

Numerical Methods

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Lecture 9

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Organisational

	To appear	Submission DL
⦿ Assignment 1	14.10	04.11
⦿ Assignment 2	04.11	23.11
⦿ Assignment 3	<u>25.11</u>	09.12

➡ Last 2 weeks – Thurs. (+Fri.) lectures

➡ 17.11/24.11 (14.30–16.10h)

Solving linear algebraic equations

- ➔ How is the matrix solution process affected by the changes of the problem?
- ➔ If the small change in the problem produces the large change in the solution:
 - is it due to small perturbation of the problem?
 - or due to the instability in the computational scheme?
- ➔ Iterative methods for linear systems
 - Iterations, Jacobi, Gauss-Seidel
 - Sparse matrix solvers

Estimating the condition number

- ➔ If the solution to a linear system changes a great deal when the problem changes only very slightly, then we suspect that the matrix is ill-conditioned
- ➔ Recall the definition (infinity norm, next slide):

$$\kappa(A) = \|A\|_{\infty} \|A^{-1}\|_{\infty}$$

- ➔ Computing $\|A\|_{\infty}$ not hard, computing $\|A^{-1}\|_{\infty}$ is! (if A is ill-conditioned, computing A^{-1} will be unreliable)

Vector and matrix norms

Definition 2: Let $\|\cdot\|$ be a given vector norm on \mathbf{R}^n . A corresponding **matrix norm** for matrices $\mathbf{A} \in \mathbf{R}^{n \times n}$ is defined by:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

→ Consequences of this definition of operator (matrix) norm:

$$\|AB\| \leq \|A\| \|B\| \quad \text{and} \quad \|Ax\| \leq \|A\| \|x\|$$

→ Not necessary to associate matrix norm with a particular vector one

- Matrix infinity norm:

$$\|A\|_{\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$

- Matrix 2-norm:

$$\|A\|_2 = \sqrt{\Lambda(A^T A)}$$

$\Lambda(B)$ is the largest (in abs. value) eigenvalue of the matrix B

Estimating condition number

➔ Algorithm for estimating the condition number, given LU factorisation of A

1. Compute $\alpha = \|A\|_\infty$
2. Take a random initial guess $y^{(0)}$
3. Compute $y^{(5)}$ in the sequence defined by

$$y^{(i+1)} = \frac{A^{-1}y^{(i)}}{\|y^{(i)}\|_\infty}, \quad i = 0, 1, \dots, 4$$

by solving the systems using the exact factorisation of A, and set $v = \|y^{(5)}\|_\infty$.

4. Set $\kappa^* = \alpha v$.

➔ Homework: write an R code determining the condition number

➔ Check with predefined R routine: `library('Matrix');`
`condnr<-1/(rcond(a,norm = c("I")))`

Iterative methods for linear systems

Theorem 1: Let $\mathbf{A} \in \mathbf{R}^{n \times n}$ be a given matrix and we define $\mathbf{T} = \mathbf{M}\mathbf{N}$, where $\mathbf{A} = \mathbf{M} - \mathbf{N}$. Then the iteration

$$x^{(n+1)} = M^{-1}Nx^{(n)} + M^{-1}b$$

converges for all initial guesses $x^{(0)}$ if and only if there exists a norm $\|\cdot\|$ such that $\|\mathbf{T}\| < 1$.

Definiton 1: Let $\mathbf{A} \in \mathbf{R}^{n \times n}$ be a given matrix. Then the **spectral radius of a matrix** \mathbf{A} , $\rho(\mathbf{A})$, is the largest (in magnitude) of all the eigenvalues of \mathbf{A} :

$$\rho(A) = \max |\lambda|$$

where the maximum is taken over all the eigenvalues of \mathbf{A} .

Theorem 2: Let $\mathbf{A} \in \mathbf{R}^{n \times n}$ be a given matrix. Then there exists a norm such that $\|\mathbf{T}\| < 1$ if and only if the spectral radius satisfies $\rho(\mathbf{A}) < 1$.

➡ The iteration converges for **all initial guesses** $x^{(0)} \Leftrightarrow \rho(\mathbf{M}^{-1}\mathbf{N}) < 1$

Jacobi iteration

$$A = D - (D - A)$$
$$x^{(n+1)} = (\mathbb{I} - D^{-1}A)x^{(n)} + D^{-1}b$$

Gauss-Seidel iteration

$$A = L - (L - A)$$
$$x^{(n+1)} = (\mathbb{I} - L^{-1}A)x^{(n)} + L^{-1}b$$

Jacobi iteration

$$A = D - (D - A)$$

$$x^{(n+1)} = (\mathbb{I} - D^{-1}A)x^{(n)} + D^{-1}b$$

```
A=t(matrix(c(4,1,0,0,1,5,1,0,0,1,6,1,1,0,1,4),
nrow=4,ncol=4))
b=c(1,7,16,14)
p=c(0,0,0,0)
xp=c(0,0,0,0)

epsilon=1e-06
max1=100
```

```
# R code for the solution of the system Ax=b using
# Jacobi iteration
#
# Input      - A is an n x n nonsingular matrix
#            - b is an n x 1 matrix (solution vector)
#            - p is an n x 1 matrix (the initial guess)
#            - epsilon is the tolerance for the
# solution P
#            - max1 is the maximum number of iterations
#
# Output     - x is an n x 1 matrix: the Jacobi
# approximation to
#             the solution of Ax = b

n = length(b);
norm_vec_infty <- function(x) max(x)

for (k in 1:max1)
{
  p=xp;
  for (j in 1:n)
    xp[j]=p[j]+(b[j]-A[j,]%*%p)/A[j,j];

  err=abs(norm_vec_infty(xp-p));
  relerr=err/(abs(norm_vec_infty(xp)));
  if (err<epsilon)  #(relerr<epsilon)
    break;
}
x=xp;
```