Numerical Methods
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Lecture 6

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Organisational (Michaelmas Term 2018)

- Make up for the lost lecture: 2nd week of Nov. (12.11, 11am-1pm)
- Following week: Wed. 21.11(no lecture) ->instead Mon.: 19.11, 11am-1pm)

	To appear	Submission DL
Assignment 1	online	**2.11 (passed)**
Assignment 2	14.11	30.11

Solving linear algebraic equations

- ➡ How is the matrix solution process affected by the changes of the problem?
- ➡ If the small change in the problem produces the large change in the solution:
  - is it due to small perturbation of the problem?
  - or due to the instability in the computational scheme?

- ➡ Iterative methods for linear systems
  - Iterations, Jacobi, Gauss-Seidel
  - Sparse matrix solvers

Estimating the condition number

- If the solution to a linear system changes a great deal when the problem changes only very slightly, then we suspect that the matrix is ill-conditioned
- Recall the definition (infinity norm, next slide):

$$\kappa(A) = ||A||_{\infty}||A^{-1}||_{\infty}$$

- Computing $||\mathbf{A}||_{\infty}$ not hard, computing $||\mathbf{A}^{-1}||_{\infty}$ is! (if A is ill-conditioned, computing A^{-1} will be unreliable)

Vector and matrix norms

<u>Definiton 2:</u> Let $||\cdot||$ be a given vector norm on **R^n**. A corresponding **matrix norm** for matrices **$A\in R^{nxn}$** is defined by:

$$||A|| = \max_{x\neq 0} \frac{||Ax||}{||x||}$$

➡ Consequences of this definition of operator (matrix) norm:

$$||AB| \leq |||A||||B|| \quad \text{and} \quad ||Ax| \leq |||A||||x||$$

➡ Not necessary to associate matrix norm with a particular vector one

- Matrix infinity norm:

$$||A||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

- Matrix 2-norm:

$$||A||_2 = \sqrt{\Lambda(A^T A)}$$

Λ(B) is the largest (in abs. value) eigenvalue of the matrix B

Estimating condition number

- Algorithm for estimating the condition number, given LU factorisation of A

1. Compute $\alpha = ||A||_\infty$
2. Take a random initial guess $\mathbf{y}^{(0)}$
3. Compute $\mathbf{y}^{(5)}$ in the sequence defined by

$$y^{(i+1)} = \frac{A^{-1}y^{(i)}}{||y^{(i)}||_\infty}, \qquad i = 0, 1, \ldots, 4$$

   by solving the systems using the exact factorisation of A, and set $v = ||\mathbf{y}^{(5)}||_\infty$.
4. Set $\kappa^* = \alpha v$.

- <u>Homework:</u> write an R code determining the condition number
- Check with predefined R routine: library('Matrix'); condnr<-1/(rcond(a,norm = c("I")))

Iterative methods for linear systems

Theorem 1: Let **$A \in R^{n \times n}$** be a given matrix and we define $T=M^{-1}N$, where **A=M-N.** Then the iteration

$$x^{(n+1)} = M^{-1}Nx^{(n)} + M^{-1}b$$

converges for all initial guesses $x^{(0)}$ if and only if there exists a norm $||\cdot||$ such that $||\mathbf{T}||<1$.

Definiton 1: Let **$A \in R^{n \times n}$** be a given matrix. Then the **spectral radius of a matrix** A, **$\rho(A)$**, is the largest (in magnitude) of all the eigenvalues of A:

$$\rho(A) = \max|\lambda|$$

where the maximum is taken over all the eigenvalues of A.

Theorem 2: Let **$A \in R^{n \times n}$** be a given matrix. Then there exists a norm such that $||\mathbf{T}||<1$ if and only if the spectral radius satisfies **$\rho(A)<1$**.

➡ The iteration converges for **all initial guesses $x^{(0)}$** ⇔ **$\rho(M^{-1}N)<1$**

Jacobi iteration

$$A = D - (D - A)$$
$$x^{(n+1)} = (\mathbb{I} - D^{-1}A)x^{(n)} + D^{-1}b$$

Gauss-Seidel iteration

$$A = L - (L - A)$$
$$x^{(n+1)} = (\mathbb{I} - L^{-1}A)x^{(n)} + L^{-1}b$$

Jacobi iteration

$$A = D - (D - A)$$

$$x^{(n+1)} = (\mathbb{I} - D^{-1}A)x^{(n)} + D^{-1}b$$

```
A=t(matrix(c(4,1,0,0,1,5,1,0,0,1,6,1,1,0,1,4),
nrow=4,ncol=4))
b=c(1,7,16,14)
p=c(0,0,0,0)
xp=c(0,0,0,0)

epsilon=1e-06
max1=100
```

```
# R code for the solution of the system Ax=b using
Jacobi iteration
#
# Input     - A is an n x n nonsingular matrix
#           - b is an n x 1 matrix (solution vector)
#           - p is an n x 1 matrix (the initial guess)
#           - epsilon is the tolerance for the
solution P
#           - max1 is the maximum number of iterations
#
# Output    - x is an n x 1 matrix: the Jacobi
approximation to
#            the solution of Ax = b

n = length(b);
norm_vec_infty <- function(x) max(x)

for (k in 1:max1)
{
    p=xp;
    for (j in 1:n)
        xp[j]=p[j]+(b[j]-A[j,]%*%p)/A[j,j];

    err=abs(norm_vec_infty(xp-p));
    relerr=err/(abs(norm_vec_infty(xp));
    if (err<epsilon)   #(relerr<epsilon)
        break;
}
x=xp;
```