INTRODUCTION TO DONALDSON–THOMAS INVARIANTS

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Abstract. These notes represent three lectures given at the International Conference on Representations of Algebras (ICRA) held at Bielefeld in August 2012. The goal of the lectures was to introduce refined Donaldson–Thomas invariants for the categories of modules over Jacobian algebras associated to quivers with potentials. We define refined Donaldson–Thomas invariants and compute them in some simple cases and then discuss their basic properties, including integrality, positivity, and wall-crossing phenomena.

1. Lecture 1

Donaldson–Thomas invariants are numbers that virtually count sheaves on 3-Calabi-Yau manifolds. One of the motivation for their study is the MNOP conjecture [11] (proved for example for toric 3-Calabi-Yau varieties) that asserts a close relationship between Donaldson–Thomas invariants and Gromov–Witten invariants of the same 3-Calabi-Yau manifold (the latter invariants virtually count curves on the manifold). In [9] Kontsevich and Soibelman developed a framework for the refined Donaldson–Thomas invariants of non-commutative 3-Calabi-Yau varieties, that is, of triangulated 3-Calabi-Yau A_∞-categories. This framework allows us to shift the study of Donaldson–Thomas invariants from 3-Calabi-Yau varieties to other sources of 3-Calabi-Yau categories. One of such sources, closely related to representation theory, consists of quivers with potentials. The goal of these lectures is to define and study refined (or quantized) Donaldson–Thomas invariants of the categories associated to quivers with potentials. According to Kontsevich and Soibelman [9], the Donaldson–Thomas invariants correspond to counting functions of BPS states in physics. Therefore it is natural to expect the integrality and positivity properties for the Donaldson–Thomas invariants. In the first lecture we will quickly introduce numerical Donaldson–Thomas invariants for one simple example and we will see how it can be quantized. In the second lecture we will introduce refined Donaldson–Thomas invariants for arbitrary quivers and we will see that the example from the first lecture corresponds to the generalized Kronecker quiver. We will also discuss various properties of refined Donaldson–Thomas invariants. In the third lecture we will introduce refined Donaldson–Thomas invariants for quivers with potentials and compute them in some situations.

1.1. **Numerical DT invariants.** Let $m > 0$ be an integer and let $\overline{\mathbb{A}} = \mathbb{Q}[\![x_1, x_2]\!]$ be a commutative algebra with multiplication

$$x_1^a x_2^b \cdot x_1^c x_2^d = (-1)^{m(ad-bc)} x_1^{a+c} x_2^{b+d}.$$

For any $(a, b) \in \mathbb{N}^2 \backslash \{0\}$, define an automorphism $T_{a,b}$ of the algebra $\overline{\mathbb{A}}$ by

$$x_1 \mapsto x_1 \cdot (1 - x_1^a x_2^b)^{-mb}, \qquad x_2 \mapsto x_2 \cdot (1 - x_1^a x_2^b)^{ma}.$$

The following result was proved by Reineke [17] and Kontsevich and Soibelman [10]

Theorem 1.1 (Integrality conjecture)**.** *There exist uniquely determined integers* $\overline{\Omega}_{a,b}$ *for* $(a,b)\in\mathbb{N}^2\backslash\{0\}$ *such that*

$$T_{1,0}\cdot T_{0,1}=\prod_{\frac{a}{b}\uparrow}T_{a,b}^{\overline{\Omega}_{a,b}},\tag{1}$$

where the product over $(a,b)\in\mathbb{N}^2\backslash\{0\}$ *is taken in increasing order of* $\frac{a}{b}$*. The integers* $\overline{\Omega}_{a,b}$ *are called numerical Donaldson–Thomas invariants.*

Example 1.2. For $m=1$ one can directly check that

$$T_{1,0}\cdot T_{0,1}=T_{0,1}\cdot T_{1,1}\cdot T_{1,0}.$$

For $m=2$ one has

$$T_{1,0}\cdot T_{0,1}=T_{0,1}\cdot T_{1,2}\cdot T_{2,3}\dots T_{1,1}^{-2}\dots T_{3,2}\dots T_{2,1}\dots T_{1,0}.$$

1.2. **Poisson algebra approach.** Let us give an alternative definition of the automorphisms $T_{a,b}$. Define a skew-symmetric form $\langle\cdot\,,\cdot\rangle$ on $\mathbb{Z}^2$ by

$$\langle(a,b),(c,d)\rangle=m(ad-bc).$$

Then for any $\alpha=(a,b)\in\mathbb{N}^2\backslash\{0\}$ the automorphism $T_\alpha=T_{a,b}$ is given by

$$x^\beta\mapsto x^\beta\cdot(1-x^\alpha)^{\langle\alpha,\beta\rangle}=x^\beta(1-(-1)^{\langle\alpha,\beta\rangle}x^\alpha)^{\langle\alpha,\beta\rangle},\qquad\beta\in\mathbb{N}^2.$$

It preserves the Poisson algebra structure on $\overline{\mathbb{A}}=\mathbb{Q}[\![x_1,x_2]\!]$ given by

$$\{x^\alpha,x^\beta\}=\langle\alpha,\beta\rangle\,x^\alpha\cdot x^\beta=(-1)^{\langle\alpha,\beta\rangle}\,\langle\alpha,\beta\rangle\,x^{\alpha+\beta}.$$

Remark 1.3. *A Poisson algebra is a commutative algebra together with a Lie bracket* $\{\cdot\,,\cdot\}$ *(called in this situation a Poisson bracket) satisfying the Leibnitz rule*

$$\{f,g\cdot h\}=\{f,g\}\cdot h+g\cdot\{f,h\}.$$

To see that T_α preserves the Poisson bracket we write T_α as an adjoint automorphism of the corresponding Lie algebra:

Lemma 1.4. *We have*

$$T_\alpha=\operatorname{Ad}\exp\Big(-\sum_{n\ge1}\frac{x^{n\alpha}}{n^2}\Big).$$

Proof. We can write

$$\begin{aligned}\operatorname{Ad}\exp\Big(\frac{-x^{n\alpha}}{n^2}\Big)(x^\beta)&=\sum_{k\ge0}\frac{1}{k!}\operatorname{ad}\Big(\frac{-x^{n\alpha}}{n^2}\Big)^k(x^\beta)\\&=\sum_{k\ge0}\frac{1}{k!}\Big(\frac{-\langle\alpha,\beta\rangle\,x^{n\alpha}}{n}\Big)^k\cdot x^\beta=\exp\Big(\frac{-\langle\alpha,\beta\rangle\,x^{n\alpha}}{n}\Big)\cdot x^\beta.\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Ad}\exp\Big(-\sum_{n\ge1}\frac{x^{n\alpha}}{n^2}\Big)(x^\beta)&=\exp\Big(-\sum_{n\ge1}\frac{x^{n\alpha}}{n}\Big)^{\langle\alpha,\beta\rangle}\cdot x^\beta\\&=(1-x^\alpha)^{\langle\alpha,\beta\rangle}\cdot x^\beta=T_\alpha(x^\beta).\end{aligned}$$

□

Remark 1.5. *The series*

$$\mathrm{Li}_2(x) = \sum_{n\ge 1} \frac{x^n}{n^2}$$

that occurred in the previous lemma is called a dilogarithm.

Given a lattice $\Gamma = \mathbb{Z}^r$ with a skew-symmetric form $\langle\cdot\,,\cdot\rangle$ we can, similarly to the above discussion, define a Poisson algebra structure on $\overline{\mathbb{A}}_\Gamma = \mathbb{Q}[\![x_1,\dots,x_r]\!]$:

$$x^\alpha \cdot x^\beta = (-1)^{\langle\alpha,\beta\rangle} x^{\alpha+\beta}, \qquad \{x^\alpha, x^\beta\} = (-1)^{\langle\alpha,\beta\rangle}\,\langle\alpha,\beta\rangle\, x^{\alpha+\beta}.$$

Similarly, for any $\alpha \in \mathbb{N}^2\backslash\{0\}$, define a Poisson algebra automorphism T_α of $\overline{\mathbb{A}}_\Gamma$ by

$$x^\beta \mapsto x^\beta \cdot (1 - x^\alpha)^{\langle\alpha,\beta\rangle}.$$

As before, we can prove that

$$T_\alpha = \mathrm{Ad}\exp\Big(-\sum_{n\ge1}\frac{x^{n\alpha}}{n^2}\Big).$$

1.3. **Quantization.** The Poisson algebra $\overline{\mathbb{A}}_\Gamma$ can be quantized [16]. This means that we can define an associative algebra structure on $\mathbb{A}_\Gamma^\circ = \mathbb{Q}[q^{\pm\frac12}][\![x_1,\dots,x_r]\!]$ (or $\mathbb{A}_\Gamma = \mathbb{Q}(q^{\frac12})[\![x_1,\dots,x_r]\!]$)

$$x^\alpha \circ x^\beta = (-q^{\frac12})^{\langle\alpha,\beta\rangle} x^{\alpha+\beta}$$

such that $\overline{\mathbb{A}}_\Gamma = \mathbb{A}_\Gamma^\circ|_{q^{\frac12}=1}$, that is,

$$\overline{\mathbb{A}}_\Gamma \simeq \mathbb{A}_\Gamma^\circ/(q^{\frac12}-1)$$

as Poisson algebras, with the Poisson bracket on the right given by

$$\{f,g\} = \frac{fg - gf}{q-1} \pmod{(q^{\frac12}-1)}.$$

Indeed

$$\begin{aligned}\{x^\alpha, x^\beta\} = \frac{(-q^{\frac12})^{\langle\alpha,\beta\rangle} - (-q^{\frac12})^{-\langle\alpha,\beta\rangle}}{q-1} x^{\alpha+\beta}\\ \equiv (-1)^{\langle\alpha,\beta\rangle}\,\langle\alpha,\beta\rangle\, x^{\alpha+\beta} \pmod{(q^{\frac12}-1)}.\end{aligned}$$

The automorphism T_α can also be quantized.

Lemma 1.6. *T_α is the specialization at $q^{\frac12}=1$ of the adjoint operator* $\mathrm{Ad}(\mathbb{E}(x^\alpha))$*, where*

$$\mathbb{E}(x) = \exp\Big(\sum_{n\ge1}\frac{1}{n}\frac{(q^{\frac12}x)^n}{1-q^n}\Big) \in \mathbb{Q}(q^{\frac12})[\![x]\!].$$

Proof. For any $\beta\in\Gamma$, define an algebra automorphism S_β of $\mathbb{A}_\Gamma$ by $S_\beta(x^\alpha) = q^{\langle\alpha,\beta\rangle}x^\alpha$. Then

$$x^\alpha \circ x^\beta = q^{\langle\alpha,\beta\rangle} x^\beta \circ x^\alpha = x_\beta \circ S_\beta(x^\alpha)$$

and more generally

$$f \circ x^\beta = x^\beta \circ S_\beta(f), \qquad f \in \mathbb{A}_\Gamma.$$

We can write

$$\mathbb{E}(x^\alpha)\circ x^\beta\circ\mathbb{E}(x^\alpha)^{-1} = x^\beta\circ S_\beta\mathbb{E}(x^\alpha)\circ\mathbb{E}(x^\alpha)^{-1} = x^\beta\circ\exp\Big(\sum_{n\ge1}\frac{1}{n}\frac{(q^{n\langle\alpha,\beta\rangle}-1)(q^{\frac12}x^\alpha)^n}{1-q^n}\Big)$$

$$\equiv x^\beta\cdot\exp\Big(\sum_{n\ge1}\frac{-x^{n\alpha}}{n}\Big)^{\langle\alpha,\beta\rangle} \equiv T_\alpha(x^\beta) \pmod{(q^{\frac12}-1)}.$$

□

Remark 1.7 (Quantum dilogaritm). *One can show, using the q-binomial theorem, that*

$$\mathbb{E}(x) = \exp\Big(\sum_{n\ge1}\frac{1}{n}\frac{(q^{\frac12}x)^n}{1-q^n}\Big) = \sum_{n\ge0}\frac{(q^{\frac12}x)^n}{(q)_n} = \sum_{n\ge0}\frac{(-q^{\frac12})^{n^2}}{|\mathrm{GL}_n(\mathbb{F}_q)|}x^n, \tag{2}$$

where $(q)_n = (1-q)\dots(1-q^n)$ and for the last equality we assume q to be the number of elements in a finite field $\mathbb{F}_q$. The series $\mathbb{E}(x)$ is called the (exponential of the) quantum dilogarithm.

Remark 1.8. *Define the plethystic exponential* $\mathrm{Exp} : \mathfrak{m} \to 1+\mathfrak{m}$ *on the maximal ideal*

$$\mathfrak{m}\subset\mathbb{Q}(q^{\frac12})[\![x_1,\dots,x_r]\!]$$

by the rules

$$\mathrm{Exp}(f+g) = \mathrm{Exp}(f)\,\mathrm{Exp}(g),\qquad \mathrm{Exp}(q^{\frac{k}{2}}x^\alpha) = \frac{1}{1-q^{\frac{k}{2}}x^\alpha}.$$

One can show that

$$\mathrm{Exp}(f) = \exp\Big(\sum_{n\ge1}\frac{1}{n}\psi_n f\Big),\qquad \psi_n f(q^{\frac12},x_1,\dots,x_r) = f(q^{\frac{n}{2}},x_1^n,\dots,x_r^n).$$

In particular

$$\mathbb{E}(x) = \mathrm{Exp}\Big(\frac{q^{\frac12}}{1-q}x\Big) = \mathrm{Exp}\left(\frac{x}{q^{-\frac12}-q^{\frac12}}\right). \tag{3}$$

If we substitute operators T_α by the elements $\mathbb{E}(x^\alpha)\in\mathbb{A}_\Gamma$ in the integrality conjecture (1), it is natural to expect an identity of the form (the lattice is $\Gamma=\mathbb{Z}^2$)

$$\mathbb{E}(x_1)\circ\mathbb{E}(x_2) = \prod_{\frac{a}{b}\uparrow}\mathbb{E}(x_1^a x_2^b)^{\Omega_{a,b}}$$

for some $\Omega_{a,b}\in\mathbb{Q}(q)$. This is indeed the case if we use the plethystic power map

$$f^g := \mathrm{Exp}(g\,\mathrm{Log}(f)), \tag{4}$$

where $\mathrm{Log} : 1+\mathfrak{m}\to\mathfrak{m}$ is inverse to the plethystic exponential Exp. More precisely, the following result is true.

Theorem 1.9 (Kontsevich-Soibelman [10]). *There exist uniquely determined polynomials $\Omega_\alpha\in\mathbb{Z}[q^{\pm\frac12}]$ for $\alpha\in\mathbb{N}^2\backslash\{0\}$ such that*

$$\mathbb{E}(x_1)\circ\mathbb{E}(x_2) = \prod_{\alpha\uparrow}\mathbb{E}(x^\alpha)^{\Omega_\alpha} = \prod_{\alpha\uparrow}\mathrm{Exp}\left(\frac{\Omega_\alpha x^\alpha}{q^{-\frac12}-q^{\frac12}}\right),$$

where the products over $\alpha=(a,b)\in\mathbb{N}^2\backslash\{0\}$ are taken in increasing order of $\frac{a}{b}$. The polynomials Ω_α are called quantized (or refined) Donaldson–Thomas invariants.

Remark 1.10. *The specialization of* $\operatorname{Ad}\mathbb{E}(x^\alpha)^{\Omega_\alpha}$ *at* $q^{\frac12}=1$ *is* $T_\alpha^{\Omega_\alpha(1)}$. *Therefore* $\overline{\Omega}_\alpha=\Omega_\alpha(1)$. *Indeed, given an element* $P\in 1+\mathfrak{m}$ *such that the limit*

$$(q-1)\log(P)|_{q^{\frac12}=1}$$

is well-defined, the specialization of $\operatorname{Ad}P$ *at* $q^{\frac12}=1$ *exists and equals*

$$\operatorname{Ad}\exp\left((q-1)\log(P)|_{q^{\frac12}=1}\right).$$

In our case we obtain

$$\begin{aligned}(q-1)\log\left(\mathbb{E}(x^\alpha)^{\Omega_\alpha}\right)&=(q-1)\log\operatorname{Exp}\left(\frac{q^{\frac12}}{1-q}\Omega_\alpha(q^{\frac12})x^\alpha\right)\\&=\sum_{n\ge1}\frac1n\frac{q-1}{1-q^n}\Omega_\alpha(q^{\frac n2})(q^{\frac12}x^\alpha)^n\equiv-\Omega_\alpha(1)\sum_{n\ge1}\frac{x^{n\alpha}}{n^2}\pmod{(q^{\frac12}-1)}.\end{aligned}$$

Therefore the specialization of $\operatorname{Ad}\mathbb{E}(x^\alpha)^{\Omega_\alpha}$ *is given by*

$$\operatorname{Ad}\exp\left(-\Omega_\alpha(1)\sum_{n\ge1}\frac{x^{n\alpha}}{n^2}\right)=T_\alpha^{\Omega_\alpha(1)}.$$

2. Lecture 2

2.1. **Quantum affine space of a quiver.** Let $Q=(Q_0,Q_1)$ be a quiver and let

$$\Gamma=\mathbb{Z}^{Q_0},\qquad \Gamma_+=\mathbb{N}^{Q_0}.$$

The category $\operatorname{Rep}(Q)$ of finite-dimensional representations of Q over a field $\Bbbk$ is hereditary, that is, $\operatorname{Ext}^i(M,N)=0$ for any $i\ge2$ and $M,N\in\operatorname{Rep}(Q)$. Define the Euler-Ringel form on Γ by

$$\chi(\alpha,\beta)=\sum_{i\in Q_0}\alpha_i\beta_i-\sum_{a:i\to j}\alpha_i\beta_j,\qquad \alpha,\beta\in\Gamma.$$

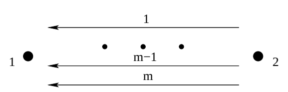
Its basic property is

$$\sum_{i\ge0}(-1)^i\dim\operatorname{Ext}^i(M,N)=\chi(\underline{\dim}M,\underline{\dim}N),\qquad M,N\in\operatorname{Rep}(Q),$$

where $\underline{\dim}M=(\dim M_i)_{i\in Q_0}\in\mathbb{N}^{Q_0}$ is the dimension vector of the representation M. Define a skew-symmetric form $\langle\cdot\,,\cdot\rangle$ on Γ by

$$\langle\alpha,\beta\rangle=\chi(\alpha,\beta)-\chi(\beta,\alpha).$$

Example 2.1. Let Q be the m-Kronecker quiver. This is the quiver with two vertices $1,2$ and m arrows from 2 to 1.



Then

$$\langle\alpha,\beta\rangle=m(ad-bc),\qquad \alpha=(a,b),\ \beta=(c,d).$$

This is the same skew-symmetric form that we seen in the first lecture.

As before, we define an associative algebra $\mathbb{A}_\Gamma = \mathbb{Q}(q^{\frac{1}{2}})[\![x_1, \dots, x_r]\!]$ (where r is the number of vertices of Q) with a product

$$x^\alpha \circ x^\beta = (-q^{\frac{1}{2}})^{\langle \alpha, \beta \rangle} x^{\alpha+\beta}.$$

This algebra is called the quantum affine space of the quiver Q.

2.2. **Stability conditions.** A stability function (see [3, §2]) on the quiver Q is a group homomorphism $Z : \Gamma \to \mathbb{C}$ such that

$$Z(\underline{\dim} M) \in \mathbb{H}_+ = \{re^{i\pi\varphi} \mid r > 0, \varphi \in (0,1]\}, \qquad 0 \neq M \in \operatorname{Rep}(Q).$$

Define the phase of $\alpha \in \Gamma_+ \backslash \{0\}$ to be the number $\varphi(\alpha) \in (0,1]$ such that $Z(\alpha) = re^{i\pi\varphi(\alpha)}$. Define $\varphi(M) = \varphi(\underline{\dim} M)$.

A representation M is called Z-semistable if for any subrepresentation $0 \neq N \subsetneq M$

$$\varphi(N) \leq \varphi(M).$$

Remark 2.2. *Let $Z = -d + ir$ for $d, r : \Gamma \to \mathbb{R}$. Define the slope $\mu(\alpha) = \frac{d(\alpha)}{r(\alpha)}$ for $\alpha \in \Gamma_+ \backslash \{0\}$. Then*

$$\varphi(\alpha) \leq \varphi(\beta) \qquad \text{iff} \qquad \mu(\alpha) \leq \mu(\beta)$$

and we can use the slope function to verify the semistability.

Theorem 2.3 (Harder-Narasimhan filtration, see e.g. [3, §2])**.** *For any representation M there exists a unique filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

with semistable M_i/M_{i-1} and

$$\varphi(M_1/M_0) > \cdots > \varphi(M_n/M_{n-1}).$$

2.3. **Harder-Narasimhan formula.** For any $\alpha \in \Gamma_+ \backslash \{0\}$, let

$$R(Q,\alpha) = \bigoplus_{e:i \to j} \operatorname{Hom}_{\Bbbk}(\Bbbk^{\alpha_i}, \Bbbk^{\alpha_j})$$

and let $R_Z^{\mathrm{sst}}(Q,\alpha) \subset R(Q,\alpha)$ be the subset of semistable representations. They admit an action of $\mathrm{GL}_\alpha(\Bbbk) = \prod_{i \in Q_0} \mathrm{GL}_{\alpha_i}(\Bbbk)$ by conjugation and $R(Q,\alpha)/\mathrm{GL}_\alpha(\Bbbk)$ parametrizes the set of isomorphism classes of Q-representations having dimension vector α.

Now let $\Bbbk = \mathbb{F}_q$ be a finite field. For any ray $l \subset \mathbb{H}_+$ (subset in $\mathbb{H}_+$ of the form $\mathbb{R}_{>0}\gamma$ for some $\gamma \in \mathbb{H}_+$), define

$$A_l^Z = 1 + \sum_{Z(\alpha) \in l} (-q^{\frac{1}{2}})^{\chi(\alpha,\alpha)} \frac{|R_Z^{\mathrm{sst}}(Q,\alpha)|}{|\mathrm{GL}_\alpha(\mathbb{F}_q)|} x^\alpha \in \mathbb{A}_\Gamma. \tag{5}$$

Theorem 2.4 (Wall-crossing formula/Harder-Narasimhan formula, see e.g. [15, Prop. 4.12])**.** *We have*

$$\prod_{l \subset \mathbb{H}_+}^{\curvearrowright} A_l^Z = \sum_{\alpha \in \Gamma_+} (-q^{\frac{1}{2}})^{\chi(\alpha,\alpha)} \frac{|R(Q,\alpha)|}{|\mathrm{GL}_\alpha(\mathbb{F}_q)|} x^\alpha =: A_Q, \tag{6}$$

where the ordered product over rays is taken in clockwise order. In particular, the product on the left is independent of the stability function Z.

We will sketch the proof of the above theorem. Let $\mathcal{A} = \operatorname{Rep}(Q, \Bbbk)$ be the category of Q-representations over $\Bbbk = \mathbb{F}_q$ (or any other exact $\Bbbk$-linear category with finite Hom and Ext^1). The Hall algebra $H_Q = H(\mathcal{A})$ of the category $\mathcal{A}$ (see e.g. [18, 19]) is spanned over $\mathbb{Q}$ by all isomorphism classes of objects in $\mathcal{A}$. Multiplication is given by

$$[M] \circ [N] = \sum_{[X]} g^X_{MN}[X], \qquad g^X_{MN} = |\{U \subset X \mid X/U \simeq M, U \simeq N\}|.$$

Similarly, we define the completed Hall algebra $\widehat{H}_Q = \prod_{[M]\in\mathcal{A}} \mathbb{Q}\cdot[M]$.

Theorem 2.5 (Ringel [18]). *The algebra H_Q (and $\widehat{H}_Q$) is associative.*

Proposition 2.6 (Reineke [15]). *The map $I : \widehat{H}_Q^{\mathrm{op}} \to \mathbb{A}_\Gamma$*

$$[M] \mapsto (-q^{\frac{1}{2}})^{\chi(\underline{\dim} M, \underline{\dim} M)} \frac{x^{\underline{\dim} M}}{|\operatorname{Aut} M|}$$

is an algebra homomorphism.

Remark 2.7. *Theorem 2.5 can be proved for any exact category. Proposition 2.6 can be proved only for hereditary categories (i.e. abelian categories with vanishing* Ext^i *for $i \geq 2$). The heredity assumption is an important limitation and is the reason why one has to develop more involved techniques to prove the wall-crossing formulas for* 3*-Calabi-Yau categories, which we will discuss in the third lecture.*

The previous result implies that relations in the Hall algebra can be translated into relations in the quantum torus. In particular, for any ray $l \in \mathbb{H}_+$ define

$$\mathcal{A}^Z_l = 1 + \sum_{\substack{[M]\ Z-\mathrm{sst} \\ \underline{\dim} M \in Z^{-1}(l)}} [M],$$

where the sum runs over all isomorphism classes of Z-semistable representations M of Q with $\underline{\dim} M \in Z^{-1}(l)$. Then, by the Harder-Narasimhan filtration 2.3, we obtain

$$\prod^{\curvearrowleft}_{l\subset\mathbb{H}_+} [\mathcal{A}^Z_l] = \sum_{[M]} [M],$$

where the sum on the right runs over all isomorphism classes of representations of Q. Applying the integration map $I : \widehat{H}_Q^{\mathrm{op}} \to \mathbb{A}_\Gamma$, we obtain the Harder-Narasimhan formula in the quantum torus (6).

2.4. **Donaldson–Thomas invariants.**

Definition 2.8. (see [10, Def. 21]) Assume that $l \subset \mathbb{H}_+$ is a ray such that

$$\langle \alpha, \beta \rangle = 0 \qquad \text{whenever} \qquad Z(\alpha), Z(\beta) \in l. \tag{7}$$

Define the Donaldson–Thomas invariants $\Omega^Z_\alpha \in \mathbb{Q}(q^{\frac{1}{2}})$ for $\alpha \in Z^{-1}(l)$ by the formula

$$A^Z_l = \prod_{Z(\alpha)\in l} \mathbb{E}(x^\alpha)^{\Omega^Z_\alpha} = \operatorname{Exp}\left(\frac{\sum_{Z(\alpha)\in l} \Omega^Z_\alpha x^\alpha}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}\right) \tag{8}$$

where $\mathbb{E}(x) = \operatorname{Exp}\left(\frac{x}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}\right)$ is the quantum dilogarithm.

Remark 2.9. *Condition* (7) *is automatically satisfied if the quiver Q is symmetric, that is, if the number of arrows from i to j equals the number of arrows from j to i for any vertices i, j in Q. In this case the bilinear form $\langle\cdot,\cdot\rangle$ is identically zero.*

Remark 2.10. *Condition* (7) *means that the subalgebra of $\mathbb{A}_\Gamma$ generated by x^α with $Z(\alpha) \in l$ is commutative, with multiplication coinciding with the usual multiplication in $\mathbb{Q}(q^{\frac{1}{2}})[\![x_1, \ldots, x_r]\!]$. That is the reason why we can talk about the plethystic exponent on this subalgebra. Although we can use equation* (8) *without the assumption* (7)*, this is unjustified for two reasons: we apply the plethystic exponential which is defined on the commutative algebra $\mathbb{Q}(q^{\frac{1}{2}})[\![x_1, \ldots, x_r]\!]$, while A_l^Z lives in the quantum affine space $\mathbb{A}_\Gamma$, but more importantly, as direct computations show, the invariants Ω_α^Z do not have any nice properties in contrast to the situation where condition* (7) *is satisfied (more on this later).*

Example 2.11. Let Q be a quiver with one vertex and no loops. Then $\operatorname{Rep} Q$ is equivalent to the category of finite-dimensional vector spaces and (see Remark 1.7)

$$A_Q = \sum_{n\ge 0} (-q^{\frac{1}{2}})^{n^2} \frac{x^n}{\#\operatorname{GL}_n(\mathbb{F}_q)} = \mathbb{E}(x),$$

There exists just one stability function up to equivalence (we say that two stability functions are equivalent if the corresponding phase functions induce the same partial preorder on $\Gamma_+\backslash\{0\}$) and we have $\Omega_1 = 1$ and $\Omega_n = 0$ for $n > 1$.

Example 2.12. Let Q be the m-Kronecker quiver as in Example 2.1. There are essentially just two non-trivial stability conditions on Q: the one with $\varphi(e_1) > \varphi(e_2)$ and the other with $\varphi(e_1) < \varphi(e_2)$

(1) Let Z_- be a stability function such that $\varphi(e_1) > \varphi(e_2)$. Let $M = (M_1, M_2, f)$ be a representation (here M_1, M_2 are vector spaces and $f = (f_i : M_2 \to M_1)_{i=1,\ldots,m}$ is an m-tuple of linear maps). Then $M' = (M_1, 0, 0)$ is a subrepresentation and we have $\varphi(M') > \varphi(M)$. This implies that if M is semistable then either M_1 or M_2 is zero. Semistable objects concentrated in the first vertex form a category equivalent to the category of vector spaces. The corresponding generating function $A_{l_1}^{Z_-}$ is equal, by the previous example, to $\mathbb{E}(x_1)$. Similarly, the generating function $A_{l_2}^{Z_-}$ of representations concentrated in the second vertex is equal to $\mathbb{E}(x_2)$. We obtain

$$\prod_{l\subset\mathbb{H}_+}^{\curvearrowright} A_l^{Z_-} = \mathbb{E}(x_1) \circ \mathbb{E}(x_2).$$

(2) Let Z_+ be a stability function such that $\varphi(e_1) < \varphi(e_2)$. Then $\varphi(a, b) > \varphi(c, d)$ if and only if $\frac{a}{b} < \frac{c}{d}$. This means that the product over rays in clockwise order corresponds to the product over pairs (a, b) taken in increasing order of $\frac{a}{b}$. By the Harder-Narasimhan formula we obtain

$$\mathbb{E}(x_1) \circ \mathbb{E}(x_2) = \prod_{\frac{a}{b}\uparrow} \mathbb{E}(x_1^a x_2^b)^{\Omega_{a,b}^{Z_+}}.$$

This is precisely the equation discussed in the first lecture.

In view of the last example we can interpret Theorem 1.9 as a statement about integrality of Donaldson–Thomas invariants for the m-Kronecker quiver. A similar statement is true for an arbitrary quiver:

Theorem 2.13 (Kontsevich-Soibelman [10])**.** *Let Z be a stability function on the quiver Q and let $l \subset \mathbb{H}_+$ be a ray satisfying condition* (7)*. Then, for any $\alpha \in Z^{-1}(l)$, we have $\Omega^Z_\alpha \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.*

Direct computations show that the Donaldson–Thomas invariants should be positive in the following sense:

Conjecture 2.14. *Under the conditions of Theorem 2.13, for any $\alpha \in Z^{-1}(l)$, the polynomial $\Omega^Z_\alpha(-q^{\frac{1}{2}})$ has non-negative integer coefficients.*

This conjecture was proved in the case of symmetric quivers (see Remark 2.9). In this case the quantum affine space is commutative and therefore the order in the product of the Harder-Narasimhan formula (6) is irrelevant. We can rewrite (6) as

$$A_Q = \prod_{l \subset \mathbb{H}_+} A^Z_l = \operatorname{Exp}\left(\frac{\sum \Omega^Z_\alpha x^\alpha}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}\right).$$

This implies that the Donaldson–Thomas invariants Ω^Z_α are independent of Z (we denote them just by Ω_α).

Theorem 2.15 (Efimov [4])**.** *Let Q be a symmetric quiver. Then, for any $\alpha \in \Gamma_+\backslash\{0\}$, the polynomial $\Omega_\alpha(-q^{\frac{1}{2}})$ has non-negative integer coefficients.*

3. Lecture 3

In this lecture we will define refined Donaldson–Thomas invariants for quivers with potential. The reason we are interested in quivers with potential is that they provide simple examples of 3-Calabi-Yau categories while Donaldson–Thomas invariants are specially designed as invariants of such categories [9]. Moreover, in some cases, 3-Calabi-Yau varieties are derived equivalent to the categories associated to quivers with potential. In this cases we can reduce the computation of Donaldson–Thomas invariants on the 3-Calabi-Yau variety to a representation-theoretic problem, which is usually easier to deal with.

3.1. Jacobian algebra.

Let $\Bbbk$ be a field, $Q = (Q_0, Q_1)$ be a quiver, and W be a potential on Q, that is, $W = \sum_u c_u u$ is a linear combination of cycles in Q. Given a cycle

$$u = a_1 \dots a_n$$

and an arrow $a \in Q_1$, we define the cyclic derivative

$$\frac{\partial u}{\partial a} = \sum_{i : a_i = a} a_{i+1} \dots a_n a_1, \dots a_{i-1}$$

as an element of the path algebra $\Bbbk Q$. We can extend cyclic derivatives to potentials by linearity. We define the Jacobian algebra $J_{Q,W}$ as the quotient algebra

$$J_{Q,W} = \Bbbk Q / \Big(\frac{\partial W}{\partial a} \mid a \in Q_1\Big).$$

Example 3.1. Let Q be the quiver with one vertex and three loops x, y, z. Let

$$W = xyz - zyx.$$

Then we have

$$\frac{\partial W}{\partial x} = [y, z], \quad \frac{\partial W}{\partial y} = [z, x], \quad \frac{\partial W}{\partial Z} = [x, y].$$

Therefore $J_{Q,W} \simeq \Bbbk[x,y,z]$, which is the structure ring of $\Bbbk^3$ and is a 3-Calabi-Yau algebra.

Example 3.2. Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite abelian subgroup. We can decompose the induced representation

$$V = \mathbb{C}^3 = \rho_1 \oplus \rho_2 \oplus \rho_3,$$

where ρ_i are irreducible one-dimensional representations satisfying $\rho_1\rho_2\rho_3 = 1$. Define the McKay quiver Q of (G, V):

(1) The set of vertices is the set $\hat{G}$ of irreducible representations of G.
(2) The set of arrows $\sigma \to \rho$ is a basis of $\mathrm{Hom}_G(\sigma, \rho \otimes V)$ for any $\sigma, \rho \in \hat{G}$.

We can parametrize arrows in Q as

$$a_i^\rho : \rho\rho_i \to \rho, \qquad \rho \in \hat{G}, i = 1, 2, 3$$

and we can parametrize 3-cycles in Q (up to cyclic shift) as

$$u_\pi^\rho : \rho \to \rho\rho_{\pi 1}\rho_{\pi_2} \to \rho\rho_{\pi 1} \to \rho, \qquad \rho \in \hat{G}, \pi \in S_3.$$

Finally, define a potential as the linear combination of 3-cycles

$$W = \sum_{u_\pi^\rho/\sim} \operatorname{sgn}\pi \cdot u_\pi^\rho.$$

Theorem 3.3. *We have $J_{Q,W} \simeq \mathbb{C}[x,y,z] \rtimes G$. In particular, $J_{Q,W}$ is a 3-Calabi-Yau algebra.*

In the case of a non-abelian subgroup $G \subset \mathrm{SL}_3(\mathbb{C})$ one can still endow the McKay quiver with a potential in a canonical way and the Jacobian algebra is Morita-equivalent to $\mathbb{C}[x,y,z] \rtimes G$ (see [6]).

3.2. **Ginzburg DG algebra.** In general, the Jacobian algebra $J_{Q,W}$ is not a 3-Calabi-Yau algebra. Nevertheless, we can always associate with (Q, W) a differential graded 3-Calabi-Yau algebra, called a Ginzburg dg algebra [6], closely related to $J_{Q,W}$.

Define a new graded quiver $\widehat{Q}$ to have the same set of vertices as Q and the following arrows:

(1) an arrow $a : i \to j$ of degree 0 for any arrow $a : i \to j$ in Q,
(2) an arrow $a^* : j \to i$ of degree -1 for any arrow $a : i \to j$ in Q,
(3) a loop $t_i : i \to i$ of degree -2 for any vertex i in Q.

Define the Ginzburg differential graded algebra $\Gamma_{Q,W}$ as the path algebra $\Bbbk Q$ with the following differential

$$da = 0, \qquad da^* = \frac{\partial W}{\partial a}, \qquad dt_i = e_i\Big(\sum_{a \in Q_1} [a, a_*]\Big)e_i.$$

It follows from our definitions that $\Gamma_{Q,W}$ is concentrated in non-negative degrees. Moreover

$$H^0(\Gamma_{Q,W}) \simeq J_{Q,W}.$$

Theorem 3.4 (Keller-Van den Bergh [8])**.** *The algebra $\Gamma_{Q,W}$ is a 3-Calabi-Yau algebra.*

Proposition 3.5 (Amiot [1])**.** *The derived category $D(\Gamma_{Q,W})$ has a natural t-structure with heart naturally equivalent to* $\mathrm{Mod}\, J_{Q,W}$.

3.3. **Donaldson–Thomas invariants.** We have seen that the category of modules over the Jacobian algebra $J_{Q,W}$ can be considered as the heart of a triangulated 3-Calabi-Yau category. In this situation Kontsevich and Soibelman [9] managed to construct the Donaldson–Thomas invariants. Here is a brief description of their approach:

(1) Construct an algebra homomorphism $I : H(J_{Q,W}) \to \mathbb{A}_\Gamma$ (this is the most difficult step).
(2) Define the generating series $A_l^Z \in \mathbb{A}_\Gamma$ counting Z-semistable modules over $J_{Q,W}$.
(3) Prove the wall-crossing formula: $\prod^{\curvearrowright}_{l\in\mathbb{H}_+} A_l^Z$ is independent of Z.
(4) Define the DT invariants of the Jacobian algebra by the formula $A_l^Z = \prod_{Z(\alpha)\in l} \mathbb{E}(x^\alpha)^{\Omega_\alpha}$.

We will define the Donaldson–Thomas invariants for quivers with potential admitting a so-called cut. A cut of (Q, W) is a subset $C \subset Q_1$ such that W is homogeneous of degree 1 with respect to the grading defined on arrows by

$$\deg a = \begin{cases} 1 & a \in C, \\ 0 & a \notin C. \end{cases}$$

The significance of a cut is that it defines an action of $\Bbbk^*$ on $R(Q, \alpha)$

$$t \cdot (M_a)_{a\in Q_1} = (t^{\deg a} M_a)_{a\in Q_1}$$

such that the map

$$w_\alpha : R(Q, \alpha) \to \Bbbk, \qquad M \mapsto \operatorname{tr}(W|M)$$

is equivariant with respect to this action.

Lemma 3.6. *The space of representations of the Jacobian algebra*

$$R(J_{Q,W}, \alpha) \subset R(Q, \alpha)$$

coincides with the degeneracy locus of the map w_α:

$$\{M \in R(Q, \alpha) \mid dw_\alpha(M) = 0\}.$$

In order to define an analogue of the series A_l^Z from the previous lecture (see formula (5)), we would like to substitute the set of Z-semistable Q-representations $R_Z^{\mathrm{sst}}(Q, \alpha)$ by the set of Z-semistable $J_{Q,W}$-representations $R_Z^{\mathrm{sst}}(J_{Q,W}, \alpha)$. Unfortunately, the space $R_Z^{\mathrm{sst}}(J_{Q,W}, \alpha)$ has, in general, singularities and is not well-suited for our purposes. As we have seen, this space can be identified with the degeneracy locus of $w_\alpha : R_Z^{\mathrm{sst}}(Q, \alpha) \to \Bbbk$. Therefore one uses certain invariant of this function, called a motivic vanishing cycle [2], in order to define an analogue of the series A_l^Z. In the presence of a cut, this invariant can be written as a difference of motivic classes of $w_\alpha^{-1}(0)$ and $w_\alpha^{-1}(1)$ (see [2, Prop. 1.10]). Therefore, for a finite field $\Bbbk = \mathbb{F}_q$, we define [13, 14]

$$A_{l,Q,W}^Z = \sum_{Z(\alpha)\in l} (-q^{\frac{1}{2}})^{\chi(\alpha,\alpha)} \frac{|w_\alpha^{-1}(0) \cap R_Z^{\mathrm{sst}}(Q, \alpha)| - |w_\alpha^{-1}(1) \cap R_Z^{\mathrm{sst}}(Q, \alpha)|}{|\mathrm{GL}_\alpha(\mathbb{F}_q)|} x^\alpha.$$

Using this definition it is not difficult to prove the wall-crossing formula by the same methods as we applied in the previous lecture. More precisely, we have [13]

$$\prod_{l\in\mathbb{H}_+}^{\curvearrowright} A^Z_{l,Q,W}=\sum_{\alpha\in\Gamma_+}(-q^{\frac12})^{\chi(\alpha,\alpha)}\frac{|w_\alpha^{-1}(0)|-|w_\alpha^{-1}(1)|}{|\mathrm{GL}_\alpha(\mathbb{F}_q)|}x^\alpha=:A_{Q,W}. \tag{9}$$

In particular, the product on the left is independent of the stability function Z. We define the Donaldson–Thomas invariants $\Omega^Z_{\alpha,Q,W}$ in the same way as in the previous lecture (under the assumption (7))

$$A^Z_{l,Q,W}=\prod_{Z(\alpha)\in l}\mathbb{E}(x^\alpha)^{\Omega^Z_{\alpha,Q,W}}=\operatorname{Exp}\left(\frac{\sum_{Z(\alpha)\in l}\Omega^Z_{\alpha,Q,W}x^\alpha}{q^{-\frac12}-q^{\frac12}}\right). \tag{10}$$

Remark 3.7. *If W is zero, then $w_\alpha^{-1}(0)=R(Q,\alpha)$ and $w_\alpha^{-1}(1)$ is empty. Therefore*

$$A^Z_{l,Q,W}=\sum_{Z(\alpha)\in l}(-q^{\frac12})^{\chi(\alpha,\alpha)}\frac{|R^{\mathrm{sst}}_Z(Q,\alpha)|}{|\mathrm{GL}_\alpha(\mathbb{F}_q)|}x^\alpha.$$

This formula coincides with the formula (5) for quivers without potential.

Conjecture 3.8. *In the same way as for quivers without potentials (see Conjecture 2.14) we conjecture that $\Omega^Z_{\alpha,Q,W}\in\mathbb{Z}[q^{\pm\frac12}]$ and $\Omega^Z_{\alpha,Q,W}(-q^{\frac12})$ has non-negative coefficients.*

Example 3.9. Let Q be a quiver and let $\widehat{Q}$ be the same quiver that we constructed from Q in the definition of the Ginzburg algebra. Define a potential on $\widehat{Q}$ by the formula

$$W=\sum_{(a:i\to j)\in Q_1}(t_jaa^*-t_ia^*a)=\sum_{i\in Q_0}t_i\cdot\sum_{a\in Q_1}[a,a^*].$$

The pair $(\widehat{Q},W)$ admits a cut:

$$C=\{t_i\mid i\in Q_0\}.$$

Therefore we can define the series $A^Z_{l,\widehat{Q},W}$, $A_{\widehat{Q},W}$ and the Donaldson–Thomas invariants using the above formulas. Note that the quiver $\widehat{Q}$ is symmetric and therefore the Donaldson–Thomas invariants $\Omega^Z_{\alpha,\widehat{Q},W}$ are independent of Z. We denote them just by Ω_α. One can show that

$$A_{\widehat{Q},W}=\sum_{\alpha\in\Gamma_+}q^{\chi_Q(\alpha,\alpha)}\frac{|R(\Pi_Q,\alpha)|}{|\mathrm{GL}_\alpha(\mathbb{F}_q)|}x^\alpha,$$

where $\Pi_Q=\Bbbk\overline{Q}/(\sum_{a\in Q_1}[a,a^*])$ is the preprojective algebra of Q ($\overline{Q}$ is a subquiver of $\widehat{Q}$ containing arrows a,a^* for $a\in Q_1$).

For example, let Q consist of one vertex and one loop. Then

$$R(\Pi_Q,n)=\{(A,B)\in\mathrm{Mat}^2_{n\times n}\mid AB=BA\}.$$

The number of points in this set was computed half a century ago by Feit and Fine [5]

$$\sum_{n\geq0}\frac{|R(\Pi_Q,n)|}{|\mathrm{GL}_n(\mathbb{F}_q)|}=\sum_{\substack{n\geq1\\k\geq0}}\frac{1}{1-q^{1-k}x^n}=\operatorname{Exp}\left(\sum_{n\geq1}\frac{qx^n}{1-q^{-1}}\right).$$

We can generalize this result to arbitrary quivers [12]

$$\sum_{\alpha\in\Gamma_+} q^{\chi_Q(\alpha,\alpha)} \frac{|R(\Pi_Q,\alpha)|}{|\mathrm{GL}_\alpha(\mathbb{F}_q)|} x^\alpha = \mathrm{Exp}\left(\frac{\sum_\alpha a_\alpha(q)x^\alpha}{1-q^{-1}}\right),$$

where $a_\alpha(q)$ is the polynomial counting absolutely indecomposable representations of Q having dimension vector α over finite fields. Comparing this formula with the definition of Donaldson–Thomas invariants (10) we obtain

$$\Omega_\alpha(q^{\frac12}) = -q^{\frac12} a_\alpha(q).$$

According to the Kac conjecture, proved recently [7], the polynomials $a_\alpha(q)$ have non-negative integer coefficients. This implies that $\Omega_\alpha \in \mathbb{Z}[q^{\pm\frac12}]$ and $\Omega_\alpha(-q^{\frac12})$ have non-negative coefficients, justifying the positivity conjecture 3.8.

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