Numerical Methods
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Lecture 8

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Organisational

	To appear	Submission DL
Assignment 1	14.10	04.11
Assignment 2	04.11	18.11
Assignment 3	25.11	09.12

➡ Last 2 weeks - Thurs. (+Fri.) lectures

➡ 17.11/24.11 (14.30-16.10h)

Solving linear algebraic equations

- How is the matrix solution process affected by the changes of the problem?
- If the small change in the problem produces the large change in the solution:
  - is it due to small perturbation of the problem?
  - or due to the instability in the computational scheme?
- Recall some linear algebra
  - Vector and Matrix norms
  - Perturbations, conditioning, stability

Vector and matrix norms

<u>Definiton 1:</u> A **vector norm** on **R^n** is any mapping $||\cdot||$, defined on **R^n**, defined on with values in **[0,∞)**, which satisfies the conditions:

1. $||x|| > 0$ for any vector **x≠0**;
2. $||ax|| = ||a||||x||$ for any scalar a;
3. $||x+y|| \leq ||x|| + ||y||$ for any two vectors **x** and **y**;

➡ Examples of vector norms:

- Infinity norm:

$$||x||_\infty = \max_{1\leq i\leq n} |x_i|$$

- Euclidean 2-norm:

$$||x||_2 = \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}}$$

Vector and matrix norms

Definiton 2: Let $||\cdot||$ be a given vector norm on $\mathbf{R}^n$. A corresponding **matrix norm** for matrices $\mathbf{A}\in\mathbf{R}^{n\times n}$ is defined by:

$$||A|| = \max_{x\neq 0} \frac{||Ax||}{||x||}$$

➡ Consequences of this definition of operator (matrix) norm:

$$||AB| \leq |||A||||B|| \quad \text{and} \quad ||Ax| \leq |||A||||x||$$

➡ Not necessary to associate matrix norm with a particular vector one

- Matrix infinity norm:

$$||A||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

- Matrix 2-norm:

$$||A||_2 = \sqrt{\Lambda(A^T A)}$$

Λ(B) is the largest (in abs. value) eigenvalue of the matrix B

Condition number

<u>Definiton 3:</u> For a given matrix $\mathbf{A} \in \mathbf{R}^{nxn}$ and a given matrix norm $||\cdot||$, the **condition number** with respect to the given norm is defined by:

$$\kappa(A) = ||A||||A^{-1}||$$

If A is singular, then we take $\boldsymbol{\kappa(A)=\infty}$.

<u>Theorem 1:</u> Let $\mathbf{A} \in \mathbf{R}^{nxn}$ be a given nonsingular matrix. Then, for any singular matrix $\mathbf{B} \in \mathbf{R}^{nxn}$ holds:

$$\frac{1}{\kappa(A)} \leq \frac{||A-B||}{||A||}$$

Effects of perturbations

➡ in **b:**

Theorem 2: Let $\mathbf{A} \in \mathbf{R}^{n \times n}$ be a given nonsingular matrix and $\mathbf{b} \in \mathbf{R}^{n}$. Let us also define $\mathbf{x} \in \mathbf{R}^{n}$ as the solution of the linear system **Ax=b** and let $\delta \mathbf{b} \in \mathbf{R}^{n}$ be a small perturbation of b. If we define $\mathbf{x}+\delta\mathbf{x} \in \mathbf{R}^{n}$ as the solution of the system $\mathbf{A}(\mathbf{x}+\delta\mathbf{x})=\mathbf{b}+\delta\mathbf{b}$ then

$$\frac{||\delta x||}{||x||} \leq \kappa(A)\frac{||\delta b||}{||b||}$$

➡ in **A:**

Theorem 3: Let $\mathbf{A} \in \mathbf{R}^{n \times n}$ be a given nonsingular matrix and let $\mathbf{E} \in \mathbf{R}^{n \times n}$ be a perturbation of **A.** Let $\mathbf{x} \in \mathbf{R}^{n}$ be a unique solution of **Ax=b.** If

then the perturbed system (A+E)xc=b has a unique solution and

$$\frac{||x - x_c||}{||x||} \leq \frac{\theta}{1-\theta}, \qquad \text{where} \quad \theta = \kappa(A)\frac{||E||}{||A||}.$$

Estimating the condition number

- If the solution to a linear system changes a great deal when the problem changes only very slightly, then we suspect that the matrix is ill-conditioned
- Recall the definition (infinity norm):

$$\kappa(A) = ||A||_\infty ||A^{-1}||_\infty$$

- Computing $||\mathbf{A}||_\infty$ not hard, computing $||\mathbf{A}^{-1}||_\infty$ is! (if A is ill-conditioned, computing A^{-1} will be unreliable)

Estimating condition number

- Algorithm for estimating the condition number, given LU factorisation of A

1. Compute $\alpha = ||A||_\infty$
2. Take a random initial guess $y^{(0)}$
3. Compute $y^{(5)}$ in the sequence defined by

$$y^{(i+1)} = \frac{A^{-1}y^{(i)}}{||y^{(i)}||_\infty}, \qquad i = 0, 1, \ldots, 4$$

   by solving the systems using the exact factorisation of A, and set $v = ||y^{(5)}||_\infty$.
4. Set $\kappa^* = \alpha v$.