QUIVER REPRESENTATIONS AND THE PATH ALGEBRA

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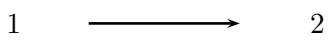
1. Quivers

Informally, a quiver is a finite directed graph, in which we allow multiple edges and vertex loops.

Definition 1.1. A quiver Q is defined as a tuple (Q_0, Q_1, t, h), where Q_0 and Q_1 are sets and $t, h : Q_1 \to Q_0$ are maps. We call Q_0 the set of vertices, Q_1 the set of arrows, $t(a)$ the start of the arrow a and $h(a)$ the end of the arrow a.

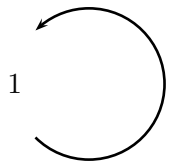
Example 1.2 (1-Subspace Quiver)**.** $Q_0 = \{1, 2\}$; $Q_1 = \{a\}$; $t(a) = 1, h(a) = 2$



Example 1.3 (2-Subspace Quiver)**.** $Q_0 = \{1, 2, 3\}$; $Q_1 = \{a, b\}$; $t(a) = 1, h(a) = 2 = h(b), t(b) = 3$



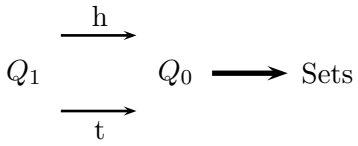
Example 1.4 (Jordan Quiver)**.** $Q_0 = \{1\}$; $Q_1 = \{a\}$; $t(a) = h(a) = 1$



Definition 1.5. A subquiver Q' of Q is a pair(Q'_0, Q'_1) such that $t(a), h(a) \in Q'_0 \ \forall a \in Q'_1$

Definition 1.6. A subquiver Q' is called full if it contains all arrows $a \in Q_1$ such that $t(a), h(a) \in Q'_0$

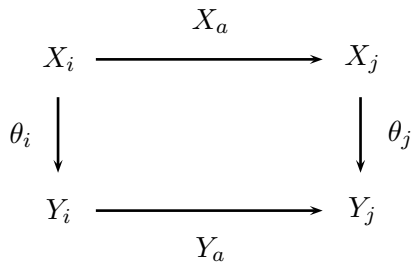
Also, note that we can consider quivers as functors from the category known as the **free quiver**, or **Kronecker quiver** to *Sets*



2. Quiver Representations and Path Algebras

Definition 2.1. Let Q be a quiver and k be a field. A representation of Q is a collection $X = ((X_i)_{i \in Q_0}, (X_a)_{a \in Q_1})$ consisting of a vector space X_i for each vertex and a linear map $X_a : X_{t(a)} \to X_{h(a)}$ for each arrow

Definition 2.2. A morphism of representations $\theta : X \to Y$ is a collection $\theta = ((\theta_i)_{i\ in Q_0}$ of linear maps $\theta_i : X_i\ to Y_i$ such that this diagram commutes $(i = t(a), j = h(a))$



This structure on quiver representations defines a category $\mathrm{Rep_k}\, Q$.

Definition 2.3. A path of length $n \geq 1$ in Q is a sequence of arrows $p = a_n a_{n-1} a_{n-2} \dots a_1$ such that $h(a_r) = t(a_{r+1})$, $1 \leq r \leq n$. We write $t(p) = t(a_n)$ and $h(p) = h(a_1)$.

Note that paths of length 1 are arrows of Q

Definition 2.4. We define the trivial path ϵ_i of length 0 for each $i \in Q_0$ to have $t(\epsilon_i) = h(\epsilon_i) = i$

Definition 2.5. Let $Q_r = \{\text{Paths of Length } r\}$. Then define the path algebra by $kQ = \bigoplus_{r \geq 0} kQ_r$ with product defined by $q.p$ is the path given by concatenation if $h(p) = t(q)$ and 0 otherwise.

In particular, note that this definition gives orthogonal idempotents ϵ_i,

$$\epsilon_i \epsilon_j = \delta_{i,j} \epsilon_i$$

and unity element $1 = \sum_{i \in Q_0} \epsilon_i$

Example 2.6 (1 Subspace Quiver)**.** kQ has a basis $\epsilon_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,
$\epsilon_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,
$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Hence $kQ \cong \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$.

Example 2.7 (Jordan Quiver)**.** kQ has a basis $\epsilon_1 = 1$, $a = T$, a^2, ... and so $kQ \cong k[T]$

Proposition 2.8. $\mathrm{Rep_k}\, Q$ *is equivalent to* $\mathrm{Mod}\, kQ$

Proof. Given a representation X of Q, define a left kQ module $F(X)$ by $F(X_i) = \bigoplus_{i \in Q_0} X_i$ with multiplication defined as a linear extension of

$$\epsilon_i x := \iota_i \pi_i(x)$$

and for $p = a_1 \dots a_n$

$$px := \iota_{h(p)} X_{a_n} \cdots X_{a_1} \pi_{t(p)}(x)$$

where

$$\iota_i : X_i \to F(X)$$

is the inclusion map and

$$\pi_i : F(X) \to X_i$$

is the projection map.

With this multiplication, $F(X)$ is a left kQ-module. Next define $F(\theta) := \bigoplus_{i \in Q_0} \theta_i$ with $(F(\theta)(x_i))_{i \in Q_0} = (\theta_i(x_i))_{i \in Q_0}$. This defines a functor $F : \mathrm{Rep_k}\, Q \to \mathrm{Mod}\, kQ$
Next we define $G : \mathrm{Mod}\, kQ \to \mathrm{Rep_k}\, Q$ by

$$G(M)_i := \epsilon_i M, i \in Q_0$$

$$G(M)_a : \epsilon_{h(a)} M \to \epsilon_{t(a)} M, m \to am, a \in Q_1$$

If $\Theta : M \to N$ is a morphism of kQ modules, define $G(\Theta) : G(M) \to G(N)$ by, given $m = \epsilon_i m \in \epsilon_i M$,

$$\Theta(m) = \Theta(\epsilon_i m) := \epsilon_i \Theta(m) \in \epsilon_i N$$

.
Then $GF = id$ on $Rep_k Q$ and $FG \cong id$ on $Mod kQ$

□

3. Standard Resolution of kQ modules

Definition 3.1. A resolution of an R-module M, is an exact sequence of R-modules

$$\cdots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{\epsilon} \mathrm{M} \longrightarrow 0$$

Definition 3.2. A resolution of M is called projective/simple, if each E_i is projective/simple

We can provide a standard projective resolution for any left kQ module, as shown below

Definition 3.3. We define $A \bigotimes_k B$ to be the direct sum of $dim B$ copies of A

Proposition 3.4. *Let X be a left kQ module. Then there is an exact sequence*

$$0 \to \bigoplus_{p \in Q_1} kQ\epsilon_{h(p)} \bigotimes_k \epsilon_{t(p)} \xrightarrow{f} \bigoplus_{i \in Q_0} kQ\epsilon_i \bigotimes_k \epsilon_i X \xrightarrow{g} X \to 0$$

where $g(s \bigotimes x) = sx$ and $f(s \bigotimes x)_a = as \bigotimes x - s \bigotimes ax$

If we define $P_i := kQ\epsilon_i$, note that, as kQ is a free module, P_i is projective, hence $P_i \bigotimes_k \epsilon_i X$ is projective.

Definition 3.5. Define $\mathrm{Ext}^i(A,B) := (R^nG)A$, where R^nG is the right derived functor of $G = \mathrm{Hom}(-,B)$

To compute $\mathrm{Ext}^i(A,B)$, we consider *any* projective resolution for A

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

Ext^i is then given by the ith homology of the cochain complex

$$0 \to \mathrm{Hom}(P_0, A) \to \mathrm{Hom}(P_1, A) \to \cdots$$

This implies that $\mathrm{Ext}(X,Y) = 0$ for any kQ modules X and Y, $i \geq 2$, as the standard resolution is a projective resolution for X, but as it gives a cochain complex of length 2, we get $\mathrm{Ext}^i = 0$ for $i \geq 2$

4. Dimension Vectors and the Euler Form

Definition 4.1. We define the dimension vector of a representation X by $\underline{\dim} X := \sum_{i \in Q_0} \dim X_i \in \mathbb{Z}^{kQ}$

Definition 4.2. We call a vector $\underline{d} = \sum_{i \in Q_0} x_i \in \mathbb{Z}^{kQ}$ a dimension vector if $x_i \geq 0 \ \forall i \in Q_0$

Definition 4.3. We define the Euler form on dimension vectors by

$$\langle \alpha, \beta \rangle := \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}$$

Definition 4.4. We can also define an Euler form on representations by $\langle X, Y \rangle := \dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y)$

Lemma 4.5. $\langle X, Y \rangle = \langle \underline{\dim} X, \underline{\dim} Y \rangle$

Proof. Apply $Hom(-,Y)$ to the standard resolution of X to get

$$0 \to \mathrm{Hom}(X,Y) \to \mathrm{Hom}(\bigoplus_{a:i \to j \in Q_1} P_j \bigotimes_k \epsilon_i X, Y) \to \mathrm{Hom}(\bigoplus_{i \in Q_0} P_i \bigotimes_k \epsilon_i X, Y) \to \mathrm{Ext}^1(X,Y) \to 0$$

This is an exact sequence. Hence, as $\sum (-1)^i \dim V_i = 0$ for an exact sequence, this implies

$$\dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) = \dim \mathrm{Hom}(\bigoplus_{i \in Q_0} P_i \bigotimes_k \epsilon_i X, Y) - \dim \mathrm{Hom}(\bigoplus_{a:i \to j \in Q_1} P_j \bigotimes_k \epsilon_i X, Y)$$

$$\dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) = \textstyle\sum_{i \in Q_0} \dim X_i \dim Y_i - \sum_{a:i \to j \in Q_1} \dim X_i \dim Y_j \quad (4.1)$$

The result follows.

□