strip.

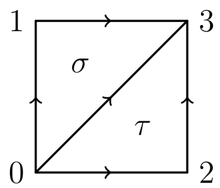


Example 1.18. Consider the real projective plane $\mathbb{R}\mathrm{P}^2 = (\mathbb{R}^3 - \{0\})/\mathbb{R}^\times \simeq S^2/\{\pm 1\}$ (see Example A.4). We can construct $\mathbb{R}\mathrm{P}^2$ by taking the upper hemisphere of S^2 (homeomorphic to the disk D^2) and identifying the opposite (antipodal) points ($x \sim -x$) on its boundary. This leads us to the following description (and triangulation) of $\mathbb{R}\mathrm{P}^2$



Example 1.19. This example generalizes the Δ-set $\Delta[n]$. For a poset P (see Definition A.1), let $\Delta(P)$ be the set of all non-empty finite chains (totally ordered subsets) $I \subset P$ or, equivalently, of (strictly) increasing maps $\sigma \colon [m] \to P$ (where $I = \sigma([m])$). This is a Δ-set, called the *order complex* or the *nerve* of P (cf. Example 1.36). The set $\Delta(P)_m$ of m-simplices consists of increasing maps $[m] \to P$. The set of vertices is $\Delta(P)_0 = P$. In particular, for $P = [n] = \{0 < \cdots < n\}$, we obtain $\Delta(P) = \Delta[n]$, the Δ-set parameterizing all non-empty subsets of $[n]$ (or faces of Δ^n, see Example 1.13).

Example 1.20. The triangulation of a square



corresponds to the Δ-set $\Delta(P)$ for the poset

$$P = \{0, 1, 2, 3 \,|\, 0 < 1 < 3,\ 0 < 2 < 3\}.$$

The 2-simplices of $\Delta(P)$ are $\sigma = \{0 < 1 < 3\}$ and $\tau = \{0 < 2 < 3\}$.

Remark 1.21. For every $0 \le i \le n$, we defined the *coface map* $\delta_i \colon [n-1] \to [n]$ to be an increasing map that misses $i \in [n]$ in the image:

$$\delta_i(k) = \begin{cases} k & k < i, \\ k+1 & k \ge i. \end{cases}$$

The corresponding map $\partial_i = \delta_i^* \colon K_n \to K_{n-1}$ is called the *face map* (or the face operator). Every increasing map $f \colon [m] \to [n]$ can be written as a composition $\delta_{i_1} \ldots \delta_{i_k}$ (such expression is unique if we require $i_1 \le \cdots \le i_k$). Therefore the map $f^* \colon K_n \to K_m$ can be written as a composition of face maps ∂_i. The maps ∂_i can not be arbitrary as they should satisfy Eq. (2) from the following lemma. This equation is necessary and sufficient for the existence of a Δ-set structure.

Lemma 1.22. *We have*

$$\delta_j \delta_i = \delta_i \delta_{j-1}, \qquad i < j. \tag{1}$$

Therefore

$$\partial_i \partial_j = \partial_{j-1} \partial_i, \qquad i < j. \tag{2}$$

Proof. To prove that first equality, we note that both compositions are increasing and miss i, j. For example, $\delta_i \delta_{j-1}(j-2) = \delta_i(j-2) < j$ and $\delta_i \delta_{j-1}(j-1) = \delta_i(j) = j+1$, hence j is not in the image. Therefore the compositions are equal. The second equality follows from the first:

$$\partial_i \partial_j = \delta_i^* \delta_j^* = (\delta_j \delta_i)^* = (\delta_i \delta_{j-1})^* = \delta_{j-1}^* \delta_i^* = \partial_{j-1} \partial_i.$$

□

Remark 1.23.* Let Δ_+ be the category with objects $[n]$ for $n \ge 0$ and (strictly) increasing maps between them. A Δ-set K can be interpreted as a functor $K \colon \Delta_+^{\mathrm{op}} \to \mathbf{Set}$, where $[n] \mapsto K_n$ and $f \colon [m] \to [n]$ is mapped to $f^* \colon K_n \to K_m$. Let $\mathcal{P}_+$ be the category of posets with (strictly) increasing maps between them. Then $\Delta_+ \subset \mathcal{P}_+$ is a subcategory and every poset P induces a functor $\mathcal{P}_+(-, P) \colon \Delta_+^{\mathrm{op}} \to \mathbf{Set}$, which is exactly the Δ-set $\Delta(P)$ described earlier.

1.3. **Geometric realization.** So far we have seen that a triangulation of a space produces a Δ-set structure. But we can also start with a Δ-set K and canonically construct a triangulated space $|K|$, called a *geometric realization* of K. This geometric realization is obtained by gluing simplices $\Delta_\sigma \simeq \Delta^n$ for $\sigma \in K_n$. Consider the space

$$\bar{K} = \bigsqcup_{n\geq 0} K_n \times \Delta^n = \bigsqcup_{\sigma\in K} \Delta_\sigma, \qquad \Delta_\sigma = \{\sigma\} \times \Delta^n,\, \sigma \in K_n,$$

with the equivalence relation generated by (see Definition A.5)

$$K_m \times \Delta^m \ni (\sigma, s) \sim (\tau, t) \in K_n \times \Delta^n \iff \sigma = f^*(\tau),\ t = f_*(s), \tag{3}$$

for some increasing $f\colon [m] \to [n]$. We will write $(\sigma, s) \xrightarrow{f} (\tau, t)$ for the above situation and call it a simple equivalence. This relation means that if $\sigma \in K_m$ is a face of $\tau \in K_n$, then the points of Δ_σ should be identified with the points of Δ_τ. Define the geometric realization

$$|K| = \bar{K}/{\sim}$$

and equip it with the quotient topology (see Section A.2) with respect to the projection map $\pi\colon \bar{K} \to |K|$. This means that $U \subset |K|$ is open $\iff$ $\pi^{-1}(U)$ is open.

Remark 1.24. For every $\sigma \in K_n$, we have a map $\pi_\sigma\colon \Delta^n \simeq \Delta_\sigma \hookrightarrow \bar{K} \to |K|$ (which is not necessarily injective) and its restriction $\mathring{\pi}_\sigma\colon \mathring{\Delta}^n \simeq \mathring{\Delta}_\sigma \to |K|$ which is injective as we will see.

Theorem 1.25. *The following map is a bijection*

$$\mathring{\pi}\colon \bar{K}^\circ = \bigsqcup_{n\geq 0} (K_n \times \mathring{\Delta}^n) \to |K|\,.$$

Proof. For any point $(\tau, t) \in K_n \times \Delta^n$, let $m \leq n$ be the minimal dimension of a face of Δ^n containing t. Then there exists an increasing map $f\colon [m] \to [n]$ such that $t = f_*(s)$ for some $s \in \Delta^m$. By the assumption on m, the point $s \in \Delta^m$ is not contained in any facet, hence $s \in \mathring{\Delta}^m$. Taking $\sigma = f^*(\tau) \in K_m$, we obtain $(\tau, t) \sim (\sigma, s) \in K_m \times \mathring{\Delta}^m$, hence $\mathring{\pi}$ is surjective.

Let us show that $\mathring{\pi}$ is injective. Consider two points $(\sigma, s), (\tau, t)$ in $\bar{K}^\circ$ and assume that they are equivalent. Then there is a chain of simple equivalences

$$(\sigma, s) = (\sigma_0, s_0) \xrightarrow{f=f_0} (\sigma_1, s_1) \xleftarrow{g=f_1} (\sigma_2, s_2) \xrightarrow{f_2} \dots \leftarrow (\tau, t),$$

with $\sigma_i \in K_{m_i}$. We will show that one can make this chain shorter. We have increasing maps $f\colon [m_0] \to [m_1]$, $g\colon [m_2] \to [m_1]$ with $s_1 = f_*(s_0) = g_*(s_2)$. As $s = s_0 \in \mathring{\Delta}^{m_0}$, we have

$$f_*(s_0) \in g_*(\Delta^{m_2}) \iff f_*(\Delta^{m_0}) \subset g_*(\Delta^{m_2}).$$

We conclude from $f_*(s_0) = g_*(s_2)$ that $f_*(\Delta^{m_0}) \subset g_*(\Delta^{m_2})$, hence $\operatorname{Im} f \subset \operatorname{Im} g$, hence there exists an increasing map $h\colon [m_0] \to [m_2]$ with $f = gh$. Then $g_*(s_2) = f_*(s_0) = g_*h_*(s_0)$, hence $s_2 = h_*(s_0)$ and we obtain $(\sigma_0, s_0) \xrightarrow{h} (\sigma_2, s_2)$. Taking its composition with f_2 we get a chain

$$(\sigma, s) \xrightarrow{f_2 g} (\sigma_3, s_3) \leftarrow \dots \leftarrow (\tau, t).$$

Finally we will get an equivalence $(\sigma, s) \xrightarrow{f} (\tau, t)$ with $t = f_*(s)$. As t is internal, we conclude that f is a bijection, hence $(\sigma, s) = (\tau, t)$. This proves injectivity. □

Corollary 1.26. *If K is a Δ-set, then $|K|$ is canonically triangulated.*

Proof. For every $\sigma \in K_n$, there is a canonical map $\pi_\sigma\colon \{\sigma\} \times \Delta^n \to \bar{K} \xrightarrow{\pi} |K|$. By the previous theorem these maps satisfy the first axiom of a triangulation. Other axioms follow from the construction of $|K|$. □

Remark 1.27. A map $f\colon K \to L$ between Δ-sets is called a *simplicial morphism* if $f(K_n) \subset L_n$ and if for every increasing $\phi\colon [m] \to [n]$ we have $\phi^* f(\sigma) = f\phi^*(\sigma)$, for all $\sigma \in K_n$. Such map induces a continuous map between geometric realizations $|f| : |K| \to |L|$ which sends $\{\sigma\} \times \Delta^n \subset |K|$ to $\{f(\sigma)\} \times \Delta^n \subset |L|$, for all $\sigma \in K_n$.

Theorem 1.28. *Let X be a space with a triangulation K. Then the map*

$$\phi\colon |K| \to X, \qquad K_n \times \Delta^n \ni (\sigma, t) \mapsto \sigma(t) \in X,$$

is a homeomorphism.

Proof. Let us show first that ϕ respects equivalence classes, that is, well-defined.

$$\begin{tikzcd}
\Delta^m \arrow[r, "f_*"] \arrow[drrr, "\sigma"'] & \Delta^n \arrow[r, hook, "i_\tau"] \arrow[drr, "\tau"] & \bar{K} \arrow[r, "\pi"] & {|K|} \arrow[d, "\phi"] \\
 & & & X
\end{tikzcd}$$

Consider a simple equivalence $(\sigma, s) \xrightarrow{f} (\tau, t)$ with $f\colon [m] \to [n]$ (3). Then $\sigma = f^*(\tau) = \tau f_*$. This implies that $\sigma(s) = \tau f_*(s) = \tau(t)$, hence the images of equivalent elements coincide.

We proved that the map $\mathring{\pi}\colon \bar{K}^\circ \to |K|$ is bijective. On the other hand the composition

$$\bar{K}^\circ \xrightarrow{\mathring{\pi}} |K| \xrightarrow{\phi} X$$

is bijective by the first axiom of a triangulation, hence $\phi\colon |K| \to X$ is also bijective.

The map $\bar{K} \to X$, $(\sigma, t) \mapsto \sigma(t)$ is continuous, hence $\phi\colon |K| \to X$ is also continuous, by the definition of the quotient topology. On the other hand, if $U \subset |K|$ is open, then $A = \phi(U) \subset X$ satisfies

$$\sigma^{-1}(A) = \pi^{-1}(U) \cap (\{\sigma\} \times \Delta^n) \qquad \forall \sigma \in K_n.$$

This set is open as π is continuous. By the third axiom of a triangulation, this implies that A is open, hence ϕ is a homeomorphism. $\square$

1.4. **Simplicial complexes.**

1.4.1. *Geometric and abstract simplicial complexes.*

Definition 1.29.

(1) For a subset $X \subset \mathbb{R}^d$, define the *convex hull*

$$\operatorname{conv}(X) = \left\{ \sum\nolimits_i t_i x_i \,\middle|\, x_i \in X,\, t_i \geq 0,\, \sum\nolimits_i t_i = 1 \right\}.$$

(2) The points $x_0, \dots, x_n \in \mathbb{R}^d$ are called *affinely independent* if any of the following equivalent conditions is satisfied

  (1) If $\sum_{i=0}^n t_i x_i = 0$ and $\sum_{i=0}^n t_i = 0$, then $t_0 = \dots = t_n = 0$.
  (2) The map $\{t \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1\} \to \mathbb{R}^d$, $t \mapsto \sum_{i=0}^n t_i x_i$, is injective.
  (3) The vectors $x_1 - x_0, \dots, x_n - x_0$ are linearly independent.

(3) Define the *standard n-simplex*

$$\Delta^n = \operatorname{conv}\{e_0, \dots, e_n\} = \left\{ t \in \mathbb{R}^{n+1} \,\middle|\, t_i \geq 0,\, \sum\nolimits_i t_i = 1 \right\} \subset \mathbb{R}^{n+1},$$

where $e_0, \dots, e_n$ is the standard basis of $\mathbb{R}^{n+1}$.

(4) For affinely independent points $x_0, \dots, x_n \in \mathbb{R}^d$, the convex hull

$$\operatorname{conv}\{x_0, \dots, x_n\} = \left\{ \sum\nolimits_{i=0}^n t_i x_i \,\middle|\, t \in \Delta^n \right\}$$

is called an *n-simplex* (or n-dimensional simplex), denoted by $\Delta(\sigma) = [x_0, \dots, x_n]$ and identified with the set $\sigma = \{x_0, \dots, x_n\}$. The points x_i are called the *vertices* of $\Delta(\sigma)$.

(5) For distinct $x, y \in \mathbb{R}^d$, define the *interval* $[x, y] = \operatorname{conv}\{x, y\} = \{tx + (1-t)y \mid t \in [0,1]\}$.

(6) A *face* of a simplex $\Delta = [x_0, \dots, x_n]$ is a simplex $[x_{i_0}, \dots, x_{i_m}]$ (of dimension $m \geq 0$) for $0 \leq i_0 < \dots < i_m \leq n$. A face of dimension $n-1$ is called a *facet* of Δ. The union $\partial\Delta$ of all facets (or all faces of dimension $< n$) is called the *boundary* of Δ. The complement $\mathring{\Delta} = \Delta \backslash \partial\Delta$ is called an *open simplex*.

Many topological spaces (for example S^1 and S^2) can be triangulated, that is, decomposed into unions of points, intervals, triangles, and higher dimensional simplices. In the next definition we will formalize this point of view.

Definition 1.30. A *geometric simplicial complex* K is a collection of simplices $(\Delta(\sigma))_{\sigma \in K}$ in $\mathbb{R}^d$ such that

(1) If $\sigma \in K$, then every face of σ is in K.
(2) The intersection of two simplices in K is a face of both of them.

The *underlying space* $|K|_u = \bigcup_{\sigma \in K} \Delta(\sigma) \subset \mathbb{R}^d$ is equipped with the induced topology. The axioms of a simplicial complex imply that $|K|_u = \bigsqcup_{\sigma \in K} \mathring{\Delta}(\sigma)$, a disjoint union of open simplices. Let K_n be the set of n-simplices of K for $n \geq 0$. The set $V \subset \mathbb{R}^d$ such that $K_0 = \{\{v\} \mid v \in V\}$ is called the set of vertices of K. Every n-simplex of K can be identified with the set of its vertices. The first axiom implies that if $\sigma \in K$ and $\varnothing \neq \tau \subset \sigma$ (as a subset of V), then $\tau \in K$. This leads us to the definition of an abstract simplicial complex.

Definition 1.31. An (abstract) *simplicial complex* (V, K) consists of a vertex set V and a collection K of non-empty finite subsets of V (called simplices) such that

(1) $\{v\} \in K$ for every $v \in V$.
(2) If $\sigma \in K$ and $\varnothing \neq \tau \subset \sigma$, then $\tau \in K$. The set τ is called a *face* of σ.

The *dimension* of a simplex $\sigma \in K$ is defined to be n if σ has $n+1$ elements. Let K_n be the set of all n-dimensional simplices of K. We identify K_0 and V. Let $\dim K = \sup\{n \geq 0 \mid K_n \neq \varnothing\}$.

The above discussion implies that a geometric simplicial complex can be interpreted as an abstract simplicial complex. We will see soon, that for an abstract simplicial complex K we can construct the corresponding geometric simplicial complex, called the geometric realization of K.

Example 1.32. A simplicial complex K of dimension ≤ 1 is called a graph. It consists of the set of vertices $K_0 = V$ and the set of edges K_1, where every edge $e = \{u, v\} \subset V$ has exactly two elements, called the vertices of e.

Example 1.33. Let $\Delta[n]$ be the set of all non-empty subsets of $[n] = \{0, \dots, n\}$. It is a simplicial complex with the set of vertices $[n]$. It parametrizes the faces of Δ^n.

Remark 1.34. Define the *power set* 2^V to be the set of all subsets of V (every $A \subset V$ induces $\chi_A \colon V \to \{0,1\}$, where $\chi_A(v) = 1$ if $v \in A$ and zero otherwise). If (V, K) is a simplicial complex, then $K \subset 2^V$ is partially ordered by inclusion of sets in V. A poset K corresponds to a simplicial complex $\iff$ for all $\sigma \in K$, the poset $K_{\le\sigma} = \{\tau \in K \mid \tau \le \sigma\}$ is isomorphic to $\Delta[n]$ (non-empty subsets of $[n]$) for some $n \ge 0$. The set of vertices is defined by $V = \min K$.

Example 1.35. For a simplicial complex K, let $\max K$ be the set of its maximal simplices. Then $\partial K = K \setminus \max K$ is a simplicial complex called the boundary of K. For example, $\partial\Delta[n]$ consists of all non-empty proper subsets $\varnothing \ne \sigma \subsetneq [n]$. It parametrizes the faces of $\partial\Delta^n$.

Example 1.36 (Order complex)**.** For a poset P, let $\Delta(P)$ be the set of all non-empty finite chains (totally ordered subsets) of P (cf. Example 1.19). Then $\Delta(P)$ is a simplicial complex with set of vertices P, called the *order complex* of P. In particular, for the poset $[n] = \{0 < \cdots < n\}$, the order complex $\Delta([n])$ coincides with $\Delta[n]$.

1.4.2. *Geometric realization.*

Definition 1.37. For an (abstract) simplicial complex (V, K), the *geometric realization* or the *simplicial polyhedron* of K is

$$\left|K\right|_p = \left\{ t \in [0,1]^V \;\middle|\; \sum\nolimits_{v\in V} t_v = 1,\ \operatorname{supp}(t) \in K \right\} \subset [0,1]^V,$$

where $\operatorname{supp}(t) = \{v \in V \mid t_v \ne 0\}$. The numbers t_v are called the *barycentric coordinates* of $t \in |K|_p$. The set $|K|_p \subset \mathbb{R}^V$ is equipped with the subspace topology.

Example 1.38. Let $K = \Delta[n]$ be the set of all non-empty subsets of $[n]$. Then

$$\left|K\right|_p = \left\{ t \in [0,1]^{n+1} \;\middle|\; \sum\nolimits_i t_i = 1 \right\} = \Delta^n.$$

Example 1.39. Let $K = \partial\Delta[n]$ be the set of all non-empty proper subsets of $[n]$. Then

$$\left|K\right|_p = \left\{ t \in [0,1]^{n+1} \;\middle|\; \sum\nolimits_i t_i = 1,\ \exists i \text{ with } t_i = 0 \right\} = \partial\Delta^n \simeq S^{n-1}.$$

Theorem 1.40. *Let K be a finite geometric simplicial complex in $\mathbb{R}^d$ and $V \subset \mathbb{R}^d$ be its vertex set. Then the map*

$$\phi\colon\ \left|K\right|_p \to \left|K\right|_u, \qquad t \mapsto \sum\nolimits_{v\in V} t_v v$$

is a homeomorphism.

Proof. For $\sigma = \{v_0, \dots, v_n\} \in K_n$, let

$$\underline{\Delta}(\sigma) = \left\{ t \in |K|_p \;\middle|\; \operatorname{supp}(t) \subset \sigma \right\} \simeq \Delta^n, \qquad \mathring{\underline{\Delta}}(\sigma) = \left\{ t \in |K|_p \;\middle|\; \operatorname{supp}(t) = \sigma \right\} \simeq \mathring{\Delta}^n.$$

The map ϕ maps $\underline{\Delta}(\sigma) \subset |K|_p$ bijectively to $\Delta(\sigma) \subset |K|_u$. Moreover, we have $|K|_p = \bigsqcup_\sigma \mathring{\underline{\Delta}}(\sigma)$ and $|K|_u = \bigsqcup_\sigma \mathring{\Delta}(\sigma)$, hence ϕ is a bijection. It is clear that ϕ is continuous. The fact that $|K|_p$ is compact and $|K|_u$ is Hausdorff, implies that ϕ is a homeomorphism. $\square$

1.4.3. *Simplicial complexes and Δ-sets.*

Remark 1.41. Note that a simplicial complex of dimension 1 is a graph and a Δ-set of dimension 1 is a directed graph (with $a \in K_1$ oriented from $v_0(a)$ to $v_1(a)$). We can equip every graph with an orientation. Conversely, for every oriented graph without loops, we obtain a non-oriented graph by forgetting the orientation. Similarly, we will relate simplicial complexes and Δ-sets by introducing or forgetting the "orientation" of simplices.

We say that a simplicial complex (V, K) is *ordered* if V is equipped with a partial order such that every simplex $\sigma \in K$ is a chain (totally ordered). In this case we denote a simplex $\{v_0 < \cdots < v_n\}$ of K by $[v_0, \ldots, v_n]$.

Lemma 1.42. *An ordered simplicial complex has a canonical structure of a Δ-set.*

Proof. Every $\sigma \in K_n$ can be written in the form $\sigma = \{v_0 < \cdots < v_n\} \subset V$. We can identify σ with the increasing map $\sigma\colon [n] \to V$, $i \mapsto v_i$. For any increasing map $f\colon [m] \to [n]$, the composition $[m] \xrightarrow{f} [n] \xrightarrow{\sigma} V$ is increasing and corresponds to a simplex in K_m, hence we obtain the map $f^*\colon K_n \to K_m$, $\sigma \mapsto \sigma f$. This means that K has a structure of a Δ-set. $\square$

Note that in a simplicial complex every simplex is uniquely determined by its set of vertices. This is not always the case for Δ-sets (for example, consider a triangulation of S^1 consisting of two vertices and two intervals between them). Let K be a Δ-set and $V = K_0$ be its set vertices. For every $0 \le i \le n$, consider the map $u_i\colon [0] \to [n]$, $0 \mapsto i$, and the corresponding map $v_i = u_i^*\colon K_n \to K_0 = V$ (cf. Remark 1.15). We call $v_i(\sigma)$ the i-th vertex of $\sigma \in K_n$. Consider the map

$$v\colon K \to 2^V, \qquad K_n \ni \sigma \mapsto \{v_i(\sigma) \mid 0 \le i \le n\} \subset V.$$

If $v\colon K \to 2^V$ is injective, we denote a simplex $\sigma \in K_n$ by $[v_0(\sigma), \ldots, v_n(\sigma)]$.

Lemma 1.43. *Let K be a Δ-set such that $v\colon K \to 2^V$ is injective (every simplex is uniquely determined by its set of vertices). Then K has a canonical structure of a simplicial complex.*

Proof. As $v\colon K \to 2^V$ is injective, we can interpret K as a subset of 2^V. For every $v \in V = K_0$, the one-point set $\{v\}$ is in K, hence the first axiom of a simplicial complex is satisfied.

We claim that, for every $\sigma \in K_n$, the subset $v(\sigma) \subset V$ has exactly $n+1$ elements. Indeed, if $v(\sigma)$ has $\le n$ elements, then $v_i(\sigma) = v_j(\sigma)$ for some $i \ne j$. But then $\tau = \delta_i^*(\sigma) \in K_{n-1}$ and σ have the same set of vertices, a contradiction. For $\sigma = \{v_0, \ldots, v_n\} \in K_n$ and $\tau \subset \sigma$, there exists an increasing map $f\colon [m] \to [n]$ such that $\tau = \left\{v_{f(0)}, \ldots, v_{f(m)}\right\}$. Then $f^*(\sigma)$ has the set of vertices τ, hence $\tau \in K_m$ and the second axiom of a simplicial complex is satisfied. $\square$

Example 1.44. Consider a triangulation of S^1 that corresponds to a simplicial complex. Such triangulation should have at least 3 vertices and 3 edges. On the other hand, there exists a triangulation of S^1 with just one vertex and one edge. Therefore Δ-sets can be more economical than simplicial complexes.

Remark 1.45. Consider the Δ-set $K = \{v_0, v_1, v_2, [v_0, v_1], [v_1, v_2], [v_2, v_0]\}$ and the corresponding simplicial complex. For any order on this simplicial complex we don't obtain the original Δ-set. Indeed, any such order would require $v_0 < v_1 < v_2 < v_0$. On the other hand, the Δ-set $K = \{v_0, v_1, v_2, [v_0, v_1], [v_1, v_2], [v_0, v_2]\}$, has the same simplicial complex and corresponds to the order $v_0 < v_1 < v_2$.

Theorem 1.46. *Let K be a finite Δ-set such that $v\colon K \to 2^V$ is injective (so that K is also a simplicial complex). Then there is a homeomorphism $\phi\colon |K| \to |K|_p$, where for $\sigma = [v_0, \dots, v_n] \in K_n$ we consider ($e_v \in \mathbb{R}^V$ are standard basis vectors)*

$$\Delta_\sigma = \{\sigma\} \times \Delta^n \xrightarrow{\phi} \Delta(\sigma) = \Big\{ t \in |K|_p \;\Big|\; \operatorname{supp}(t) \subset \sigma \Big\},$$
$$(t_0, \dots, t_n) \mapsto \sum\nolimits_i t_i e_{v_i} \in \mathbb{R}^V,$$

Proof. The map $\bar{\phi}\colon \bar{K} = \bigsqcup_{n\geq 0} K_n \times \Delta^n \to |K|_p$ described above is continuous and respects equivalence classes. Therefore it induces a continuous map $\phi\colon |K| \to |K|_p$. For $\sigma \in K_n$, the open simplex $\mathring{\Delta}_\sigma$ is mapped bijectively to $\mathring{\Delta}(\sigma) = \Big\{ t \in |K|_p \;\Big|\; \operatorname{supp}(t) = \sigma \Big\}$. We proved earlier that $|K| \simeq \bigsqcup_{\sigma\in K} \mathring{\Delta}_\sigma$ and it is clear that $|K|_p = \bigsqcup_{\sigma\in K} \mathring{\Delta}(\sigma)$. Therefore ϕ is a bijection.

Since K is finite, the space $|K|$ is compact. Since $|K|_p \subset [0,1]^V$, it is Hausdorff. Therefore the bijective continuous map $\phi\colon |K| \to |K|_p$ is a homeomorphism. $\square$

1.5. Product triangulation.

Theorem 1.47 (Triangulation of the product). *The product $\Delta^m \times \Delta^n$ has a triangulation parametrized by the order complex $\Delta(P)$ (see Example 1.36), where $P = [m] \times [n]$ is equipped with the partial order $(i,j) \le (i',j')$ if $i \le i'$ and $j \le j'$. Namely, there is a homeomorphism*

$$|\Delta(P)| \xrightarrow{\sim} \Delta^m \times \Delta^n, \qquad \sum_{(i,j)\in P} t_{ij} e_{ij} \mapsto \sum_{(i,j)\in P} t_{ij}(e_i, e_j) \in \mathbb{R}^{m+n+2},$$

where $(e_{ij})_{(i,j)\in P}$ is the standard basis of $\mathbb{R}^P$.

Proof. We will consider only the case $P = [n] \times [1]$ and the corresponding map

$$f\colon |\Delta(P)| \to \Delta^n \times [0,1].$$

We have $P = \{v_0, \dots, v_n, w_0, \dots, w_n\}$, where $v_i = (i,0)$, $w_i = (i,1)$ and $v_i \le v_j$, $v_i \le w_j$, $w_i \le w_j$ for $i \le j$. The maximal simplices of $\Delta(P)$ have dimension $n+1$ and are of the form $\sigma_i = (v_0, \dots, v_i, w_i, \dots, w_n)$ for $0 \le i \le n$. A point

$$(t_0, \dots, t_{i-1}, t_i', t_i'', t_{i+1}, \dots, t_n) \in \Delta(\sigma_i) \simeq \Delta^{n+1}$$

is mapped to

$$(t_0, \dots, t_{i-1}, t_i = t_i' + t_i'', t_{i+1}, \dots, t_n; s) \in \Delta^n \times [0,1], \qquad s = t_i'' + t_{i+1} + \dots + t_n.$$

Let us introduce new coordinates on Δ^n by sending $t \in \Delta^n$ to

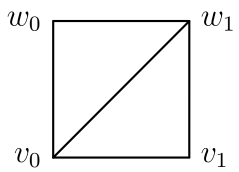
$$x = (x_1, \dots, x_n) \in \mathbb{R}^n, \qquad x_j = t_0 + \dots + t_{j-1},\ 1 \le j \le n.$$

In these coordinates Δ^n is given by $0 \le x_1 \le \dots \le x_n \le 1$. Note that $1 - s = t_0 + \dots + t_{i-1} + t_i'$, hence the above point in $\Delta^n \times [0,1]$ satisfies

$$0 \le x_1 \le \dots \le x_i \le 1 - s \le x_{i+1} \le \dots \le x_n.$$

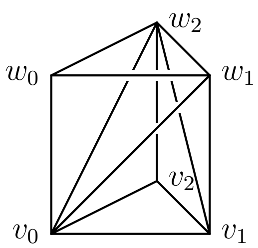
For any point in $\Delta^n \times [0,1]$, we can find an appropriate i such that this point is contained in the image of $\Delta(\sigma_i)$. This implies that f is surjective. Injectivity follows from the same analysis. ☐

Example 1.48. For $n = 1$, consider the triangulation of the square $\Delta^1 \times \Delta^1$.



The 1-simplex $[v_0, w_1]$ divides the square into triangles $[v_0, w_0, w_1]$, $[v_0, v_1, w_1]$. They correspond to the maximal chains of $P = [1] \times [1] = \{v_0, v_1, w_0, w_1\}$, where $v_i = (i,0)$ and $w_i = (i,1)$.

Example 1.49. For $n = 2$, consider the triangulation of the cylinder $\Delta^2 \times \Delta^1$.

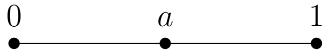


The 2-simplices $[v_0, w_1, w_2]$, $[v_0, v_1, w_2]$ divide the cylinder into three tetrahedrons, corresponding to the maximal chains $[v_0, w_0, w_1, w_2]$, $[v_0, v_1, w_1, w_2]$, $[v_0, v_1, v_2, w_2]$ of $P = [2] \times [1]$.

Remark 1.50. More generally, for two posets P_1, P_2, the product $|\Delta(P_1)| \times |\Delta(P_2)|$ is homeomorphic to $|\Delta(P)|$ for the product poset $P = P_1 \times P_2$.

1.6. **Barycentric subdivision.** Let K be a simplicial complex with the set of vertices V. Then $K \subset 2^V$ is a poset ordered by inclusion, and its order complex $\Delta(K)$ (consisting of the chains in K, see Example 1.36), is called the *barycentric subdivision* of K. It turns out that the geometric realizations of K and $\Delta(K)$ are homeomorphic.

Example 1.51. Let us denote a chain $\sigma = \{i_0 < \cdots < i_m\}$ by $[i_0, \dots, i_m]$ or just $i_0 \dots i_m$. For $K = \Delta[1] = \{0, 1, a = 01\}$, we have $\Delta(K) = \{0, 1, a, [0, a], [1, a]\}$, where $0, 1, a$ are the vertices of $\Delta(K)$. The geometric realization of $\Delta(K)$ is

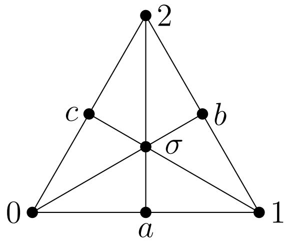


which is an interval, homeomorphic to $|K| = \Delta^1$.

Let $K = \Delta[2]$ (nonempty subsets of $[2]$) so that $|K| = \Delta^2$. Then

$$K = \{0,\ 1,\ 2,\ a = 01,\ b = 12,\ c = 02,\ \sigma = 012\}$$

is ordered by inclusion: $0, 1 < a < \sigma$ etc. Consider a triangulation



where we identify the simplices of K with their barycenters (centers of gravity), for example, $a = 01$ is identified with the barycenter of $\{0, 1\}$ and $\sigma = 012$ is identified with the barycenter of $\{0, 1, 2\}$. The simplices of $\Delta(K)$ (which are chains in K) can be identified with the simplices of the above triangulation. For example

(1) $\{0 < a < \sigma\} \subset K$ is identified with the triangle $[0, a, \sigma]$.
(2) $\{0, \sigma\}] \subset K$ is identified with the interval $[0, \sigma]$.
(3) $\{b\} \subset K$ is identified with the vertex b.

This implies that the geometric realization of $\Delta(K)$ can be identified with the geometric realization of K (both are given by the large triangle).

Theorem 1.52. *For a finite simplicial complex (V, K), there is a canonical homeomorphism*

$$\phi\colon\ |\Delta(K)| \to |K|\,, \qquad t = (t_\sigma)_{\sigma \in K} \mapsto \sum_{\sigma \in K} t_\sigma b_\sigma \in |K| \subset \mathbb{R}^V,$$

where b_σ is the barycenter (center of gravity) of $\Delta(\sigma)$ (with $e_v \in \mathbb{R}^V$ standard basis vectors)

$$b_\sigma = \tfrac{1}{n+1} \sum_{v \in \sigma} e_v \in \Delta(\sigma) \subset |K| \subset \mathbb{R}^V, \qquad \sigma \in K_n.$$

Proof. Let $L = \Delta(K)$ so that $L_0 = K$. Let $t = (t_\sigma)_{\sigma \in K} \in |L| \subset \mathbb{R}^K$. If $\operatorname{supp}(t) = \{\sigma_0 < \cdots < \sigma_n\} \in L$, then $\operatorname{supp}(\phi(t)) = \sigma_n$, hence $\phi(t) \in \mathring{\Delta}(\sigma_n) \subset |K|$. One can show that ϕ induces a bijection between $t \in |L|$ with $\max \operatorname{supp}(t) = \sigma_n$ and $\mathring{\Delta}(\sigma_n) \subset |K|$. Therefore $\phi\colon\ |L| \to |K|$ is a bijection (and a homeomorphism if K is finite). $\square$

1.7. **Simplicial homology.** Consider a Δ-set K (or a triangulated space (X,K)). For every $0 \le i \le n$, consider the coface map $\delta_i \colon [n-1] \to [n]$ which is an increasing map that misses $i \in [n]$ in the image. Consider the corresponding face map (see Remark 1.21)

$$\partial_i = \delta_i^* \colon K_n \to K_{n-1}, \qquad 0 \le i \le n.$$

By Lemma 1.22 we have

$$\partial_i \partial_j = \partial_{j-1} \partial_i, \qquad i < j.$$

Definition 1.53. Let K be a Δ-set.

(1) Define the *group of n-chains* $C_n = C_n(K)$ to be the free abelian group with a basis consisting of n-simplices of K. An element of $C_n(K)$, called an *n-chain*, is a finite sum $\sum_{\sigma \in K_n} n_\sigma \sigma$, where $n_\sigma \in \mathbb{Z}$.

(2) Define the *boundary operator* (also called the *differential*)

$$d = d_n \colon C_n \to C_{n-1}, \qquad d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \partial_i(\sigma).$$

(3) The sequence of homomorphisms between abelian groups

$$\dots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \dots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0 \to \dots$$

satisfies the equation $d_n d_{n+1} = 0$ for all n (see the next Lemma) and is called the *chain complex* of K, denoted as $C_\bullet(K)$, $C_\bullet$ or $(C_\bullet, d_\bullet)$.

(4) Given a triangulated space (X,K), define $C_\bullet^K(X) = C_\bullet(K)$.

Example 1.54. For a simplex $\sigma = [v_0, \dots, v_n]$, we have

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n].$$

For example $d[v_0, v_1] = v_1 - v_0$ and $d[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$.

Lemma 1.55. *The composition $C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$ is zero (also written as $d^2 = 0$).*

Proof. We have

$$\begin{aligned} d_n d_{n+1}(\sigma) &= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} \partial_i \partial_j(\sigma) = \sum_{i<j} (-1)^{i+j} \partial_i \partial_j(\sigma) + \sum_{i \ge j} (-1)^{i+j} \partial_i \partial_j(\sigma) \\ &= \sum_{i<j} (-1)^{i+j} \partial_{j-1} \partial_i(\sigma) + \sum_{i \ge j} (-1)^{i+j} \partial_i \partial_j(\sigma) \\ &= \sum_{i \le j} (-1)^{i+j+1} \partial_j \partial_i(\sigma) + \sum_{i \ge j} (-1)^{i+j} \partial_i \partial_j(\sigma) = 0 \end{aligned}$$

where we used the fact that $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$. $\square$

Note that the equation $d_n d_{n+1} = 0$ implies

$$\operatorname{Im} d_{n+1} \subset \operatorname{Ker} d_n.$$

Definition 1.56. Let K be a Δ-set and $C_\bullet = C_\bullet(K)$ be the corresponding chain complex.

(1) Define the n-th *homology group* of K (or of the complex $C_\bullet$) to be

$$H_n(K) = H_n(C) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}.$$

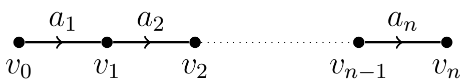
(2) The elements of $Z_n(C) = \operatorname{Ker} d_n \subset C_n$ are called *n-cycles* and the elements of $B_n(C) = \operatorname{Im} d_{n+1} \subset C_n$ are called *n-boundaries*.

(3) If (X,K) is a triangulated space, define the n-th *simplicial homology group* of X to be $H_n^K(X) = H_n(K)$ which is the n-homology group of $C_\bullet^K(X)$.

Remark 1.57. More generally, for any ring R, we define the chain complex $C_\bullet(K,R)$ with $C_n(K,R) = R \otimes_{\mathbb{Z}} C_n(K)$ (the free R-module with the basis K_n) and simplicial homology groups (or R-modules) $H_n(K,R) = H_n(C_\bullet(K,R))$. Note that $H_n(K,R) \not\simeq R \otimes_{\mathbb{Z}} H_n(K)$ in general, although this is true if R is a field of characteristic zero (or does not have torsion over $\mathbb{Z}$).

Remark 1.58. We will show later that the simplicial homology groups $H_n^K(X)$ are independent of the triangulation K. But this statement will require a lot of work.

Example 1.59. Consider a sequence of 1-simplices (called a path)

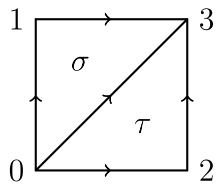


and a 1-chain $c = a_1 + \cdots + a_n \in C_1$ corresponding to this path. Then

$$d(c) = \sum_{i=1}^{n} (v_i - v_{i-1}) = v_n - v_0.$$

This means that c is a 1-cycle if and only if $v_n = v_0$, that is, the above path is an actual cycle.

Example 1.60. The signs are used to take orientation into account. Consider a triangulation of a square



with $\sigma = [0,1,3]$ (clockwise) and $\tau = [0,2,3]$ (anti-clockwise). Then

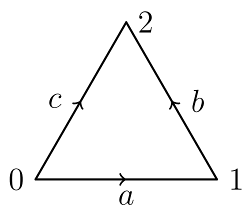
$$d(\sigma) = [1,3] - [0,3] + [0,1] = [0,1] + [1,3] - [0,3]$$

which corresponds to the boundary of σ going clockwise along the cyclic path $0 \to 1 \to 3 \to 0$. This is why $d(\sigma)$ is called a 1-boundary. On the other hand

$$d(\sigma - \tau) = [0,1] + [1,3] - [2,3] - [0,2]$$

corresponds to the boundary of the square going clockwise along the cyclic path $0 \to 1 \to 3 \to 2 \to 0$.

Example 1.61. Consider the Δ-set $K = \partial\Delta[2]$ which is a triangulation of S^1



Then

$$d(a) = [1] - [0], \qquad d(b) = [2] - [1], \qquad d(c) = [2] - [0]$$

Therefore $d(a + b - c) = 0$, hence $a + b - c$ is a 1-cycle. It corresponds to the cyclic path $0 \to 1 \to 2 \to 0$, hence the name. Let us compute homologies. We have $C_0 = \mathbb{Z}^3$, $C_1 = \mathbb{Z}^3$ with $d = d_1 \colon C_1 \to C_0$ described above. Then $\operatorname{Ker} d_1 = (a+b-c)\mathbb{Z}$, hence $H_1(C) = \operatorname{Ker} d_1 / \operatorname{Im} d_2 \simeq \mathbb{Z}$.

We have $n_0[0]+n_1[1]+n_2[2] \in \operatorname{Im} d_1 \iff n_0+n_1+n_2=0$. Therefore $H_0(C)=\mathbb{Z}^3/\operatorname{Im} d_1 \simeq \mathbb{Z}$, where $n_0[0]+n_1[1]+n_2[2] \mapsto n_0+n_1+n_2$. We conclude that

$$H_0^K(S^1) \simeq \mathbb{Z}, \qquad H_1^K(S^1) \simeq \mathbb{Z}.$$

Example 1.62. Consider a triangulation of S^1 consisting of a vertex v and a segment $e=[v,v]$. Then $C_0=\mathbb{Z}v$, $C_1=\mathbb{Z}e$ and

$$d\colon C_1 \to C_0, \qquad de = v - v = 0$$

with all other differentials equal zero. This implies that

$$H_1(C)=\operatorname{Ker} d/0=\mathbb{Z}, \qquad H_0(C)=\mathbb{Z}/\operatorname{Im} d=\mathbb{Z}$$

and all other homology groups are zero. We conclude that $H_0^K(S^1)=H_1^K(S^1)=\mathbb{Z}$.

Let us consider a different triangulation of S^1, with two vertices v_0, v_1 and two segments $e=[v_0,v_1]$, $e'=[v_1,v_0]$ between them. Then $C_0=\mathbb{Z}^2$, $C_2=\mathbb{Z}^2$ and $d(e)=v_0-v_1$, $d(e')=v_1-v_0$ can be written as a matrix

$$d\colon C_1 \to C_0, \qquad d=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then $\operatorname{Ker} d=\mathbb{Z}(1,1)$, $\operatorname{Im} d=\mathbb{Z}(1,-1)$ and

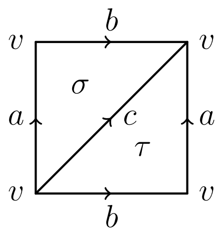
$$H_1(C)=\operatorname{Ker} d/0 \simeq \mathbb{Z}, \qquad H_0(C)=\mathbb{Z}^2/\operatorname{Im} d \simeq \mathbb{Z}.$$

We see that the homology groups are the same as before, although our chain complex is different. We will see later that this is a general phenomenon. Therefore it is enough to find the simplest possible triangulation of a space in order to compute its homologies.

Example 1.63. $H_n(\Delta^1)=\mathbb{Z}$ for $n=0$ and zero otherwise. $H_n(\Delta^2)=\mathbb{Z}$ for $n=0$ and zero otherwise. More generally, one can show that $H_n(\Delta^k)=\mathbb{Z}$ for $n=0$ and zero otherwise. The reason is that Δ^n is homotopic to a point.

Example 1.64. We can triangulate S^2 by taking two copies of Δ^2 and identifying them along the boundary. Similarly, S^n is obtained by taking two copies σ,τ of Δ^n and identifying them along the boundary. We can show that $H_n(S^n)\simeq\mathbb{Z}$. Let us compute homology groups for S^2.

Example 1.65. Consider the following triangulation of the torus T



with vertices of σ going along a,b and vertices of τ going along b,a. Then $d_2(\sigma)=b-c+a$, $d_2(\tau)=a-c+b$, $d_1(a)=d_1(b)=d_1(c)=0$

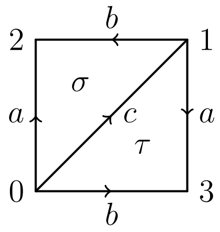
$$\mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z}.$$

Therefore $H_0(T)\simeq\mathbb{Z}$, $H_1(T)=\mathbb{Z}^3/\operatorname{Im} d_2\simeq\mathbb{Z}^2$, $H_2(T)=\operatorname{Ker} d_2\simeq\mathbb{Z}$.

Example 1.66. Define an n-dimensional projective space

$$\mathbb{R}P^n=(\mathbb{R}^{n+1}\backslash\{0\})/\mathbb{R}^* \simeq S^n/\mathbb{Z}_2,$$

where $\mathbb{R}^*$ acts on $\mathbb{R}^{n+1}$ by multiplication and $\mathbb{Z}_2$ acts on S^n by the antipodal map $x\mapsto -x$. For example, we can describe $\mathbb{R}P^2$ as the quotient of the disc D^2 with antipodal points on the boundary identified. We construct a triangulation of $\mathbb{R}P^2$ by identifying arrows in the following picture, where antipodal points on the boundary are identified.



Let $v = 0 \sim 1$, $w = 2 \sim 3$. We consider $\sigma = [012]$ and $\tau = [013]$ so that

$$d_2(\sigma) = b - a + c, \quad d_2(\tau) = a - b + c, \qquad d_1(a) = d_1(b) = w - v, \quad d_1(c) = 0.$$

The corresponding chain complex is

$$\begin{array}{ccccc} C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 \\ \| & & \| & & \| \\ \mathbb{Z}^2 & \xrightarrow{d_2} & \mathbb{Z}^3 & \xrightarrow{d_1} & \mathbb{Z}^2 \end{array} \qquad d_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}, \; d_1 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

We have

$$\operatorname{Im}(d_2) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \mathbb{Z}, \qquad \operatorname{Ker} d_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathbb{Z}.$$

Therefore $H_1(K) = \operatorname{Ker} d_1 / \operatorname{Im} d_2 \simeq \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$. The map d_2 is injective, hence $H_2(K) = \operatorname{Ker} d_2 = 0$. Finally, $H_0(K) = \mathbb{Z}^2 / \operatorname{Im} d_1 \simeq \mathbb{Z}$. Note that over rational numbers we have $H_1(K, \mathbb{Q}) \simeq \mathbb{Q}/2\mathbb{Q} = 0$.

Example 1.67. Let us assume that we have a triangulated surface (X, K). Consider its chain complex $C_n = C_n^K(X, \mathbb{Q})$ with rational coefficients

$$\cdots \to 0 \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0 \to \dots$$

and the corresponding homology groups H_0, H_1, H_2. It's a simple exercise in linear algebra to show that

$$e(X) = \#K_0 - \#K_1 + \#K_2 = \dim C_0 - \dim C_1 + \dim C_2 = \dim H_0 - \dim H_1 + \dim H_2.$$

As we will see later, the vector spaces H_n are independent of a triangulation. In particular, the above number $e(X)$, called the *Euler characteristic* of X, is independent of a triangulation. Compare this with Example 1.1, where we considered a triangulation of S^2. Of course all the arguments work also for higher dimensional spaces.

2. Homological algebra

Let R be a ring. Recall that an R-module M is an abelian group equipped with a map

$$R \times M \to M, \qquad (a, m) \mapsto am,$$

such that, for all $a, b \in R$ and $x, y \in M$,

(1) $a(bx) = (ab)x$.
(2) $1_R x = x$.
(3) $a(x + y) = ax + ay$.
(4) $(a + b)x = ax + ay$.

If $R = \mathbb{Z}$, then an R-module is just an abelian group. If R is a field (for example $\mathbb{R}$ or $\mathbb{C}$), then an R-module is a vector space over R. A map $f\colon M \to N$ between two R-modules is called a homomorphism (or an R-linear map) if for all $a \in R$ and $x, y \in M$

(1) $f(ax) = af(x)$.
(2) $f(x + y) = f(x) + f(y)$.

Definition 2.1.

(1) A *(chain) complex* $C_\bullet$ (or $(C_\bullet, d_\bullet)$) is a sequence of R-modules and homomorphisms

$$\dots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \dots$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$ (also written as $d^2 = 0$). The map d_n is called a *differential* (or a *boundary operator*).

(2) The elements of C_n are called *n-chains*. The elements of

$$Z_n(C) = \operatorname{Ker} d_n, \qquad B_n(C) = \operatorname{Im} d_{n+1}$$

are called *n-cycles* and *n-boundaries* respectively. Note that $d^2 = 0$ implies $B_n(C) \subset Z_n(C)$. The module

$$H_n(C) = Z_n(C)/B_n(C) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$$

is called the n-th *homology group* (or homology module). Its elements are called the n-th *homology classes*.

Definition 2.2.

(1) A sequence of homomorphisms $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ is called *exact* at M_2 if $\operatorname{Im} f = \operatorname{Ker} g$.
(2) A complex $C_\bullet$ is called *exact* (or *acyclic*) if it is exact at every C_n. Equivalently, $H_n(C) = 0$ for all n.

Lemma 2.3.

(1) A sequence $0 \to M_1 \xrightarrow{f} M_2$ is exact at M_1 $\iff$ f is a monomorphism.
(2) A sequence $M_1 \xrightarrow{f} M_2 \to 0$ is exact at M_2 $\iff$ f is an epimorphism.
(3) A sequence $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ with $gf = 0$ is exact (at all components) $\iff$ f is a monomorphism, g is an epimorphism and $\operatorname{Im} f = \operatorname{Ker} g$. Such sequence is called a short exact sequence.
(4) If $M' \subset M$ is a submodule, then $0 \to M' \to M \to M/M' \to 0$ is a short exact sequence.

Definition 2.4. A *chain map* $f\colon A_\bullet \to B_\bullet$ between complexes $A_\bullet, B_\bullet$ is a collection of homomorphisms $f_n\colon A_n \to B_n$ for $n \in \mathbb{Z}$ such that $d_n f_n = f_{n-1} d_n$ $\forall n$. This means that we have a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & B_{n+1} \arrow[r, "d_{n+1}"] & B_n \arrow[r, "d_n"] & B_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

We will always assume that a map between complexes is a chain map unless otherwise stated.

Lemma 2.5. *A chain map $f\colon A_\bullet \to B_\bullet$ induces homomorphisms $f_* = H_n f\colon H_n(A) \to H_n(B)$ between homology modules.*

Proof. Given $x \in Z_n(A) = \operatorname{Ker}(d\colon A_n \to A_{n-1})$, we have $df(x) = fd(x) = 0$, hence $f(x) \in Z_n(B) = \operatorname{Ker}(d\colon B_n \to B_{n-1})$. We define then $f_*(x) = [f(x)] \in H_n(B) = Z_n(B)/B_n(B)$. We need to show that this definition is independent of the choice of x in its homology class, that is, $f_*(x + dy) = f_*(x)$ for $y \in A_{n+1}$. But $f_*(dy) = [fd(y)] = [df(y)] = 0$ as $df(y) \in B_n(B)$. □

Definition 2.6.

(1) Let $f, g\colon A_\bullet \to B_\bullet$ be two chain maps. A *chain homotopy* s between f and g (or from f to g, written as $s\colon f \sim g$) is a sequence of homomorphisms $s_n\colon A_n \to B_{n+1}$ such that

$$d_{n+1}s_n + s_{n-1}d_n = g_n - f_n, \qquad \forall n \in \mathbb{Z}$$

$$\begin{tikzcd}
\dots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] \arrow[dl, "s_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"'] & \dots \\
\dots \arrow[r] & B_{n+1} \arrow[r, "d_{n+1}"] & B_n \arrow[r, "d_n"] & B_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

Chain maps f, g are called *homotopic* (written as $f \sim g$) if such s exists.

(2) A chain map $f\colon A_\bullet \to B_\bullet$ is called *null-homotopic* if $f \sim 0$.

(3) A chain map $f\colon A_\bullet \to B_\bullet$ is called a *homotopy equivalence* if there exists a chain map $g\colon B_\bullet \to A_\bullet$ such that $fg \sim 1_B$ and $gf \sim 1_A$. In this case g is called a *homotopy inverse* of f and $A_\bullet, B_\bullet$ are called *homotopy equivalent complexes*.

(4) A chain complex $A_\bullet$ is called *contractible* (or *null-homotopic*) if $1_A \sim 0$.

Remark 2.7. A chain complex $A_\bullet$ is null-homotopic $\iff$ $A_\bullet$ is homotopy equivalent to the zero complex. Indeed, consider the unique chain maps $f = 0\colon A_\bullet \to 0$ and $g = 0\colon 0 \to A_\bullet$. If $1_A \sim 0$, then $gf = 0 \sim 1_A$ and $fg = 0 = 1_0$. Therefore f is a homotopy equivalence. Conversely, if $f\colon A_\bullet \to 0$ has a homotopy inverse $g\colon 0 \to A_\bullet$, then $0 = gf \sim 1_A$.

Lemma 2.8.

(1) The homotopy relation is an equivalence relation.

(2) The homotopy relation is compatible with composition: if $f, g\colon A_\bullet \to B_\bullet$ are homotopic and $f', g'\colon B_\bullet \to C_\bullet$ are homotopic, then $f'f \sim g'g$.

Proof. cl 1 It is clear that $f \sim f$ and if $s\colon f \sim g$, then $-s$ is a homotopy between g and f: $f - g = d(-s) + (-s)d$. We need to show transitivity: if $s\colon f \sim g$ and $t\colon g \sim h$, then $f \sim h$. We have $h - f = (g - f) + (h - g) = d(s + t) + (s + t)d$, hence $s + t$ is a homotopy between f and h.

cl 2 Let us show that $f'f \sim f'g$. Similarly one can show that $f'g \sim g'g$ and then by transitivity $f'f \sim g'g$. If $s\colon f \sim g$, then

$$f'g - f'f = f'(ds + sd) = d(f's) + (f's)d,$$

where we used the property $f'd = df'$. This implies that $f's$ is a homotopy between $f'f, f'g$. □

Lemma 2.9. *If $f, g\colon A_\bullet \to B_\bullet$ are homotopic, then they induce the same morphisms of homologies: $f_* = g_*\colon H_n(A) \to H_n(B)$.*

Proof. Let $s\colon f \sim g$. Given $x \in Z_n(A) = \operatorname{Ker}(d_n\colon A_n \to A_{n-1})$, we need to show that $f(x), g(x) \in Z_n(B)$ coincide modulo $B_n(B) = \operatorname{Im}(d_{n+1}\colon B_{n+1} \to B_n)$. But $gx - fx = (g - f)(x) = dsx + sdx = dsx \in \operatorname{Im} d_{n+1}$, where we used the fact that $dx = 0$. □

Corollary 2.10. *A homotopy equivalence induces an isomorphism between homology groups.*

Proof. Let $f\colon A_\bullet \to B_\bullet$ have a homotopy inverse $g\colon B_\bullet \to A_\bullet$. Then $fg \sim 1_B$ implies $f_*g_* = 1$ and $gf \sim 1_A$ implies $g_*f_* = 1$. This means that $f_*\colon H_n(A) \to H_n(B)$ is an isomorphism. □

Theorem 2.11 (Snake lemma)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
 & \operatorname{Ker} f \arrow[r] \arrow[d] & \operatorname{Ker} g \arrow[r] \arrow[d] & \operatorname{Ker} h \arrow[d] \arrow[dddll, bend left, "\partial"] & \\
 & X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "p"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r, "j"] \arrow[d] & Y' \arrow[r, "q"] \arrow[d] & Z' \arrow[d] & \\
 & \operatorname{Coker} f \arrow[r] & \operatorname{Coker} g \arrow[r] & \operatorname{Coker} h &
\end{tikzcd}$$

with exact middle rows. Then there is an exact sequence

$$\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \xrightarrow{\partial} \operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h,$$

where ∂ is defined by $j^{-1}gp^{-1}$.

Proof. We will show exactness only at $\operatorname{Ker} g$ and $\operatorname{Ker} h$. First let us show that ∂ is well defined. For $z \in \operatorname{Ker} h$ choose $y \in Y$ with $py = z$. Then $qgy = hpy = hz = 0 \implies gy = jx'$ for a unique $x' \in X'$. Then we define $\partial(z) = [x'] = x' + \operatorname{Im} f$. If $\bar{y} \in Y$ is another element with $p\bar{y} = z \implies p(\bar{y} - y) = 0 \implies \bar{y} - y = ix$ for some $x \in X \implies g\bar{y} - gy = gix = jfx \implies g\bar{y} = j(x' + fx)$ and we note that x' and $x' + fx$ are equivalent modulo $\operatorname{Im} f$.

To prove exactness at $\operatorname{Ker} g$, consider $y \in \operatorname{Ker} g$ with $py = 0$. Then $y = ix$ for some $x \in X$ $\implies jfx = gix = gy = 0 \implies fx = 0 \implies x \in \operatorname{Ker} f$.

To prove exactness at $\operatorname{Ker} h$, we first note that $\partial p = 0$. Indeed, if $z = py$ for some $y \in \operatorname{Ker} g$, then by the above construction $\partial(z) = [x']$, where $jx' = gy = 0$, hence $x' = 0$. This implies that $\partial p(y) = \partial(z) = 0$. On the other hand, assume that $z \in \operatorname{Ker} \partial$. Using the above notation we conclude that $x' \in \operatorname{Im} f \implies x' = fx$ for some $x \in X$. Then $gy = jx' = jfx = gix \implies \bar{y} = y - ix \in \operatorname{Ker} g$ and $p\bar{y} = py - pix = py = z$. This implies that $z \in \operatorname{Im}(p\colon \operatorname{Ker} g \to \operatorname{Ker} h)$. □

Definition 2.12. A sequence of chain maps

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

is called a short exact sequence of complexes if

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

are short exact sequences of modules for all n.

Theorem 2.13 (Long exact sequence)**.** *Consider a short exact sequence of complexes*

$$0 \to A_\bullet \xrightarrow{f} D_\bullet \xrightarrow{g} C_\bullet \to 0.$$

Then there is a long exact sequence of homology groups:

$$\cdots \to H_n(A) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f_*} H_{n-1}(D) \xrightarrow{g_*} H_{n-1}(C) \to \dots$$

Proof. We will apply the Snake lemma to the following diagram

$$\begin{tikzcd}
 & A_n/B_n(A) \arrow[r, "f"] \arrow[d, "d"] & D_n/B_n(D) \arrow[r, "g"] \arrow[d, "d"] & C_n/B_n(C) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f"] & Z_{n-1}(D) \arrow[r, "g"] & Z_{n-1}(C) &
\end{tikzcd}$$

We need to check that the rows are exact. Let us show that the bottom row is exact. It is clear that $Z_{n-1}(A) \to Z_{n-1}(D)$ is injective as $A_{n-1} \to D_{n-1}$ is injective. Let us show exactness at $Z_{n-1}(D)$. Assume that $x \in Z_{n-1}(D)$ is such that $g(x) = 0$. Then there exists $y \in A_{n-1}$ such that $f(y) = 0$. Then $fd(y) = df(y) = 0$, hence $d(y) = 0$ as f is injective. This implies that $y \in Z_{n-1}(A)$, hence the bottom sequence is exact at $Z_{n-1}(D)$.

Note that

$$\operatorname{Ker}(d\colon A_n/B_n(A) \to Z_{n-1}(A)) \sim Z_n(A)/B_n(A) = H_n(A),$$
$$\operatorname{Coker}(d\colon A_n/B_n(A) \to Z_{n-1}(A)) \sim Z_{n-1}(A)/B_{n-1}(A) = H_{n-1}(A)$$

and the same applies to other vertical maps. From the Snake lemma we obtain an exact sequence

$$H_n(A) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f_*} H_{n-1}(D) \xrightarrow{g_*} H_{n-1}(C)$$

□

Theorem 2.14 (Five-lemma). *Consider a commutative diagram with exact rows:*

$$\begin{tikzcd}
A_1 \arrow[r, "d"] \arrow[d, "f_1"] & A_2 \arrow[r, "d"] \arrow[d, "f_2"] & A_3 \arrow[r, "d"] \arrow[d, "f_3"] & A_4 \arrow[r, "d"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "d"] & B_2 \arrow[r, "d"] & B_3 \arrow[r, "d"] & B_4 \arrow[r, "d"] & B_5
\end{tikzcd}$$

If f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is also an isomorphism.

Proof. We will show that f_3 is injective using the so-called diagram chasing. Let $x \in A_3$ and $f_3x = 0 \implies 0 = df_3x = f_4dx \implies dx = 0 \implies x = dy$ for some $y \in A_2 \implies 0 = f_3x = f_3dy = df_2y \implies f_2y = dz$ for some $z \in B_1$. Let $z = f_1z'$ for some $z' \in A_1$. Then $f_2y = df_1z' = f_2dz' \implies y = dz' \implies x = dy = d^2z' = 0$. This means that $\operatorname{Ker} f_3 = 0$. □

3. Singular homology

3.1. Definition and basic properties.

Definition 3.1. Let X be a space.

(1) A continuous map $\sigma\colon \Delta^n \to X$ is called a *singular n-simplex* of X.
(2) Let $S_n(X)$ be the set of singular n-simplices of X. For any increasing $f\colon [m] \to [n]$, define

$$f^*\colon S_n(X) \to S_m(X), \qquad [\Delta^n \xrightarrow{\sigma} X] \mapsto [\Delta^m \xrightarrow{\Delta^f} \Delta^n \xrightarrow{\sigma} X].$$

In particular, define $\partial_i = \delta_i^*\colon S_n(X) \to S_{n-1}(X)$, where $\delta_i\colon [n-1] \to [n]$ is the i-th facet. We obtain a Δ-set $S(X)$, called a *singular Δ-set*.
(3) Define the *singular chain complex* $C(X)$ with the group of *n-chains* $C_n(X)$ equal the free abelian group generated by singular n-simplices and *differential* (also called a *boundary operator*),

$$d\colon C_n(X) \to C_{n-1}(X), \qquad d(\sigma) = \sum_{i=0}^{n} (-1)^i \partial_i(\sigma).$$

(4) For any abelian group R, define the chain complex $C(X, R)$ with the group of n-chains

$$C_n(X, R) = C_n(X) \otimes_{\mathbb{Z}} R.$$

If R is a ring, then $C_n(X, R)$ is equal to the free R-module generated by singular n-simplices. We have $C_n(X, \mathbb{Z}) = C_n(X)$.

Remark 3.2. Note that $C(X)$ is the chain complex associated with the Δ-set $S(X)$, hence we automatically have $d^2 = 0$ by Lemma 1.55.

Remark 3.3.* Let **ssSet** be the category of Δ-sets and Top be the category of topological spaces. Then the above construction produces a functor $S\colon \mathrm{Top} \to \mathbf{ssSet}$, $X \mapsto S(X)$. On the other hand we have the geometric realization functor $\mathbf{ssSet} \to \mathrm{Top}$, $K \mapsto |K|$. One can show that $\mathrm{Top}(|K|, X) \simeq \mathbf{ssSet}(K, S(X))$, meaning that these functors are adjoint to each other.

Definition 3.4. Define the *singular homology groups* of X:

$$H_n(X) = H_n(C(X)), \qquad n \geq 0.$$

More generally, define $H_n(X, R) = H_n(C(X, R))$ for any abelian group R.

Proposition 3.5. *Let $X = \bigsqcup_\alpha X_\alpha$ be a decomposition of X into its connected components. Then $H_n(X) \simeq \bigoplus_\alpha H_n(X_\alpha)$.*

Proof. Every singular n-simplex $\sigma\colon \Delta^n \to X$ has its image in one connected component. This implies that $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$. The differential d takes $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$, hence there is a similar splitting of $\operatorname{Ker} d$ and $\operatorname{Im} d$, for example,

$$\operatorname{Ker}\Big(C_n(X_\alpha) \xrightarrow{d} C_{n-1}(X_\alpha)\Big) = \bigoplus_\alpha \operatorname{Ker}\Big(C_n(X_\alpha) \xrightarrow{d} C_{n-1}(X_\alpha)\Big).$$

This implies that the same is true for homology groups, hence $H_n(X) \simeq \bigoplus_\alpha H_n(X_\alpha)$. $\square$

Proposition 3.6. *If $X \neq \varnothing$ is path-connected, then $H_0(X) \simeq \mathbb{Z}$. Generally, if $X = \bigsqcup_\alpha X_\alpha$ is a decomposition into connected components, then $H_0(X) \simeq \bigoplus_\alpha \mathbb{Z}$.*

Proof. The singular chain complex has the form

$$\dots \to C_1(X) \xrightarrow{d_1} C_0(X) \to 0 \to \dots,$$

hence $H_0(X) = C_0(X)/\operatorname{Im} d_1$. We can identify Δ^0 with a point and Δ^1 with an interval $[0,1]$. Then a 0-simplex can be identified with a point $x \in X$ and a 1-simplex can be identified with a path $\gamma\colon [0,1] \to X$. Under this identification $d_1(\gamma) = \gamma(1) - \gamma(0) \in C_0(X)$. Consider a homomorphism

$$\varepsilon\colon C_0(X) \to \mathbb{Z}, \qquad \sum n_i x_i \mapsto \sum n_i.$$

This map is surjective as $X \neq \varnothing$. We will show that $\operatorname{Ker}\varepsilon = \operatorname{Im} d_1$ and this would imply that $\mathbb{Z} \simeq C_0(X)/\operatorname{Ker}\varepsilon \simeq C_0(X)/\operatorname{Im} d_1$ as required.

We have $\varepsilon d_1(\gamma) = \varepsilon(\gamma(1) - \gamma(0)) = 0$, hence $\operatorname{Im} d_1 \subset \operatorname{Ker}\varepsilon$. Assume that $\sum_i n_i x_i \in \operatorname{Ker}\varepsilon$, which means that $\sum_i n_i = 0$. Choose a basepoint $x_0 \in X$ and a path $\gamma_i\colon [0,1] \to X$ from x_0 to x_i (meaning that $\gamma(0) = x_0$ and $\gamma(1) = x_i$) for every i. Then $d_1(\sum n_i\gamma_i) = \sum n_i(x_i - x_0) = \sum n_i x_i - (\sum n_i)x_0 = \sum n_i x_i$. This implies that $\operatorname{Ker}\varepsilon \subset \operatorname{Im} d_1$. □

Proposition 3.7. *If X is a point, then*

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n > 0. \end{cases}$$

Proof. If X is a point, there is a unique singular n-simplex $\sigma_n\colon \Delta^n \to X$. Moreover, we have $\partial_i(\sigma_n) = \sigma_{n-1}$, hence $d_n(\sigma_n) = \sum_{i=0}^n (-1)^i \sigma_{n-1} = 0$ for n odd and equals σ_{n-1} for n even. This means that the singular chain complex $C(X)$ has the form

$$\dots \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

We see that for $n > 0$ we have $H_n(X) = \operatorname{Ker}(0)/\operatorname{Im}(\mathrm{id}) = \mathbb{Z}/\mathbb{Z} = 0$ if n is odd and $H_n(X) = \operatorname{Ker}(\mathrm{id})/\operatorname{Im}(0) = 0$ if n is even. We also have $H_0(X) = \mathbb{Z}$. □

Definition 3.8. Given a nonempty space X, define the *augmented chain complex*

$$\dots \to C_2(X) \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon(\sum_i n_i x_i) = \sum_i n_i$. Its homology groups $\tilde{H}_n(X)$ are called *reduced homology groups*.

Remark 3.9. We have $\tilde{H}_n(X) = H_n(X)$ for $n > 0$. On the other hand $\varepsilon_*\colon H_0(X) \to \mathbb{Z}$ is surjective and $\tilde{H}_0(X) = \operatorname{Ker}\varepsilon_*$. If X is path-connected, then $\tilde{H}_0(X) = 0$. In general there is a (non-canonical) isomorphism $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$.

Example 3.10. For a 1-point set $\{x\}$, we have $\tilde{H}_n(\{x\}) = 0$ for all n.

3.2. **Homotopy invariance.** Let $f\colon X \to Y$ be a continuous map between two spaces. It induces a map $f_*\colon S(X) \to S(Y)$ between Δ-sets, where a simplex $\sigma\colon \Delta^n \to X$ is sent to $f_\sharp(\sigma) = f \circ \sigma\colon \Delta^n \to Y$. It also induces a map $f_\sharp\colon C(X) \to C(Y)$ between chain complexes.

Proposition 3.11. *The map $f_\sharp\colon S(X) \to S(Y)$ is a simplicial morphism 1.27.*

Proof. Given an increasing map $\phi\colon [m] \to [n]$, we need to show that the following diagram commutes

$$\begin{array}{ccc} S_n(X) & \xrightarrow{\phi^*} & S_m(X) \\ \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} \\ S_n(Y) & \xrightarrow{\phi^*} & S_m(Y) \end{array}$$

For any $\sigma\colon \Delta^n \to X$, we have $f_\sharp\phi^*(\sigma) = f_\sharp(\sigma\Delta^\phi) = f \circ (\sigma\Delta^\phi)$ and $\phi^* f_\sharp(\sigma) = \phi^*(f\sigma) = (f\sigma)\circ\Delta^\phi$, where $\Delta^\phi\colon \Delta^m \to \Delta^n$. $\square$

Proposition 3.12. *The map $f_\sharp\colon C(X) \to C(Y)$ is a chain map 2.4. Therefore it induces a homomorphism $f_* = H_n(f_\sharp)\colon H_n(X) \to H_n(Y)$ for all $n \geq 0$.*

Proof. By the previous result we have $f_\sharp\partial_i = \partial_i f_\sharp$, where $\partial_i = \delta_i^*$ and $\delta_i\colon [n-1] \to [n]$ is the i-th face. This implies that $f_\sharp d_n = d_n f_\sharp$, where $d_n = \sum_{i=0}^n (-1)^i \partial_i$.

$$\begin{array}{ccc} C_n(X) & \xrightarrow{d_n} & C_{n-1}(X) \\ \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} \\ C_n(Y) & \xrightarrow{d_n} & C_{n-1}(Y) \end{array}$$

$\square$

Definition 3.13. Let X, Y be two spaces.

(1) Two maps $f, g\colon X \to Y$ are called *homotopic* (written as $f \sim g$) if there exists a (continuous) map $F\colon X \times I \to Y$ such that $F_0 = f$ and $F_1 = g$, where $F_t(x) = F(x,t)$. The map F as above is called a *homotopy* between f and g.
(2) A map $f\colon X \to Y$ is called *null-homotopic* if it is homotopic to a constant map.
(3) A map $f\colon X \to Y$ is called a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $gf \sim \mathbb{1}_X$ and $fg \sim \mathbb{1}_Y$. The spaces X, Y are called *homotopy equivalent* in this case.
(4) A space X is called *contractible* (or *null-homotopic*) if $\mathbb{1}_X$ is null-homotopic, meaning that there exists a (continuous) map $F\colon X \times I \to X$ such that $F_0 = \mathbb{1}_X$ and F_1 is a constant map. Equivalently, X is homotopy equivalent to a point.

Theorem 3.14. *If two continuous maps $f, g\colon X \to Y$ are homotopic, then they induce the same homomorphisms $f_* = g_*\colon H_n(X) \to H_n(Y)$.*

Proof. For every $t \in I = \Delta^1$, consider the map $\eta^t\colon X \to X \times I$, $x \mapsto (x,t)$. In particular, consider chain maps $\eta^0_\sharp, \eta^1_\sharp\colon C(X) \to C(X \times I)$. We will construct a chain homotopy $P\colon \eta^0_\sharp \sim \eta^1_\sharp$, that is, maps $P\colon C_n(X) \to C_{n+1}(X \times I)$, called prism operators, satisfying

$$dP + Pd = \eta^1_\sharp - \eta^0_\sharp.$$

If $F\colon X \times I \to Y$ is a homotopy between f, g, then $f = F\eta^0$ and $g = F\eta^1$. Therefore we obtain a chain homotopy $F_\sharp P$ between $f_\sharp = F_\sharp \eta^0_\sharp$ and $g_\sharp = h_\sharp \eta^1_\sharp$. This implies that $f_* = g_*$.

A singular simplex $\sigma\colon \Delta^n \to X$ induces a map

$$(\sigma \times \mathbb{1})_\sharp\colon C_{n+1}(\Delta^n \times I) \to C_{n+1}(X \times I).$$

We will construct

$$P(\sigma) = (\sigma \times \mathbb{1})_\sharp(a_n) \in C_{n+1}(X \times I),$$

where $a_n \in C_{n+1}(\Delta^n \times I)$ is a certain chain corresponding to a triangulation of $\Delta^n \times I$ (the prism). Then the equation

$$dP = \eta^1_\sharp - \eta^0_\sharp - Pd$$

can be interpreted as a statement that the boundary of $\Delta^n \times I$ can be represented as a combination of $\Delta^n \times \{1\}$, $\Delta^n \times \{0\}$ and the sides $\partial\Delta^n \times I$.

We will use the triangulation of $\Delta^n \times I$ constructed in Theorem 1.47. In particular, the parametrization of maximal simplices of $\Delta^n \times I$ that appeared there. Consider the vectors $v_i = (e_i, 0)$ and $w_i = (e_i, 1)$ in $\Delta^n \times I$ and the singular $n+1$-simplices $[v_0, \dots, v_i, w_i, \dots, w_n] \in C_{n+1}(\Delta^n \times I)$ (cf. Theorem 1.47). Define

$$a_n = \sum_{i=0}^{n} (-1)^i [v_0, \dots, v_i, w_i, \dots, w_n] \in C_{n+1}(\Delta^n \times I)$$

and $P(\sigma) = (\sigma \times \mathbb{1})_\sharp(a_n)$. Then

$$\begin{aligned} dP(\sigma) = \sum_{j \le i} &(-1)^{i+j} (\sigma \times \mathbb{1})|[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n] \\ &+ \sum_{j \ge i} (-1)^{i+(j+1)} (\sigma \times \mathbb{1})|[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]. \end{aligned}$$

On the other hand

$$\begin{aligned} Pd(\sigma) = \sum_j (-1)^j P(\partial_j \sigma) &= \sum_j (-1)^j (\partial_j \sigma \times \mathbb{1})_\sharp(a_{n-1}) \\ &= \sum_{j<i} (-1)^{j+(i-1)} (\sigma \times \mathbb{1})|[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n] \\ &\quad + \sum_{j>i} (-1)^{j+i} (\sigma \times \mathbb{1})|[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]. \end{aligned}$$

The sum $dP(\sigma) + Pd(\sigma)$ contains only summands with $i = j$ and all of them cancel except the following two:

$$(\sigma \times \mathbb{1})|[\hat{v}_0, w_0, \dots, w_n] - (\sigma \times \mathbb{1})|[v_0, \dots, v_n, \hat{w}_n] = \eta^1\sigma - \eta^0\sigma.$$

We conclude that $dP + Pd = \eta^1_\sharp - \eta^0_\sharp$. □

Corollary 3.15. *We have*

(1) If $f\colon X \to Y$ is a homotopy equivalence, then $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism (the same is true for the reduced homologies).
(2) If X is contractible, then $\tilde{H}_n(X) = 0$ for all n.

Proof. cl 1 Let $g\colon Y \to X$ be homotopy inverse to f, meaning that $gf \sim \mathbb{1}_X$ and $fg \sim \mathbb{1}_Y$. This implies that $(gf)_* = \mathbb{1}\colon H_n(X) \to H_n(X)$ and $(fg)_* = \mathbb{1}\colon H_n(Y) \to H_n(Y)$ by the previous theorem. Therefore $f_*\colon H_n(X) \to H_n(Y)$ has the inverse $g_*\colon H_n(Y) \to H_n(X)$, hence f_* is an isomorphism.

cl 2 If X is contractible, then X is homotopy equivalent to a point, hence $\tilde{H}_n(X) \simeq \tilde{H}_n(\mathbf{pt})$ by the previous statement. But we know that $\tilde{H}_n(\mathbf{pt}) = 0$ for all n, hence $\tilde{H}_n(X) = 0$ for all n. □

Example 3.16. The space D^n is contractible, hence $\tilde{H}_i(D^n) \simeq \tilde{H}_i(\mathbf{pt}) = 0$ for all i. The same is true for $\mathbb{R}^n$.

3.3. **Relative homology groups.**

Definition 3.17.
(1) A *pair* (X, A) of topological spaces consists of a space X and a subspace $A \subset X$.
(2) A map of pairs $f\colon (X, A) \to (Y, B)$ is a continuous map $f\colon X \to Y$ such that $f(A) \subset B$.

Given pair (X, A), define

$$C_n(X, A) = C_n(X)/C_n(A), \qquad n \geq 0.$$

The differential $d\colon C_n(X) \to C_{n-1}(X)$ maps $C_n(A)$ to $C_{n-1}(A)$, hence it induces a map $d\colon C_n(X, A) \to C_{n-1}(X, A)$. We obtain a chain complex

$$\cdots \to C_{n+1}(X, A) \xrightarrow{d} C_n(X, A) \xrightarrow{d} C_{n-1}(X, A) \to \dots$$

Its homology groups $H_n(X, A)$ are called *relative homology groups*.

Theorem 3.18. *For any pair (X, A) there is a long exact sequence*

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(X) \to H_{n-1}(X, A) \xrightarrow{\partial} \dots$$
$$\cdots \to H_0(X) \to H_0(X, A) \to 0$$

Proof. We have a short exact sequence of chain complexes

$$0 \to C(A) \to C(X) \to C(X, A) \to 0$$

by the construction of the complex $C(X, A)$. Applying Theorem 2.13 we obtain the required long exact sequence. □

Remark 3.19. If $A \neq \varnothing$, then we can use reduced homology groups:

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X, A) \to \tilde{H}_{n-1}(A) \to \tilde{H}_{n-1}(X) \to H_{n-1}(X, A) \to \dots$$
$$\cdots \to \tilde{H}_0(X) \to H_0(X, A) \to 0$$

Example 3.20. Given a point $x \in X$, we have $\tilde{H}_n(x) = 0$ for all n, hence we obtain from the long exact sequence

$$H_n(X, x) \simeq \tilde{H}_n(X) \qquad \forall n.$$

Example 3.21. Consider a disc D^n and its boundary $\partial D^n = S^{n-1}$. We have an exact sequence

$$\cdots \to \tilde{H}_i(D^n) \to H_i(D^n, S^{n-1}) \to \tilde{H}_{i-1}(S^{n-1}) \to \tilde{H}_{i-1}(D^n) \to \dots$$

As D^n is contractible, we obtain that $\tilde{H}_i(D^n) = 0$ for all i. Therefore there is an isomorphism

$$H_i(D^n, S^{n-1}) \simeq \tilde{H}_{i-1}(S^{n-1})$$

Remark 3.22. A subspace $A \subset X$ is called a retract of X if there exists a map $r\colon X \to A$ such that $ri = \mathbb{1}_A$, where $i\colon A \to X$ is the embedding. In this case the complex

$$0 \to C(A) \xrightarrow{i_\sharp} C(X) \to C(X, A) \to 0$$

splits by the map $r_\sharp$ (we get a direct sum decomposition $C(X) = C(A) \oplus C(X, A)$). Therefore there is a short exact sequence

$$0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0$$

which splits by r_*. This means that the boundary map $\partial\colon H_n(X, A) \to H_{n-1}(A)$ in the above long exact sequence is zero.

3.4. **Excision and Mayer-Vietoris.** Let X be a space and $\mathcal{U} = (U_i)_{i\in I}$ be a collection of subspaces, such that $X = \bigcup_i U_i^\circ$. Let $C_n^{\mathcal{U}}(X) \subseteq C_n(X)$ be the group generated by simplices $\sigma\colon \Delta^n \to X$ such that $\sigma(\Delta^n) \subseteq U_i$ for some $i \in I$. Then we obtain a chain subcomplex $C^{\mathcal{U}}(X) \subseteq C(X)$. We denote its homology groups by $H_n^{\mathcal{U}}(X)$ for $n \in \mathbb{Z}$.

Theorem 3.23. *The inclusion $j\colon C^{\mathcal{U}}(X) \to C(X)$ is a chain homotopy equivalence. In particular, it induces isomorphisms $H_n^{\mathcal{U}}(X) \simeq H_n(X)$.*

Theorem 3.24 (Excision theorem)**.** *Let $Z \subset A \subset X$ be subspaces such that $\bar{Z} \subset A^\circ$. Then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$H_n(X - Z, A - Z) \simeq H_n(X, A) \qquad \forall n.$$

Equivalently, for any subspaces $A, B \subset X$ with $X = A^\circ \cup B^\circ$ (we take $B = X - Z$), the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(B, A \cap B) \simeq H_n(X, A) \qquad \forall n.$$

Proof. Let $A, B \subset X$ be such that $X = A^\circ \cup B^\circ$. We consider the covering $\mathcal{U} = (A, B)$. Then we have subgroups $C_n(A) \subset C_n(X)$, $C_n(B) \subset C_n(X)$ such that

$$C_n(A \cap B) = C_n(A) \cap C_n(B), \qquad C_n^{\mathcal{U}}(X) = C_n(A) + C_n(B).$$

Therefore
$$\frac{C_n(B)}{C_n(A \cap B)} = \frac{C_n(B)}{C_n(A) \cap C_n(B)} \simeq \frac{C_n(A) + C_n(B)}{C_n(A)} = \frac{C_n^{\mathcal{U}}(X)}{C_n(A)}.$$
We claim that the homology groups of $C^{\mathcal{U}}(X)/C(A)$ and $C(X)/C(A)$ are isomorphic, hence

$$H_n(B, A \cap B) \simeq H_n(C^{\mathcal{U}}(X)/C(A)) \simeq H_n(C(X)/C(A)) = H_n(X, A)$$

as required. To prove the claim, we consider a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C(A) \arrow[r] \arrow[d] & C^{\mathcal{U}}(X) \arrow[r] \arrow[d] & C^{\mathcal{U}}(X)/C(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(A) \arrow[r] & C(X) \arrow[r] & C(X)/C(A) \arrow[r] & 0
\end{tikzcd}$$

Taking the long exact sequences of both rows we get

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d] & H_n^{\mathcal{U}}(X) \arrow[r] \arrow[d] & H_n(C^{\mathcal{U}}(X)/C(A)) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}^{\mathcal{U}}(X) \arrow[d] \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(C(X)/C(A)) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

The outer arrows are isomorphisms by the previous theorem. Therefore the middle arrow is also an isomorphism by the 5-lemma. □

Example 3.25. We know that $\tilde{H}_i(S^n) \simeq H_i(S^n, \mathbf{pt})$. Let $D_-^n \subset S^n$ be the lower hemisphere. Then $\tilde{H}_i(D_-^n) = \tilde{H}_i(\mathbf{pt}) = 0$ and we obtain $H_i(S^n, \mathbf{pt}) \simeq H_i(S^n, D_-^n)$ using the 5-lemma. We apply the excision theorem to $X = S^n$, $A = D_-^n$ and $Z \subset D_-^n$ a slightly smaller disc. Note that $X - Z$ is homotopic to D_+^n (the upper hemisphere) and $A - Z$ is homotopic to S^{n-1}. We obtain

$$H_i(S^n, D_-^n) \simeq H_i(X - Z, A - Z) \simeq H_i(D_+^n, S^{n-1}) \simeq \tilde{H}_{i-1}(S^{n-1}),$$

where the last isomorphism follows from the long exact sequence for the pair (D^n, S^{n-1}). We obtain
$$\tilde{H}_i(S^n) \simeq H_i(S^n, \mathbf{pt}) \simeq H_i(S^n, D_-^n) \simeq \tilde{H}_{i-1}(S^{n-1}).$$

Note that $S^0 = \{x \in \mathbb{R} \mid |x| = 1\} = \{\pm 1\}$, hence $\tilde{H}_i(S^0) = 0$ for $i \neq 0$ and $\tilde{H}_0(S^0) = \mathbb{Z}$. We obtain inductively

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$

Theorem 3.26 (Mayer-Vietoris sequence)**.** *Let $A, B \subset X$ be subspaces such that $A^\circ \cup B^\circ = X$. Then there is a long exact sequence*

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0$$

If $A \cap B \neq \varnothing$, then we have a similar exact sequence of reduced homology groups.

Proof. Let us consider the covering $\mathcal{U} = (A, B)$ of X. We have

$$C_n(A \cap B) = C_n(A) \cap C_n(B), \qquad C_n^{\mathfrak{U}}(X) = C_n(A) + C_n(B)$$

and short exact sequences

$$0 \to C_n(A) \cap C_n(B) \to C_n(A) \oplus C_n(B) \xrightarrow{f} C_n(A) + C_n(B) \to 0, \qquad f(x, y) = x - y$$

inducing a short exact sequence of complexes $0 \to C(A \cap B) \to C(A) \oplus C(B) \to C^{\mathfrak{U}}(X) \to 0$. The corresponding long exact sequence of homology groups is

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n^{\mathfrak{U}}(X) \to \dots$$

Now we use the fact that $H_n^{\mathfrak{U}}(X) \simeq H_n(X)$. $\square$

Example 3.27. Let $X = S^n$ and let $A \supset D_+^n$ and $B \supset D_-^n$ be open neighborhoods. Then $A \cap B$ is homotopic to S^{n-1} and A, B are contractible. We obtain an exact sequence

$$\tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^{n-1}) \to \tilde{H}_{i-1}(A) \oplus \tilde{H}_{i-1}(B)$$

The outer groups are trivial, hence $\tilde{H}_i(S^n) \simeq \tilde{H}_{i-1}(S^{n-1})$. This allows us to compute $\tilde{H}_i(S^n)$ in the same way as before.

Example 3.28. Define the suspension

$$\Sigma X = \frac{X \times I}{(x, 0) \sim (y, 0), (x, 1) \sim (y, 1)}.$$

One can think about it as a union of two cones $CX = X \times I / X \times \{0\}$ glued along the boundary X. For example $C(S^{n-1}) \simeq D^n$ and $\Sigma(S^{n-1})$ is a union of two discs D^n glued along the boundary, hence $\Sigma(S^{n-1}) \simeq S^n$. Let $A, B \subset \Sigma X$ be the above cones (more precisely, their open neighborhoods). Then $A, B \simeq CX$ are contractible (prove this) and $A \cap B$ is homotopy equivalent to X. We obtain an exact sequence

$$\tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(\Sigma X) \to \tilde{H}_{i-1}(A \cap B) \to \tilde{H}_{i-1}(A) \oplus \tilde{H}_{i-1}(B)$$

The outer groups are zero, hence

$$\tilde{H}_i(\Sigma X) \simeq \tilde{H}_{i-1}(A \cap B) \simeq \tilde{H}_{i-1}(X).$$

This generalizes the formula $\tilde{H}_i(S^n) \simeq \tilde{H}_{i-1}(S^{n-1})$.

3.5. Long exact sequence for good pairs.

Definition 3.29. Let $A \subset X$ be a subspace and $i\colon A \hookrightarrow X$ be an inclusion. Then A is called

(1) A *retract* if there exists a map $r\colon X \to A$ such that $ri = \mathbb{1}_A$. The map r as above is called a *retraction*.
(2) A *deformation retract* if there exists a map $r\colon X \to A$ such that $ri = \mathbb{1}_A$ and $ir \sim \mathbb{1}_X$. Equivalently, there exists a homotopy $h\colon X \times I \to X$ such that $h_0 = \mathbb{1}_X$, $h_1(X) \subset A$ and $h_1|_A = \mathbb{1}_A$ (we define $h_t(x) = h(x,t)$ and $r = h_1\colon X \to A$). The map h as above is called a *deformation retraction*. In this case $i_*\colon H_\bullet(A) \to H_\bullet(X)$ is an isomorphism.
(3) A *strong deformation retract* if there exists a homotopy $h\colon X \times I \to X$ such that $h_0 = \mathbb{1}_X$, $h_1(X) \subset A$ and $h_t|_A = \mathbb{1}_A$ for all $t \in I$. The map h as above is called a *strong deformation retraction*.

Example 3.30. Consider $A = S^{n-1} \subset X = \mathbb{R}^n \backslash \{0\}$ and define

$$h\colon X \times I \to X, \qquad (x,t) \mapsto (1 - t + t/\|x\|)x.$$

This is a strong deformation retract, as $h_0(x) = x$ for all $x \in \mathbb{R}^n\backslash\{0\}$, $h_1(x) = x/\|x\| \in A$ for all $x \in \mathbb{R}^n\backslash\{0\}$, $h_t(x) = (1-t+t)x = x$ for all $x \in S^{n-1}$. Therefore $H_\bullet(S^{n-1}) \to H_\bullet(\mathbb{R}^n\backslash\{0\})$ is an isomorphism.

Definition 3.31. A pair (X, A) is called *good* if $A \subset X$ is nonempty, closed and is a strong deformation retract of its neighborhood $V \subset X$ (this means that $A \subset V^\circ$).

Theorem 3.32. *If (X, A) is a good pair, then $H_n(X, A) \simeq \tilde{H}_n(X/A)$.*

Proof. Consider the quotient map $q\colon (X, A) \to (X/A, A/A)$. The space A/A has just one point, hence $H_n(X/A, A/A) \simeq \tilde{H}_n(X/A)$ by Example 3.20. Therefore we need to show that $q_*\colon H_n(X, A) \to H_n(X/A, A/A)$ is an isomorphism.

Let $A \subset V \subset X$ be a neighborhood such that A is a deformation retract of V. Consider the diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "q_*"] & H_n(X-A, V-A) \arrow[l] \arrow[d, "q_*"] \\
H_n(X/A, A/A) \arrow[r] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l]
\end{tikzcd}$$

We need to show that the left vertical arrow is an isomorphism. For this we will show that all horizontal arrows and the right vertical arrow are isomorphisms.

As $A \subset V$ is a deformation retract, we obtain that $H_n(A) \to H_n(V)$ are isomorphisms, hence $H_n(X, A) \to H_n(X, V)$ are isomorphisms (by the long exact sequence and 5-lemma). The same argument applies to the left bottom arrow: if $h\colon V \times I \to V$ is a strong deformation retraction of $A \subset V$, then it induces a homotopy $h\colon V/A \times I \to V/A$ which is a strong deformation retraction of $A/A \subset V/A$. Then we conclude that $H_n(X/A, A/A) \simeq H_n(X/A, V/A)$ as before.

The right horizontal arrows are isomorphisms by excision. The right vertical arrow is an isomorphism because it is induced by a homeomorphism between pairs. $\square$

Theorem 3.33. *Let (X, A) be a good pair. Then there is a long exact sequence*

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{\pi_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{\pi_*} \dots$$
$$\dots \xrightarrow{\pi_*} \tilde{H}_0(X/A) \to 0$$

where $i\colon A \to X$ is the inclusion and $\pi\colon X \to X/A$ is the quotient map.

Proof. By Theorem 3.18 we have a similar long exact sequence for the relative homology groups $H_n(X, A)$ (note that we can consider reduced homology groups $\tilde{H}_n(X)$ and $\tilde{H}_n(A)$ there)

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \to H_n(X, A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \to \dots$$
$$\dots \xrightarrow{\pi_*} H_0(X, A) \to 0$$

By the previous theorem we have $H_n(X, A) \simeq \tilde{H}_n(X/A)$. □

Lemma 3.34. *We have*

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$

Proof. Choose a good pair $(X, A) = (D^n, S^{n-1})$ for $n \geq 1$. Then $X/A \simeq S^n$. We have $\tilde{H}_i(D^n) = 0$ for all $i \geq 0$ as D^n is contractible. From the long exact sequence for the pair (X, A) we obtain

$$0 = \tilde{H}_i(D^n) \to \tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \to \tilde{H}_{i-1}(D^n) = 0,$$

Hence $\tilde{H}_i(S^n) \simeq \tilde{H}_{i-1}(S^{n-1})$ for $i > 0$ and $\tilde{H}_0(S^n) = 0$. Now the result follows by induction. In the case of S^0, we have $S^0 = \{\pm 1\} \subset \mathbb{R}$, hence $H_0(S^0) \simeq \mathbb{Z}^2$ and $\tilde{H}_0(S^0) \simeq \mathbb{Z}$. We also have $\tilde{H}_i(S^0) = 0$ for $i > 0$. □

Corollary 3.35. *$\mathbb{R}^n$ are not homeomorphic to each other for different n.*

Proof. We have seen in Example 3.30 that $S^{n-1} \subset \mathbb{R}^n \backslash \{0\}$ is a deformation retract. This implies $\tilde{H}_i(\mathbb{R}^n \backslash \{0\}) \simeq \tilde{H}_i(S^{n-1})$ and we have seen that spheres of different dimension have different homology groups. □

Remark 3.36. Similarly, if $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are nonempty open and homeomorphic, then $m = n$. Indeed, we have $H_i(U, U - \{x\}) \simeq H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ by excision: consider $Z = \mathbb{R}^m - U \subset A = \mathbb{R}^m - \{x\}$. From the long exact sequence for the pair $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ and the fact that $\tilde{H}_i(\mathbb{R}^m) = 0$, we obtain

$$H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \simeq \tilde{H}_i(\mathbb{R}^m - \{x\}) \simeq \tilde{H}_i(S^{m-1}).$$

We conclude that $\tilde{H}_i(S^{m-1}) \simeq \tilde{H}_i(S^{n-1})$, hence $m = n$.

Corollary 3.37 (Brouwer fixed point theorem). *A continuous map $f \colon D^n \to D^n$ has a fixed point.*

Proof. Suppose that $f(x) \neq x$ for all $x \in D^n$. Define a map $r \colon D^n \to \partial D^n = S^{n-1}$ by letting $r(x)$ to be the point where the ray going from $f(x)$ to x intersects ∂D^n. Then $r(x) = x$ for $x \in \partial D^n$, hence $ri = \mathbb{1}$, where $i \colon \partial D^n \hookrightarrow D^n$ (r is a retraction). But this implies that

$$\tilde{H}_{n-1}(\partial D^n) \to \tilde{H}_{n-1}(D^n) \to \tilde{H}_{n-1}(\partial D^n)$$

is an identity. However, $\tilde{H}_{n-1}(\partial D^n) \simeq \tilde{H}_{n-1}(S^{n-1}) \simeq \mathbb{Z}$ while $\tilde{H}_{n-1}(D^n) = 0$ as D^n is contractible. This is a contradiction. □

3.6. **Equivalence of simplicial and singular homologies.** Let (X, K) be a triangulated space and $C^K(X) = C(K)$ be its simplicial chain complex. Every simplex $\sigma\colon \Delta^n \to X$ from the triangulation can be considered as a singular n-simplex. Therefore we have an embedding $i\colon C^K(X) \hookrightarrow C(X)$ to the singular chain complex. It induces maps between homology groups

$$i_*\colon H_n^K(X) \to H_n(X)$$

and we are going to prove that these maps are isomorphisms. This would imply that simplicial homologies are independent of the triangulation and coincide with singular homologies.

More generally, assume that $A \subset X$ is a union of simplices of the triangulation. Then we can construct the simplicial chain complex of the pair (X, A)

$$C^K(X, A) = C^K(X)/C^K(A)$$

and the corresponding homology groups $H_n^K(X, A)$. As before, there is a map $i\colon C^K(X, A) \to C(X, A)$ which induces a map between homology groups

$$i_*\colon H_n^K(X, A) \to H_n(X, A)$$

and we want to show that these maps are isomorphisms.

Theorem 3.38. *The homomorphisms $H_n^K(X, A) \to H_n(X, A)$ are isomorphisms for all n.*

Proof. We will assume that $A = \varnothing$. Let $X^i \subset X$ be the i-skeleton of X consisting of simplices of dimension $\le i$. Consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
H_{n+1}^K(X^i, X^{i-1}) & \longrightarrow & H_n^K(X^{i-1}) & \longrightarrow & H_n^K(X^i) & \longrightarrow & H_n^K(X^i, X^{i-1}) & \longrightarrow & H_{n-1}^K(X^{i-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X^i, X^{i-1}) & \longrightarrow & H_n(X^{i-1}) & \longrightarrow & H_n(X^i) & \longrightarrow & H_n(X^i, X^{i-1}) & \longrightarrow & H_{n-1}(X^{i-1})
\end{array}$$

We can assume that the second and the fifth vertical maps are isomorphisms by induction on i. Let us show that the forth (hence also the first) vertical map is an isomorphism. The group of n-chains $C_n^K(X^i, X^{i-1})$ is zero for $n \ne i$, and is free abelian with the basis consisting of n-simplices of X for $n = i$. Therefore the same is true for the homology groups $H_n^K(X^i, X^{i-1})$. Consider the characteristic maps $\Phi_\alpha\colon \Delta_\alpha^i = \Delta^i \to X$ for i-simplices of X, and the corresponding map $\Phi\colon \bigsqcup_\alpha (\Delta_\alpha^i, \partial\Delta_\alpha^i) \to (X^i, X^{i-1})$. It induces a homeomorphism

$$\bigsqcup_\alpha \Delta_\alpha^i / \bigsqcup_\alpha \partial\Delta_\alpha^i \to X^i/X^{i-1},$$

hence isomorphisms of singular homology groups. But Δ^i is homeomorphic to D^i and $\partial\Delta^i$ is homeomorphic to $\partial D^i = S^{i-1}$ and we know by Lemma 3.34

$$\tilde{H}_n(D^i/S^{i-1}) = \tilde{H}_n(S^i) = \begin{cases} \mathbb{Z} & n = i, \\ 0 & n \ne i. \end{cases}$$

By Theorem 3.32 we have $H_n(X^i, X^{i-1}) \simeq \tilde{H}_n(X^i/X^{i-1})$. Hence we conclude that it is zero for $n \ne i$ and is a free abelian group with a basis given by n-simplices of X for $n = i$. We conclude that the forth vertical map is indeed an isomorphism. By the 5-lemma 2.14 the third vertical map is also an isomorphism. $\square$

3.7. **CW-complexes and cellular homology.** Define an *n-cell* to be a space homeomorphic to $\mathring{D}^n$. A *CW-complex* (or *cell complex*) is a space X equipped with a decomposition into cells $(e_\alpha)_{\alpha\in K}$ such that, if $X^n \subset X$ is the union of cells up to dimension n (called the *n-skeleton*), then

(1) X^0 is a discrete space.
(2) For every n-cell e_α, there is a continuous map $\Phi_\alpha\colon D^n \to X^n$ which induces a homeomorphism $\mathring{D}^n \to e_\alpha$ and sends $S^{n-1} = \partial D^n$ to X^{n-1}.

Definition 3.39. A *CW-decomposition* of a space X is a chain

$$\varnothing = X^{-1} \subset X^0 \subset X^1 \subset \ldots \subset X$$

of spaces such that

(1) The space X^n is obtained from X^{n-1} by attaching n-cells via *attaching maps*

$$\phi_\alpha\colon S^{n-1}_\alpha \to X^{n-1},$$

meaning that X^n is the quotient space $(X^{n-1} \coprod_\alpha D^n_\alpha)/\sim$ under the identifications $\phi_\alpha(x) \sim x$ for $x \in S^{n-1}_\alpha$. The map $\Phi_\alpha\colon D^n_\alpha \to X^n$ is called the *characteristic map* of the n-cell $e^n_\alpha = \Phi_\alpha(\mathring{D}^n_\alpha)$. There is a commutative (pushout) diagram

$$\begin{tikzcd} \coprod_\alpha S^{n-1}_\alpha \arrow[r, "\phi"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\ \coprod_\alpha D^n_\alpha \arrow[r, "\Phi"] & X^n \end{tikzcd}$$

(2) $X = \bigcup_{n\geq 0} X^n$ has the weak topology: $U \subset X$ is open $\iff$ $U \cap X^n \subset X^n$ is open $\forall n$.

The space X equipped with a CW-decomposition is called a *CW-complex* (or *cell complex*). The space X^n is called the *n-skeleton* of X. A space is called a *CW-space* if it has some CW-decomposition.

Theorem 3.40. *We have*

(1) $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$ and is free with a basis given by the n-cells if $k = n$.
(2) $H^k(X^n) = 0$ for $k > n$.
(3) The inclusion $i\colon X^n \hookrightarrow X$ induces isomorphisms $i_*\colon H_k(X^n) \to H_k(X)$ for $k < n$.

Proof. cl 1 We can identify D^n/S^{n-1} with a pointed sphere S^n. The quotient X^n/X^{n-1} can be identified with $(\coprod_\alpha D^n_\alpha)/(\coprod_\alpha S^{n-1}_\alpha) \simeq \bigvee_\alpha S^n_\alpha$ (wedge sum). We have $\tilde{H}_n(\bigvee_\alpha S^n_\alpha) \simeq \bigoplus_\alpha \tilde{H}_n(S^n_\alpha)$. Therefore

$$H_n(X^n, X^{n-1}) \simeq \tilde{H}_n(X^n/X^{n-1}) \simeq \bigoplus_\alpha \tilde{H}_n(S^n_\alpha) \simeq \bigoplus_\alpha \mathbb{Z}.$$

cl 2 We have an exact sequence

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1})$$

By the previous statement the first and the fourth components are zero for $k > n$. By induction we can assume that $H_k(X^{n-1}) = 0$, hence we conclude that $H_k(X^n) = 0$.

cl 3 From the previous argument we obtain $H_k(X^{n-1}) \simeq H_k(X^n)$ for $k \neq n, n-1$. Therefore $H_k(X^n) \simeq H^k(X^{n+1})$ for $k \neq n, n+1$ and in particular, for $k < n$. This implies $H_k(X^n) \simeq H^k(X)$ (if X is finite-dimensional). $\square$

The long exact sequence for the pair (X^n, X^{n-1}) gives in particular

$$0 \to H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \xrightarrow{p_n} H_{n-1}(X^n) \to 0$$

Let us organize these maps into a commutative diagram, with $d_n = j_{n-1}\partial_n$,

$$\begin{tikzcd}[column sep=small] & & H_n(X^n) \arrow[dr, "j_n"] & & H_{n-1}(X^{n-1}) \arrow[dr, "j_{n-1}"] & & \\ \dots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[ur, "\partial_n"] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \dots \end{tikzcd}$$

Note that $d_n d_{n+1} = 0$ as $\partial_n j_n = 0$. The above complex is called a *cellular chain complex* and its homology groups are called *cellular homology groups*, denoted by $H_n^{CW}(X)$.

Theorem 3.41. *We have $H_n^{CW}(X) \simeq H_n(X)$.*

Proof. We have $\operatorname{Ker} d_n = \operatorname{Ker} \partial_n \simeq H_n(X^n)$ as j_{n-1} is injective. Therefore

$$\operatorname{Ker} d_n / \operatorname{Im} d_{n+1} \simeq H_n(X^n)/\operatorname{Im} d_{n+1} = H_n(X^n)/\operatorname{Im} \partial_{n+1} \simeq H_n(X^{n+1}) \simeq H_n(X).$$

□

Recall that $H_n(X^n, X^{n-1})$ is a free abelian group with a basis given by n-cells e_α^n. For any such cell we have the attaching map $S_\alpha^{n-1} \xrightarrow{\phi_\alpha} X^{n-1}$ as well as the quotient map

$$X^n \xrightarrow{q} X^n/X^{n-1} \simeq \bigvee_\alpha (D_\alpha^n / S_\alpha^{n-1}) \xrightarrow{q_\alpha} D_\alpha^n/S_\alpha^{n-1} \simeq S_\alpha^n$$

Theorem 3.42. *The differential $d_n\colon \mathbb{Z}^{K_n} \to \mathbb{Z}^{K_{n-1}}$ is given by $d_n(e_\alpha^n) = \sum_{\beta \in K_{n-1}} d_{\alpha\beta} e_\beta^{n-1}$, where $d_{\alpha\beta}$ is the degree of the composition $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$.*

4. Applications of Homology theory

4.1. **Degree.** Given a map $f\colon S^n \to S^n$, consider the induced map

$$f_*\colon H_n(S^n) \to H_n(S^n).$$

We know that $H_n(S^n) \simeq \mathbb{Z}$, hence f_* is of the form $\sigma \mapsto d\sigma$ for some $d \in \mathbb{Z}$. This integer is called the *degree* of f, denoted by $\deg f$.

Proposition 4.1. *We have*

(1) $\deg \mathbb{1} = 1$.
(2) $f \simeq g \implies \deg f = \deg g$.
(3) $\deg fg = \deg f \cdot \deg g$.
(4) If f is a reflection with respect to some hyperplane in $\mathbb{R}^{n+1}$, then $\deg f = -1$.
(5) If $f = -\mathbb{1}$, the antipodal map, then $\deg f = (-1)^{n+1}$.
(6) If $f\colon S^n \to S^n$ is not surjective, then $\deg f = 0$.

Proof. (1) We have $\mathbb{1}_* = \mathbb{1}$, the identity map $H_n(S^n)$.
(2) If $f \simeq g$, then $f_* = g_*$.
(3) $(fg)_* = f_* g_*$.
(4) Let $S^{n-1} \subset S^n$ be fixed by the reflection. Consider a triangulation of S^n with two hemispheres σ, τ as its n-simplices. Then $H_n(S^n)$ has a generator $\sigma - \tau$. The reflection f interchanges σ and τ, hence $f_*(\sigma - \tau) = \tau - \sigma$. This means that $\deg f = -1$.
(5) The antipodal map can be represented as a composition of $n+1$ reflections, each changing the sign of one coordinate.
(6) If f is not surjective then it is a composition of $S^n \to S^n - \{x\} \to S^n$ for some $x \in S^n - \operatorname{Im} f$. But $S^n - \{x\}$ is contractible, hence $f_*\colon H_n(S^n) \to H_n(S^n)$ is zero.

□

Proposition 4.2. *If $f\colon S^n \to S^n$ does not have fixed points, then* $\deg f = (-1)^{n+1}$.

Proof. If $f(x) \neq x$ for all $x \in S^n$, then $g_t(x) = tf(x) - (1-t)x \neq 0$ for $t \in [0,1]$. Define the homotopy $h_t\colon S^n \to S^n$, $h_t(x) = g_t(x)/\|g_t(x)\|$ between $h_0 = -\mathbb{1}$ and $h_1 = f$. Then $\deg f = \deg(-\mathbb{1}) = (-1)^{n+1}$. □

In order to compute the degree of a map one introduces local degrees. Recall that if $U \subset \mathbb{R}^n$ is open and $x \in U$, then

$$H^n(U, U-x) \simeq H^n(\mathbb{R}^n, \mathbb{R}^n - x) \simeq \tilde{H}_{n-1}(\mathbb{R}^n - x) \simeq \tilde{H}_{n-1}(S^{n-1}) \simeq \mathbb{Z},$$

where the first isomorphism follows from excision, the second from the long exact sequence

$$\tilde{H}_n(\mathbb{R}^n) \to H^n(\mathbb{R}^n, \mathbb{R}^n - x) \to \tilde{H}_{n-1}(\mathbb{R}^n - x) \to \tilde{H}_{n-1}(\mathbb{R}^n)$$

and the third follows from the fact that $\mathbb{R}^n - x$ is homotopic to S^{n-1}.

Assume now that we have $f\colon S^n \to S^n$ and a point $y \in S^n$ such that the preimage $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$. We can choose disjoint neighborhoods $U_1, \dots, U_m$ of these points which are mapped to a neighborhood V of y. Then we obtain a homomorphism

$$f_*\colon \mathbb{Z} \simeq H_n(U, U - x_i) \to H_n(V, V - y) \simeq \mathbb{Z}$$

which is a multiplication by an integer, called the *local degree* of f at x_i, denoted by $\deg f|_{x_i}$. Note that if f maps U_i homeomorphically to V, then the above homomorphism is an isomorphism, hence $\deg f|_{x_i} = \pm 1$.

Theorem 4.3. *We have* $\deg f = \sum_i \deg f|_{x_i}$.

Proof. Consider $X = S^n$, $B = \bigcup_i U_i$ and $A = S^n - f^{-1}(y)$. Then by excision $H_n(X, A) \simeq H_n(B, A \cap B)$ and we have $A \cap B = \bigcup_i (U_i - x_i)$. Therefore

$$H_n(S^n, S^n - f^{-1}(y)) \simeq \bigoplus_i H_n(U_i, U_i - x_i) \simeq \mathbb{Z}^m.$$

To prove the theorem we consider

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*^{(i)}"] \arrow[dl, "\sim"'] \arrow[d, "j_i"'] & H_n(V, V - y) = \mathbb{Z} \arrow[d, "\wr"] \\
H_n(S_n, S_n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l] \arrow[r, "f'_*"] & H_n(S^n, S^n - y) = \mathbb{Z} \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\sim"] \arrow[u, "j"] & H_n(S^n) = \mathbb{Z} \arrow[u, "\wr"']
\end{tikzcd}$$

and observe that $j(1) = \sum_i j_i(1)$. But this means that $f_*(1) = \deg f$ can be identified with $f'_* j(1) = \sum_i f'_* j_i(1)$ or with $\sum_i f_*^{(i)}(1) = \sum_i \deg f|_{x_i}$. $\square$

4.2. **Hedgehog theorem.** For any point $x \in S^n \subset \mathbb{R}^{n+1}$, its *tangent space* $T_x S^n$ consists of vectors in $\mathbb{R}^{n+1}$ perpendicular to x. A (continuous) *vector field* is a continuous map $f\colon S^n \to \mathbb{R}^{n+1}$ such that $f(x) \in T_x S^n$, a tangent vector at $x \in S^n$.

The following statement says that one can not comb a hairy ball without a cowlick:

Theorem 4.4 (Hedgehog theorem)**.** *There is no continuous tangent vector field on S^2 that is nowhere zero.*

We can generalize this statement to any even dimension:

Theorem 4.5 (2.28)**.** *If n is even, there is no continuous tangent vector field on S^n that is nowhere zero.*

Proof. Let $f\colon S^n \to \mathbb{R}^{n+1}$ be a vector field. If $f(x) \neq 0$ for all x, we can normalize it by taking $f(x)/\|f(x)\|$, hence we can assume that $f(x) \in S^n$. The vectors x and $f(x)$ are orthogonal to each other in $\mathbb{R}^{n+1}$ (the tangent space of S^n at x is orthogonal to x). The vectors

$$h_t(x) = \cos(t)x + \sin(t)f(x), \qquad t \in [0, \pi]$$

are contained in S^n. Therefore $h_t\colon S^n \to S^n$ is a homotopy between $h_0(x) = \mathbb{1}$ and $h_1 = -\mathbb{1}$ (the antipodal map). Therefore $\deg(-\mathbb{1}) = \deg \mathbb{1}$, hence $(-1)^{n+1} = 1$. A contradiction as n is even. $\square$

Remark 4.6. Note that there is a non-vanishing vector field on S^1. Indeed, represent $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ and define $f(z) = iz$, which is orthogonal to z. In real coordinates this means $z = x_1 + ix_2 \mapsto -x_2 + ix_1$, hence $(x_1, x_2) \mapsto (-x_2, x_1)$. More generally, for any odd $n = 2k - 1$, consider the map

$$\mathbb{R}^{2k} \ni x \mapsto f(x) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}).$$

Then $f(x)$ is orthogonal to x, hence f defines a non-vanishing vector field on S^n.

4.3. **Jordan curve theorem.**

Theorem 4.7 (Jordan curve theorem)**.** *A subspace of $\mathbb{R}^2$ homeomorphic to S^1 separates $\mathbb{R}^2$ into two connected components.*

We can consider S^2 instead of $\mathbb{R}^2$ as one obtains $\mathbb{R}^2$ from S^2 be removing a point and this does not affect connectedness. The result then follows from a more general statement.

Theorem 4.8. *Given an embedding $f\colon S^k \to S^n$ with $k < n$, we have*

$$\tilde{H}_i(S^n - f(S^k)) = \begin{cases} \mathbb{Z} & i = n - k - 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof of Theorem 4.7. Taking $k = 1$ and $n = 2$, we obtain $\tilde{H}_0(S^2 - f(S^1)) \simeq \mathbb{Z}$. Therefore $H_0(S^2 - f(S^1)) \simeq \mathbb{Z}^2$, hence there are two connected components in the complement. □

Proof. If $k = 0$, then $S^n - f(S^0) \simeq \mathbb{R}^n - \{0\}$ which is homotopic to S^{n-1}. But we know how to compute $\tilde{H}_i(S^n)$. For the induction on k, write S^k as a union of hemispheres D^k_+, D^k_- intersecting at S^{k-1}. Let $A = S^n - f(D^k_+)$, $B = S^n - f(D^k_-)$. Then

$$A \cap B = S^n - (f(D^k_+) \cup f(D^k_-)) = S^n - f(S^k), \qquad A \cup B = S^n - f(S^{k-1}).$$

The spaces A, B have trivial reduced homologies by Theorem 4.9, hence by the Meyer-Vietoris 3.26 we have $\tilde{H}_i(A \cup B) \simeq \tilde{H}_{i-1}(A \cap B)$. Therefore

$$\tilde{H}_{i-1}(S^n - f(S^k)) \simeq \tilde{H}_i(S^n - f(S^{k-1}))$$

and we apply induction. □

Theorem 4.9. *Given an embedding $f\colon D^k \to S^n$, we have $\tilde{H}_i(S^n - f(D^k)) = 0$ for all i.*

Proof. For $k = 0$, we have $S^n - f(D^0) \simeq \mathbb{R}^n$ and the statement is obvious. Generally, we can identify D^k with a cube I^k. Consider a decomposition $I^k = I' \cup I''$, where $I' = I^{k-1} \times [0, \frac{1}{2}]$ and $I'' = I^{k-1} \times [\frac{1}{2}, 1]$. Let $A = S^n - f(I')$ and $B = S^n - f(I'')$. Then

$$A \cup B = S^n - f(I' \cap I'') = S^n - f(I^{k-1}), \qquad A \cap B = S^n - f(I' \cup I'') = S^n - f(I^k).$$

By induction on k we have $\tilde{H}_i(S^n - f(I^{k-1})) = 0$, hence by the Meyer-Vietoris we have

$$\tilde{H}_i(S^n - f(I^k)) \simeq \tilde{H}_i(A) \oplus \tilde{H}_i(B),$$

induced by the embeddings $S^n - f(I_k) \to A$ and $S^n - f(I_k) \to B$. Assume there exists a chain $c \in Z_i(S^n - f(I^k))$ which is not in the boundary. Then its image in $Z_i(A)$ or $Z_i(B)$ is not in the boundary. Continuing this process, we find a sequence of shrinking intervals

$$[0, 1] = I_0 \supset I_1 \supset I_2 \supset \dots$$

such that c as a chain over $U_m = S^n - f(I^{k-1} \times I_m)$ is not in the boundary, for all m.

We have $U_1 \subset U_2 \subset \dots$ and $U = \bigcup_m U_m = S^n - f(I^{k-1} \times \{p\})$, where $\{p\} = \bigcap_m I_m$. Then the chain c over $U = S^n - f(I^{k-1} \times \{p\})$ is in the boundary by induction. Let $a \in C_{i+1}(U)$ be such that $c = da$ and let $Z \subset U$ be its support. Then $(S^n - Z) \cup \bigcup_m U_m$ is an open cover of S^n, hence it contains a finite subcover and we conclude that $(S^n - Z) \cup U_m = S^n$ for some m. Then $Z \subset U_m$, hence $a \in C_{i+1}(U_m)$ and we conclude that $c = da \in B_i(U_m)$, a contradiction. □

Corollary 4.10. *One cannot embed S^n in $\mathbb{R}^n$.*

Proof. Given such an embedding, we would get a non-surjective embedding $f\colon S^n \to S^n$. Using notation from the previous theorem, consider $A = S^n - f(D^n_+)$ and $B = S^n - f(D^n_-)$. Assuming if $A \cap B = S^n - f(S^n) \neq \varnothing$, then MV gives an exact sequence

$$\tilde{H}_0(A) \oplus \tilde{H}_0(B) \to \tilde{H}_0(A \cup B) = \tilde{H}_0(S^n - f(S^{n-1})) \to 0$$

But the left group is zero, while the right group is nonzero by the previous theorem. We conclude that $A \cap B = \varnothing$, hence $f(S^n) = S^n$ and f is sujrective, a contradiction. □

4.4. **Invariance of domain.**

Theorem 4.11. *Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^n$ be injective continuous. Then $f(U)$ is open in $\mathbb{R}^n$ and f maps U homeomorphically onto $f(U)$.*

Proof. We can embed $\mathbb{R}^n \subset S^n$ using a one-point compactification and consider $f\colon U \to S^n$. It is enough to show that $f(U) \subset S^n$ is open. Given $a \in U$, choose $\varepsilon > 0$ such that $D = \{x \in \mathbb{R}^n \mid \|x-a\| \le \varepsilon\} \subset U$. It suffices to show that $f(D^\circ)$ is open. We have $\partial D \simeq S^{n-1}$ and by the previous results $S^n - f(\partial D)$ has two path-connected components. Note that $f(D - \partial D)$ is path-connected as $D - \partial D$ is path connected, and $S^n - f(D)$ is path-connected as $\tilde{H}_0(S^n - f(D)) = 0$ by the previous results. This means that these two sets are path-connected components of $S^n - f(\partial D)$. Such components are closed, hence open as there are finitely many of them. We conclude that $f(D^\circ) = f(D - \partial D)$ is open. □

Definition 4.12. Define a *topological n-manifold* to be a Hausdorff space locally homeomorphic to $\mathbb{R}^n$ (this means that every point has an open neighborhood homeomorphic to an open subset $U \subset \mathbb{R}^n$).

Corollary 4.13. *If M, N are topological n-manifolds, M is compact and N is connected, then any embedding $f\colon M \to N$ is a homeomorphism.*

Proof. By the previous result $f(M)$ is open. But it is also closed as M is compact. As N is connected, we conclude that $f(M) = N$, hence $f\colon M \to N$ is a bijection. It is an open/closed map, hence a homeomorphism. □

4.5. **Algebraic applications.**

Lemma 4.14. *We have $H_1(\mathbb{R}P^n) \simeq \mathbb{Z}_2$ for $n \ge 2$. In particular, $\mathbb{R}P^n \not\simeq S^n$ for $n \ge 2$.*

Proof. The space $\mathbb{R}P^n \simeq (\mathbb{R}^{n+1}\backslash\{0\})/\mathbb{R}^* \simeq S^n/\mathbb{Z}_2$ can be constructed as a quotient of D^n, where we identify the antipodal points of $\partial D^n = S^{n-1}$. Under this identification we get $S^{n-1}/\mathbb{Z}_2 \simeq \mathbb{R}P^{n-1}$. In this way we obtain $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$ such that

$$\mathbb{R}P^n/\mathbb{R}P^{n-1} \simeq D^n/S^{n-1} \simeq S^n.$$

Therefore we have a long exact sequence

$$\dots \to \tilde{H}_{i+1}(S^n) \to \tilde{H}_i(\mathbb{R}P^{n-1}) \to \tilde{H}_i(\mathbb{R}P^n) \to \tilde{H}_i(S^n) \to \dots$$

This implies $\tilde{H}_1(\mathbb{R}P^n) \simeq \tilde{H}_1(\mathbb{R}P^{n-1})$ for $n \ge 3$. Now we use the fact $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}_2$. □

Theorem 4.15. *The only finite field extensions of $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$.*

Proof. Let $K = \mathbb{R}^n$ be equipped with a field structure. Let $\|x\|$ denote the Euclidean norm of $x \in \mathbb{R}^n$. Consider a continuous map

$$f\colon S^{n-1} \to S^{n-1}, \qquad x \mapsto x^2/\left\|x^2\right\|.$$

If $f(x) = f(y)$, then $x^2 = a^2y^2$, where $a^2 = \|x^2\|/\|y^2\| \in \mathbb{R}$. Therefore $(x-ay)(x+ay) = 0$, hence $x = \pm ay$. As $x, y \in S^{n-1}$, we conclude that $x = \pm y$. Therefore we get an injective continuous map

$$\bar{f}\colon \mathbb{R}P^{n-1} = S^{n-1}/\{\pm 1\} \to S^{n-1}.$$

By 4.13, $\bar{f}$ is a homeomorphism. But this is not possible if $n \ge 3$.

Let us show that for $n = 2$, we have $K \simeq \mathbb{C}$. Let $x \in K\backslash\mathbb{R}$ and $x^2 = a + 2bx$ for some $a, b \in \mathbb{R}$. Then $(x-b)^2 = a + b^2$. If $a + b^2 \ge 0$, then $x \in \mathbb{R}$, a contradiction. If $a + b^2 = -c^2 < 0$, then $y = (x-b)/c$ satisfies $y^2 = -1$, hence $K \simeq \mathbb{C}$. □

Remark 4.16. It was proved by Frobenius that the only finite-dimensional division algebras over $\mathbb{R}$ (not necessarily commutative) are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (the algebra of quaternions).

Theorem 4.17 (Fundamental theorem of algebra)**.** *The field $\mathbb{C}$ is algebraically closed (every non-constant polynomial has a root).*

Proof. By the previous result every finite field extension of $\mathbb{C}$ coincides with $\mathbb{C}$. But this implies that $\mathbb{C}$ is algebraically closed: if $f(x)$ is an irreducible polynomial, then $K = \mathbb{C}[x]/(f)$ is a field, hence $K = \mathbb{C}$ and $\deg f = 1$. Therefore f has a root. $\square$

4.6. **Borsuk-Ulam theorem.** Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be such that $-A = A$ and $-B = B$. We call a map $f\colon A \to B$ *odd* if $f(-x) = -f(x)$ for all $x \in A$.

Theorem 4.18. *The following statements are equivalent (all maps are continuous)*

(1) For any map $f\colon S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.
(2) For any odd map $f\colon S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = 0$.
(3) There does not exist an odd map $f\colon S^n \to S^{n-1}$.
(4) There does not exist a map $f\colon D^n \to S^{n-1}$ which is odd on the boundary S^{n-1}.
(5) An odd map $f\colon S^{n-1} \to S^{n-1}$ is not null homotopic.
(6) Let $S^n = F_1 \cup \cdots \cup F_{n+1}$ with closed F_i. Then at least one F_i has antipodal points.

Proof. **(1) $\Longrightarrow$ (2).** There exists $x \in S^n$ with $f(x) = f(-x)$. If f is antipodal, then $f(-x) = -f(x)$, hence $f(x) = -f(x)$ and $f(x) = 0$.
(2) $\Longrightarrow$ (3). If $f\colon S^n \to S^{n-1} \subset \mathbb{R}^n$ is odd, then there exists $x \in S^n$ with $f(x) = 0$, a contradiction.
(3) $\Longrightarrow$ (4). Consider hemispheres $D^n_\pm = \{(x_0, \ldots, x_n) \in S_n \mid \pm x_0 \geq 0\}$ and the projection $p\colon D^n_+ \to D^n$ onto the last n coordinates which is a homeomorphism. If $f\colon D^n \to S^{n-1}$ is odd on the boundary, define a new map

$$g\colon S^n \to S^{n-1}, \qquad g(x) = \begin{cases} fp(x) & x \in D^n_+ \\ -fp(-x) & x \in D^n_- \end{cases}$$

Note that if $x \in S^{n-1} = D^n_+ \cap D^n_-$, then $-fp(-x) = -f(-px) = f(px)$ as f is odd on the boundary. This implies that g is well-defined and continuous. It is odd by definition and this contradicts the assumption.
(4) $\Longrightarrow$ (5). If f is null homotopic, consider the corresponding homotopy $h\colon S^{n-1} \times I \to S^{n-1}$. It is constant on $S^{n-1} \times \{1\}$, hence factors through the cone

$$\bar{h}\colon CS^{n-1} = (S^{n-1} \times I)/(S^{n-1} \times \{1\}) \to S^{n-1}.$$

But $CS^{n-1} \simeq D^n$ with the boundary $\partial D^n = S^{n-1} \times \{0\}$. But f is odd on S^{n-1}, hence $\bar{h}$ is odd on the boundary. This contradicts our assumption.
(5) $\Longrightarrow$ (1). Assume that $f\colon S^n \to \mathbb{R}^n$ is such that $f(x) \neq f(-x)$ for all $x \in S^n$. Then the map

$$g\colon S^n \to S^{n-1}, \qquad x \mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is antipodal. Its restriction $g\colon S^{n-1} \to S^{n-1}$ is odd and null homotopic (as it is defined on the upper hemisphere of S^n), a contradiction.
(1) $\Longrightarrow$ (6). If some of the F_i are empty, we make them 1-point sets. Consider the map

$$f\colon S^n \to \mathbb{R}^n, \ldots, x \mapsto (d(x, F_1), \qquad , d(x, F_n))$$

using the Euclidean distance d. By assumption, there exists $x \in S^n$ with $f(x) = f(-x) = y$. If $y_i = 0$, then $d(x, F_i) = d(-x, F_i)$, hence $x, -x \in F_i$ as F_i is compact. If $y_i \neq 0$ for all i, then $x, -x \notin \bigcup_{i=1}^n F_i$, hence $x, -x \in F_{n+1}$.
(6) $\implies$ (3). Assume there exists an odd map $f\colon S^n \to S^{n-1}$. There exists a closed covering $F_1, \dots, F_{n+1}$ of S^{n-1} such that no F_i contains antipodal points. Indeed, consider a projection of $\partial\Delta^n$ onto S^{n-1} and take the images of the facets of $\partial\Delta^n$ (there are $n+1$ facets). If $x, -x \in f^{-1}(F_i)$, then $-f(x) = f(-x) \in F_i$, hence F_i contains antipodal points, a contradiction. This means that the closed covering $f^{-1}(F_1), \dots, f^{-1}(F_{n+1}$ of S^n contradicts our assumption. □

Remark 4.19. In particular, if S^2 is covered by 3 closed sets F_1, F_2, F_3 then at least one of them has antipodal points. The above proof shows that there exists a covering of S^2 by 4 closed sets such that neither of them has antipodal points.

We will prove that statement (5) of the above theorem is satisfied.

Theorem 4.20. *If $f\colon S^n \to S^n$ is odd, then* $\deg f$ *is odd. In particular, f is not null homotopic.*

Proof. Recall that $f_*\colon H_n(S^n) \to H_n(S^n)$ is a multiplication by $\deg f$. Assuming that $\deg f$ is even, we consider homology groups with coefficients in $\mathbb{Z}_2$ and obtain that the map

$$f_*\colon H_n(S^n, \mathbb{Z}_2) \to H_n(S^n, \mathbb{Z}_2)$$

is zero. We will show that f_* is actually an isomorphism. Let us write $H_n(X)$ for $H_n(X, \mathbb{Z}_2)$ and $C_n(X)$ for $C_n(X, \mathbb{Z}_2)$ from now on.

Consider the projection $p\colon S^n \to P^n = \mathbb{R}\mathrm{P}^n$ which is a $2\colon 1$ covering. As $f\colon S^n \to S^n$ is odd, it induces $\bar{f}\colon P^n \to P^n$ such that $\bar{f}p = pf$.

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ P^n & \xrightarrow{\bar{f}} & P^n \end{array}$$

For any simplex $\sigma\colon \Delta^k \to P^n$ there exist exactly 2 lifts $\sigma_1, \sigma_2\colon \Delta^k \to S^n$ (as Δ^k is simply-connected). Considering chain complexes with coefficients in $\mathbb{Z}_2$, we have an exact sequence

$$0 \to C_k(P^n) \xrightarrow{\tau} C_k(S^n) \xrightarrow{p_\sharp} C_k(P^n) \to 0$$

where $\tau(\sigma) = \sigma_1 + \sigma_2$ (note that $p_\sharp(\sigma_1 + \sigma_2) = 2\sigma = 0$ because of the coefficient ring $\mathbb{Z}_2$). The above maps commute with differentials, hence there is a long exact sequence

$$\begin{aligned} 0 \to H_n(P^n) \xrightarrow[\sim]{\tau_*} H_n(S^n) \xrightarrow{p_*=0} H_n(P_n) \xrightarrow[\sim]{} H_{n-1}(P^n) \to 0 \to \dots \\ \dots \to 0 \to H_i(P_n) \xrightarrow[\sim]{} H_{i-1}(P^n) \to 0 \to \dots \\ \dots \to 0 \to H_1(P_n) \xrightarrow[\sim]{} H_0(P^n) \xrightarrow{0} H_0(S^n) \xrightarrow[\sim]{} H_0(P_n) \to 0 \end{aligned}$$

To see that $\tau_*\colon H_n(P^n, \mathbb{Z}_2) \to H_n(S^n, \mathbb{Z}_2)$ is an isomorphism, we represent the generator of $H_n(S^n, \mathbb{Z}) \simeq \mathbb{Z}$ as $\sigma_1 - \sigma_2$, where σ_1, σ_2 are hemispheres in S^n. They both are mapped to the same $\sigma \in H_n(P^n, \mathbb{Z})$. Therefore $\tau_*\colon H_n(P^n, \mathbb{Z}_2) \to H_n(S^n, \mathbb{Z}_2)$ is surjective. We also note that $H_i(S^n) = 0$ for $0 < i < n$. We conclude from the diagram that $\partial\colon H_i(P^n) \to H_{i-1}(P^n)$ are isomorphisms for $1 \leq i \leq n$.

Next we note that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_k(P^n) & \xrightarrow{\tau} & C_k(S^n) & \xrightarrow{p_\sharp} & C_k(P^n) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \bar{f}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \bar{f}} & & \\ 0 & \longrightarrow & C_k(P^n) & \xrightarrow{\tau} & C_k(S^n) & \xrightarrow{p_\sharp} & C_k(P^n) & \longrightarrow & 0 \end{array}$$

It induces a commutative diagram with long exact sequences as rows. In particular, we have

$$\begin{array}{ccccccc} H_k(P^n) & \xrightarrow{\tau_*} & H_k(S^n) & \xrightarrow{p_*} & H_k(P^n) & \xrightarrow{\sim} & H_{k-1}(P^n) \\ \big\downarrow{\scriptstyle \bar{f}_*} & & \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle \bar{f}_*} & & \big\downarrow{\scriptstyle \bar{f}_*} \\ H_k(P^n) & \xrightarrow{\tau} & H_k(S^n) & \xrightarrow{p_*} & H_k(P^n) & \xrightarrow{\sim} & H_{k-1}(P^n) \end{array}$$

Starting with the fact that $\bar{f}_*\colon H_0(P^n) \to H_0(P_n)$ is an isomorphism, we inductively prove that $\bar{f}_*\colon H_n(P^n) \to H_n(P_n)$ is an isomorphism. But then also $f_*\colon H_n(S^n) \to H_n(S_n)$. This is what we wanted. ☐

4.7. **Lefschetz fixed point theorem.** Let X be a space such that all homology groups $H_n(X, \mathbb{Q})$ are finite-dimensional and only finitely many of them are non-zero. Given a continuous map $f\colon X \to X$, define the *Lefschetz number*

$$\tau(f) = \sum_n (-1)^n \operatorname{tr}(f_*\colon H_n(X) \to H_n(X)).$$

Note that if $f = \mathbb{1}$ is the identity, then $\operatorname{tr}(\mathbb{1}_*\colon H_n(X) \to H_n(X)) = \dim H_n(X)$, hence

$$\tau(\mathbb{1}) = \sum_n (-1)^n \dim H_n(X) = \chi(X),$$

the Euler number of X.

Theorem 4.21. *If X is a finite simplicial complex and $f\colon X \to X$ satisfies $\tau(f) \neq 0$, then f has a fixed point.*

Proof. Assume that $f\colon X \to X$ does not have fixed points. We use without a proof the fact that in this case there exists a subdivision L of X, a subdivision K of L and a simplicial map $g\colon K \to L$ homotopic to f such that $g(\sigma) \cap \sigma = \varnothing$ for all σ (we mix abstract simplices and their geometric realizations here). Let $K^n \subset |K| = X$ and $L^n \subset |L| = X$ be the n-skeletons. Then $L^n \subset K^n$ as K is a subdivision of L. Therefore $g(K^n) \subset K^n$ (we use the geometric realization of g here). It induces the chain map of the cellular chain complex $C_n = H_n(K^n, K^{n-1})$. Then

$$\tau(g) = \sum_n (-1)^n \operatorname{tr}(g_*\colon C_n \to C_n).$$

But the map $g_*\colon H_n(K^n, K^{n-1}) \to H_n(K^n, K^{n-1})$ has trace zero since $H_n(K^n, K^{n-1})$ has a basis consisting of n-simplices and $g(\sigma) \cap \sigma = \varnothing$ by our assumption. We conclude that $\tau(f) = \tau(g) = 0$. ☐

Corollary 4.22 (Brouwer fixed point theorem 3.37)**.** *A continuous map $f\colon D^n \to D^n$ has a fixed point.*

Proof. The space D^n is homotopic to a point, hence f is homotopic to the identity map. Therefore $\tau(f) = \chi(D^n) = 1 \neq 0$, hence f has a fixed point. ☐

Lemma 4.23. *Given a map $f\colon S^n \to S^n$, we have $\tau(f) = 1 + (-1)^n \deg f$.*

Appendix A. Preliminaries

A.1. Relations.

Definition A.1. Let X be a set.

1. A (binary) *relation* on X is a subset $R \subset X \times X$. We write xRy if $(x, y) \in R$.
2. A relation $\sim$ on X is called an *equivalence relation* if we have
   1. $x \sim x$ (reflexivity).
   2. $x \sim y \implies y \sim x$ (symmetry).
   3. $x \sim y$, $y \sim z \implies x \sim z$ (transitivity).
3. A relation $\leq$ on X is called a *partial order* if we have
   1. $x \leq x$ (reflexivity).
   2. $x \leq y$, $y \leq x \implies x = y$ (anti-symmetry).
   3. $x \leq y$, $y \leq z \implies x \leq z$ (transitivity).

   A set X equipped with a partial order is called a *poset* (*partially ordered set*). We write $x < y$ if $x \leq y$ and $x \neq y$.
4. A poset X is called a *chain* (or a *totally ordered set*) if for any $x, y \in X$ we have $x \leq y$ or $y \leq x$.
5. A map $f \colon X \to Y$ between posets is called *order-preserving* (or monotone) if
   $$x \leq y \implies f(x) \leq f(y).$$
   It is called (strictly) *increasing* (or strictly monotone) if
   $$x < y \implies f(x) < f(y).$$
   By an increasing map we will usually mean a strictly increasing map.

Definition A.2. Given an equivalence relation $\sim$ on X, we define an *equivalence class* of $x \in X$ to be $[x] = \{y \in X \mid y \sim x\}$. The set of all equivalence classes of X is denoted by $X/{\sim}$, called the *quotient set* of X by $\sim$.

Example A.3. Let $X = [0, 1]$ and let $x \sim y$ if $x = y$ or $\{x, y\} = \{0, 1\}$. Then $X/{\sim}$ is a circle.

Example A.4. Let G be a group acting on a set X, that is, we have a map (called an *action*)
$$G \times X \to X, \qquad (g, x) \mapsto gx$$
such that $g(hx) = (gh)x$ and $ex = x$ for $g, h \in G$, $x \in X$ and the identity $e \in G$. Define an equivalence relation on X to be $x \sim y$ if $y = gx$ for some $g \in G$. The equivalence classes $[x] = \{gx \mid g \in G\} = Gx$ are called *orbits*. The quotient set $X/{\sim}$ is denoted by X/G, called the *orbit space*.

Definition A.5. Given a relation $R \subset X \times X$, define the *equivalence relation generated by* R to be the minimal equivalence relation $\sim$ that contains R (the intersection of equivalence relations that contain R). We have $x \sim y \iff$ there exist elements $x_0, \ldots, x_n$ with $x_0 = x$, $x_n = y$ and $x_{i-1}Rx_i$ or x_iRx_{i-1} for $1 \leq i \leq n$. The quotient set $X/{\sim}$ is also denoted by X/R.

Remark A.6. Similarly, we can define a partial order generated by a relation $R \subset X \times X$. For its existence we require that whenever $x_0Rx_1R \ldots Rx_nRx_0$, we have $x_0 = x_1 = \cdots = x_n$ (so that anti-symmetry is satisfied).

Example A.7. Given $A \subset X$, let X/A denote the quotient set $X/(a \sim b : a, b \in A)$. It can be decomposed as $X/A = (X - A) \cup \{*\}$, where $*$ is the equivalence class of all elements in A.

Example A.8. For example, let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. Then $S^{n-1} \subset D^n$ and $D^n/S^{n-1} \simeq S^n$. In particular, $S^0 = \{\pm 1\} \subset D^1 = [-1,1]$ and we have $D^1/S^0 = [-1,1]/\{\pm 1\} \simeq S^1$. Similarly $D^2/S^1 \simeq S^2$.

A.2. **Quotient topology.** By a space we always mean a topological space.

Definition A.9. Let X be a set and $\tau, \tau' \subset 2^X$ be two topologies on X. If $\tau \subset \tau'$, then τ is called *weaker* (coarser, smaller) than τ', and τ' is called *stronger* (finer, larger) than τ.

Definition A.10. Let X be a space and $f\colon X \to Y$ be a surjective map onto a set Y. Define the *quotient topology* on Y with respect to f to be

$$\tau = \big\{U \subset Y \,\big|\, f^{-1}(U) \text{ is open in } X\big\}.$$

It is the strongest topology on Y such that $f\colon X \to Y$ is continuous. The space Y equipped with the quotient topology is called the *quotient space*. A surjective map $f\colon X \to Y$ between spaces is called the *quotient map* if Y has the quotient topology.

Lemma A.11. *Let $f\colon X \to Y$ be a quotient map between spaces. Then a map $g\colon Y \to Z$ into a space Z is continuous $\iff$ gf is continuous.*

Example A.12. If X is a topological space equipped with an equivalence relation $\sim$, then we equip $X/\sim$ with the quotient topology with respect to the projection $p\colon X \to X/\sim$. In particular, if $A \subset X$ is a subspace, then we equip X/A with the quotient topology.

Example A.13. Let $f\colon A \to X$ and $g\colon A \to Y$ be two continuous maps. Define their *pushout* to be the quotient space

$$X \sqcup_A Y = X \sqcup Y/(f(a) \sim g(a) : a \in A).$$

The maps $i_1\colon X \to X \sqcup_A Y$ and $i_2\colon Y \to X \sqcup_A Y$ have the following universal property. If continuous $j_1\colon X \to Z$ and $j_2\colon Y \to Z$ satisfy $j_1 f = j_2 g$, then there exists a unique continuous $u\colon X \sqcup_A Y \to Z$ making the following diagram commute

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "i_1"] \arrow[drr, bend right, "j_1"'] & X \sqcup_A Y \arrow[dr, dashed, "u"] & \\
& & Z
\end{tikzcd}$$

Example A.14. In particular, let $A \subset Y$ be a subspace and $\phi\colon A \to X$ be a continuous map. We consider the inclusion map $i\colon A \to Y$ and the corresponding pushout

$$X \cup_\phi Y = X \sqcup Y/(\phi(a) \sim a : a \in A),$$

called the *adjunction space* (or *attaching space*) obtained by attaching Y to X via ϕ

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "i"'] & X \arrow[d, "j"] \\
Y \arrow[r, "\Phi"] & X \cup_\phi Y
\end{tikzcd}$$

There is a decomposition $X \cup_\phi Y = X \sqcup (Y - A)$. The map ϕ is called the *attaching map*. The map Φ is called the *characteristic map*.

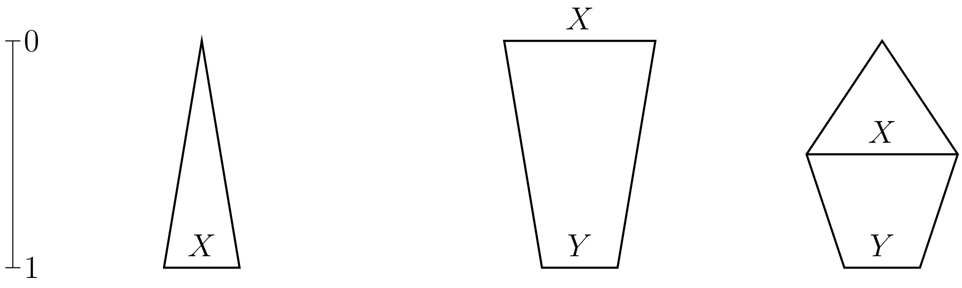


FIGURE 1. Cone, mapping cylinder and mapping cone.

Example A.15. Given a space X, define its *cone* to be the quotient space ($I = [0, 1]$)

$$CX = \frac{X \times I}{X \times \{0\}}.$$

Example A.16. Given a continuous map $f\colon X \to Y$, define its *mapping cylinder* to be the quotient space

$$M_f = \frac{X \times I \sqcup Y}{(x, 1) \sim f(x)}.$$

Define its *mapping cone* to be the quotient space

$$C_f = \frac{CX \sqcup Y}{(x, 1) \sim f(x)} = M_f / X \times \{0\}.$$

Definition A.17. Define a *pointed space* (X, x_0) to be a pair consisting of a space X and a distinguished point $x_0 \in X$. Define the *wedge sum* of pointed spaces (X, x_0) and (Y, y_0) to be the quotient space

$$X \vee Y = X \sqcup Y / (x_0 \sim y_0)$$

with the distinguished point $[x_0] = [y_0]$.

Remark A.18. Note that if $A \subset X$ is a subspace, then X/A has a distinguished point – the equivalence class of points in A.

Remark A.19. The wedge sum can be interpreted as a coproduct in the category of pointed spaces.

Definition A.20. Let $(X_i)_{i \in I}$ be a family of spaces, $X = \prod_i X_i$ and let $p_i\colon X \to X_i$ be projections. The *product topology* on X is the weakest topology that contains sets $p_i^{-1}(U)$ for open $U \subset X_i$. This topology consists of (arbitrary) unions of sets

$$\bigcap_{i \in J} p_i^{-1}(U_i) = \prod_{i \in J} U_i \times \prod_{i \in I \setminus J} X_i,$$

where $J \subset I$ is finite and $U_i \subset X_i$ is open, for $i \in J$.

Lemma A.21. *The product topology on $X = \prod_i X_i$ is the weakest topology such that $p_i\colon X \to X_i$ are continuous. For any family of continuous maps $f_i\colon Z \to X_i$, there exists a unique continuous map $f\colon Z \to X$ such that $p_i f = f_i$, for all i*

$$\begin{tikzcd} Z \arrow[r, dashed, "f"] \arrow[rr, bend left, "f_i"] & \prod_{i \in I} X_i \arrow[r, "p_i"] & X_i. \end{tikzcd}$$

Appendix B. Categories and functors

Definition B.1. A *category* $\mathcal{A}$ consists of the following data

(1) A family $\operatorname{Ob}\mathcal{A}$, whose elements are called objects of $\mathcal{A}$.
(2) For all objects X, Y of $\mathcal{A}$, a set $\operatorname{Hom}(X,Y) = \operatorname{Hom}_{\mathcal{A}}(X,Y)$, whose elements are called morphisms from X to Y.
(3) For all objects X, Y, Z of $\mathcal{A}$, a map

$$\operatorname{Hom}(X,Y) \times \operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z), \qquad (f,g) \mapsto g \circ f,$$

called the composition map.

This data should satisfy

(1) $\forall X \in \operatorname{Ob}\mathcal{A}$, $\exists 1_X \in \operatorname{Hom}(X,X)$ s.t. $1_Y \circ f = f \circ 1_X = f$ for any $f \in \operatorname{Hom}(X,Y)$.
(2) The composition of morphisms is associative.

Remark B.2.

(1) The element 1_X is unique for every $X \in \operatorname{Ob}\mathcal{A}$.
(2) We write $f\colon X \to Y$ for $f \in \operatorname{Hom}(X,Y)$.
(3) A morphism $f\colon X \to Y$ is called an isomorphism if $\exists g\colon Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.

Example B.3.

(1) The category $\operatorname{Mod} A$ of modules over a ring A and homomorphisms between them.
(2) The category **Set** of sets and all maps between them.
(3) The category Top of topological spaces and continuous maps between them.
(4) The category Com of commutative rings and ring homomorphisms.
(5) The category Grp of groups and group homomorphisms.
(6) The category $A_\bullet$ of abelian groups and group homomorphisms. It can be identified with $\operatorname{Mod}\mathbb{Z}$.

Definition B.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. A *(covariant) functor* F from $\mathcal{A}$ to $\mathcal{B}$ consists of the following data

(1) A map $F\colon \operatorname{Ob}(\mathcal{A}) \to \operatorname{Ob}(\mathcal{B})$.
(2) A map $F\colon \operatorname{Hom}_{\mathcal{A}}(X,Y) \to \operatorname{Hom}_{\mathcal{B}}(FX,FY)$ for all objects $X, Y \in \operatorname{Ob}(\mathcal{A})$.

This data should satisfy

(1) $F(1_X) = 1_{FX}$ for all $X \in \operatorname{Ob}(\mathcal{A})$.
(2) $F(g \circ f) = F(g) \circ F(f)$.

Definition B.5.

(1) Given a category $\mathcal{A}$, we define the *opposite category* $\mathcal{A}^{\mathrm{op}}$ using the data

$$\operatorname{Ob}(\mathcal{A}^{\mathrm{op}}) = \operatorname{Ob}(\mathcal{A}), \qquad \operatorname{Hom}_{\mathcal{A}^{\mathrm{op}}}(X,Y) = \operatorname{Hom}_{\mathcal{A}}(Y,X).$$

(2) A functor from $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{B}$ is called a *contravariant functor* from $\mathcal{A}$ to $\mathcal{B}$.

Example B.6.

(1) Given a category $\mathcal{A}$ and an object X, there is a (covariant) functor

$$\operatorname{Hom}(X,-)\colon \mathcal{A} \to \mathbf{Set}, \qquad Y \mapsto \operatorname{Hom}(X,Y).$$

There is also a contravariant functor

$$\operatorname{Hom}(-,X)\colon \mathcal{A} \to \mathbf{Set}, \qquad Y \mapsto \operatorname{Hom}(Y,X).$$

(2) For the category $\operatorname{Mod} A$ and an A-module M, we have similar functors $\operatorname{Hom}(M, -)$ and $\operatorname{Hom}(-, M)$ from $\operatorname{Mod} A$ to $\operatorname{Mod} A$.

(3) For any A-module N, there is a functor

$$- \otimes N \colon \operatorname{Mod} A \to \operatorname{Mod} A, \qquad M \mapsto M \otimes N.$$

Definition B.7. Let F, G be two functors from $\mathcal{A}$ to $\mathcal{B}$. A *morphism* (or *natural transformation*) ϕ from F to G consists of the data

(1) Morphism $\phi_X \colon FX \to GX$ for every object $X \in \operatorname{Ob}(\mathcal{B})$

such that for every $f \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$ the following diagram commutes

$$\begin{array}{ccc} FX & \xrightarrow{F(f)} & FY \\ {\scriptstyle \phi_X}\downarrow & & \downarrow{\scriptstyle \phi_Y} \\ GX & \xrightarrow{G(f)} & GY \end{array}$$

Definition B.8. Let $f \colon A \to B$ be a ring homomorphism.

(1) Given a B-module M, we can consider it as an A-module by setting $ax = f(a)x$ for $a \in A$, $x \in M$. In this way we obtain a functor $\operatorname{Mod} B \to \operatorname{Mod} A$, called a *restriction of scalars*.

(2) Given an A-module M, we consider a B-module

$$M_B = B \otimes_A M, \qquad b(b' \otimes x) = bb' \otimes x, \qquad b, b' \in B,\ x \in M.$$

In this way we obtain a functor $B \otimes_A - \colon \operatorname{Mod} A \to \operatorname{Mod} B$, called an *extension of scalars*.

Definition B.9. Two functors $F \colon \mathcal{A} \to \mathcal{B}$, $G \colon \mathcal{B} \to \mathcal{A}$ are called *adjoint* if there exist natural bijections

$$\operatorname{Hom}_{\mathcal{B}}(F(X), Y) \simeq \operatorname{Hom}_{\mathcal{A}}(X, G(Y)) \qquad \forall\ X \in \operatorname{Ob}(\mathcal{A}),\ Y \in \operatorname{Ob}(\mathcal{B}).$$

In this case F is called a *left adjoint* functor to G and G is called a *right adjoint* functor to F.

Example B.10. There is a Tensor-Hom adjunction

$$\operatorname{Hom}(L \otimes M, N) \simeq \operatorname{Hom}(L, \operatorname{Hom}(M, N)).$$

for A-modules L, M, N. It implies that the functors

$$F \colon \operatorname{Mod} A \to \operatorname{Mod} A, \qquad L \mapsto L \otimes M,$$
$$G \colon \operatorname{Mod} A \to \operatorname{Mod} A, \qquad N \mapsto \operatorname{Hom}(M, N)$$

are adjoint.

Appendix C. Simplicial approximation

A *simplicial map* $f\colon K \to L$ between two simplicial complexes is a map $f\colon K_0 \to L_0$ such that, for every simplex $\sigma \in K$ (which is a subset of K_0), the set $f(\sigma) \subset L_0$ is a simplex in L. Such map induces a continuous map $|f|: |K| \to |L|$ between geometric realizations.

Recall, that for every simplicial complex K, we have its barycentric subdivision $N(K)$ which consists of all finite non-empty chains of simplices in K. There is a homeomorphism $|N(K)| \to |K|$.

Definition C.1. A *simplicial approximation* of a map $F\colon |K| \to |L|$ is a simplicial map $f\colon K \to L$ such that, for every simplex $\tau \in L$,

$$x \in |K|, \, F(x) \in \Delta(\tau) \implies |f|(x) \in \Delta(\tau).$$

Lemma C.2. *The above definition is equivalent to the requirement that, for every $v \in K_0$,*

$$F\Big(\bigcup_{\sigma \ni v} \mathring{\Delta}(\sigma)\Big) \subset \bigcup_{\tau \ni f(v)} \mathring{\Delta}(\tau)$$

Proof. Assume that f is a simplicial approximation of F. Let $x \in \mathring{\Delta}(\sigma)$ for some $\sigma \ni v$. Then $\tau = f(\sigma)$ is a simplex in L and $|f|(x) \in \mathring{\Delta}(\tau)$. By assumption, if $F(x) \in \mathring{\Delta}(\tau')$ (there always exists such simplex), then $|f|(x) \in \Delta(\tau')$, hence $\tau \subset \tau'$. But $f(v) \in \tau$, hence $f(v) \in \tau'$. Conversely, we can assume that $F(x) \in \mathring{\Delta}(\tau)$ and we need to show that $|f|(x) \in \Delta(\tau)$. Let $x \in \mathring{\Delta}(\sigma)$ for some $\sigma \in K$. By assumption, for every $v \in \sigma$, we have $F(x) \in \mathring{\Delta}(\tau')$ for some $\tau' \ni f(v)$. But such simplex is unique and we get $\tau' = \tau$, hence $f(v) \in \tau$. We conclude that $f(\sigma) \subset \tau$, hence $|f|(x) \in |f|(\Delta(\sigma)) \subset \Delta(\tau)$.

□

Proposition C.3. *If $f\colon K \to L$ is a simplicial approximation to $F\colon |K| \to |L|$, then $|f|$ is homotopic to F.*

Theorem C.4 (Simplicial approximation)**.** *For any continuous map $F\colon |K| \to |L|$, there exists $r \geq 1$ and a simplicial approximation $f\colon N^r(K) \to L$ to F.*