Numerical Methods 5633

Lecture 5

Michaelmas Term 2017

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Organisational (Michaelmas Term 2018)

- Make up for the lost lecture: 2nd week of Nov. (12.11: 11am-1pm)

	To appear	Submission DL
Assignment 1	**online**(start early!)	2.11
Assignment 2	14.11	30.11

Solving linear algebraic equations

- How is the matrix solution process affected by the changes of the problem?
- If the small change in the problem produces the large change in the solution:
  - is it due to small perturbation of the problem?
  - or due to the instability in the computational scheme?
- Recall some linear algebra
  - Vector and Matrix norms
  - Perturbations, conditioning, stability

Vector and matrix norms

Definiton 1: A **vector norm** on $\mathbf{R}^n$ is any mapping $||\cdot||$, defined on $\mathbf{R}^n$, defined on with values in $[0,\infty)$, which satisfies the conditions:

1. $||x|| > 0$ for any vector $\mathbf{x} \neq \mathbf{0}$;
2. $||ax|| = ||a||||x||$ for any scalar a;
3. $||x+y|| \leq ||x|| + ||y||$ for any two vectors $\mathbf{x}$ and $\mathbf{y}$;

➡ Examples of vector norms:

- Infinity norm:

$$||x||_\infty = \max_{1 \leq i \leq n} |x_i|$$

- Euclidean 2-norm:

$$||x||_2 = \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}}$$

Vector and matrix norms

Definiton 2: Let $||\cdot||$ be a given vector norm on $\mathbf{R}^n$. A corresponding **matrix norm** for matrices $\mathbf{A}\in\mathbf{R}^{n\times n}$ is defined by:

$$||A|| = \max_{x\neq 0} \frac{||Ax||}{||x||}$$

➡ Consequences of this definition of operator (matrix) norm:

$$||AB| \leq ||A||||B||$$ and $$||Ax| \leq ||A||||x||$$

➡ Not necessary to associate matrix norm with a particular vector one

- Matrix infinity norm:

$$||A||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

- Matrix 2-norm:

$$||A||_2 = \sqrt{\Lambda(A^T A)}$$

Λ(B) is the largest (in abs. value) eigenvalue of the matrix B

Condition number

<u>Definiton 3:</u> For a given matrix **A**$\in$**R**^{nxn} and a given matrix norm $||\cdot||$, the **condition number** with respect to the given norm is defined by:

$$\kappa(A) = ||A||||A^{-1}||$$

If A is singular, then we take **κ(A)=∞.**

<u>Theorem 1:</u> Let **A**$\in$**R**^{nxn} be a given nonsingular matrix. Then, for any singular matrix **B**$\in$**R**^{nxn} holds:

$$\frac{1}{\kappa(A)} \leq \frac{||A-B||}{||A||}$$

Effects of perturbations

- in **b**:

<u>Theorem 2:</u> Let **A**$\in$**R**$^{n\times n}$ be a given nonsingular matrix and **b**$\in$**R**^{n}. Let us also define **x**$\in$**R**^{n} as the solution of the linear system **Ax=b** and let **δb**$\in$**R**^{n} be a small perturbation of b. If we define **x+δx**$\in$**R**^{n} as the solution of the system **A(x+δx)=b+δb** then

$$\frac{||\delta x||}{||x||} \leq \kappa(A) \frac{||\delta b||}{||b||}$$

- in **A**:

<u>Theorem 3:</u> Let **A**$\in$**R**$^{n\times n}$ be a given nonsingular matrix and let **E**$\in$**R**$^{n\times n}$ be a perturbation of **A**. Let **x**$\in$**R**^{n} be a unique solution of **Ax=b**. If

then the perturbed system (A+E)xc=b has a unique solution and

$$\frac{||x - x_c||}{||x||} \leq \frac{\theta}{1-\theta}, \qquad \text{where} \quad \theta = \kappa(A)\frac{||E||}{||A||}.$$

Estimating the condition number

- If the solution to a linear system changes a great deal when the problem changes only very slightly, then we suspect that the matrix is ill-conditioned
- Recall the definition (infinity norm):

$$\kappa(A) = ||A||_\infty ||A^{-1}||_\infty$$

- Computing $||\mathbf{A}||_\infty$ not hard, computing $||\mathbf{A}^{-1}||_\infty$ is! (if A is ill-conditioned, computing A^{-1} will be unreliable)

Estimating condition number

➡ Algorithm for estimating the condition number, given LU factorisation of A

1. Compute **$\alpha=||A||_\infty$**
2. Take a random initial guess **$y^{(0)}$**
3. Compute **$y^{(5)}$** in the sequence defined by

$$y^{(i+1)} = \frac{A^{-1}y^{(i)}}{||y^{(i)}||_\infty}, \qquad i = 0, 1, \ldots, 4$$

   by solving the systems using the exact factorisation of A, and set $v=||y^{(5)}||_\infty$.
4. Set **$\kappa^*=\alpha v$.**

➡ Homework: write an R code determining the condition number

➡ Check with predefined R routine: library('Matrix'); condnr<-1/(rcond(a,norm = c("I")))