IS2 Problem Sheet I

Question 1
Translate the following compound propositions into symbols:
(i) Napoleon is dead and the world is rejoicing.
(ii) If all eggs are not square, then all eggs are oval.
(iii) If the barometer falls then either it will rain or it will snow.
(iv) If there is low humidity and the sun is shining, then I will play tennis this afternoon.
(v) The sum of two numbers is even if and only if either both numbers are even or both numbers are odd.

Question 2
Write out the truth tables for the following compound expressions:
(i) $(\neg p) \vee (\neg q)$
(ii) $(p \to q) \to (\neg(q \to p))$
(iii) $(p \wedge q) \to r$
(iv) $(p \leftrightarrow (\neg q)) \vee q$
(v) $(((\neg p) \wedge q) \to ((\neg q) \wedge r))$

Question 3
Which of the following propositions are tautologies?
(i) $p \to (q \to p)$
(ii) $(q \vee r) \to ((\neg r) \to q)$
(iii) $(p \wedge (\neg q)) \vee ((q \wedge (\neg r)) \vee (r \wedge (\neg p)))$
(iv) $(p \to (q \to r)) \to ((p \wedge (\neg q)) \vee r)$

Question 4
Write the converse, inverse, contrapositive and negation of the following statement:
"If Sandra finishes her work, then she will go to the cinema"

Question 5
Show that the following pairs of statements are logically equivalent
(i) $p \to q,\ (\neg q) \to (\neg p)$
(ii) $((\neg p) \wedge (\neg q)) \to (\neg r),\ r \to q \vee p$
(iii) $((\neg p) \vee q) \to r,\ (p \wedge (\neg q)) \vee r$

Question 6
Use De Morgan's laws to find and simplify the negation of:
(i) $(\neg a) \wedge b$
(ii) $a \vee (\neg b)$
(iii) $a \vee (b \wedge c)$
(iv) $a \vee ((\neg b) \wedge c)$
(v) $(\neg a) \vee (b \wedge c)$
(vi) $((\neg a) \wedge b) \vee ((\neg d) \wedge c)$
(vii) $(a \wedge b) \vee (\neg(d \wedge c))$

Question 7
Test the validity of the following arguments:
(a) If the function f is not continuous then the function g is not differentiable. g is differentiable. Therefore f is continuous.
(b) If there is oil in Polygonia then either the experts are right or the government is lying. There is no oil in Polygonia, or else the experts are wrong. Therefore the government is not lying.
(c)If U is a subspace of V then U is a subset of V, U contains the zero vector and U is closed. U is a subset of V, and if U is closed then U contains the zero vector. Therefore if U is closed then U is a subspace of V.
(d) If Rachel gets the supervisor's position and works hard, then she'll get a raise. If she gets the raise, then she'll buy a new car. She has not purchased a new car. Therefore either Rachel did not get the supervisor's position or she did not work hard.

Question 8
In the following question take the domain of discourse to be the integers.
Let $p(x), q(x), r(x), s(x)$ and $t(x)$ be the following predicates:

$$\begin{aligned} p(x) &: x > 0 \\ q(x) &: \text{x is even} \\ r(x) &: \text{x is a perfect square} \\ s(x) &: \text{x is (exactly) divisible by 4} \\ t(x) &: \text{x is (exactly) divisible by 5} \end{aligned}$$

Write the following statements in symbolic form:
(i) There exists a positive integer that is even.
(ii) If x is even, then x is not divisible by 5.
(iii) Every even integer is not divisible by 5.
(iv) There exists an even integer divisible by 5
(v) If x is even and x is a perfect square, then x is divisble by 4.

Question 9 Negate and simplify each of the following:
(i) $\exists x[p(c) \vee q(x)]$
(ii) $\forall[p(x) \wedge \neg q(x)]$
(iii) $\forall x[p(x) \rightarrow q(x)]$
(iv) $\exists x[(p(x) \vee q(x)) \rightarrow r(x)]$
(v) $\exists x[p(x) \rightarrow (q(x) \wedge r(x))]$

Answers

Question 1

(i) $p \wedge q$
(ii) $(\neg p) \rightarrow q$
(iii) $p \rightarrow (q \vee r)$
(iv) $(p \wedge q) \rightarrow r$
(v) $p \leftrightarrow (q \vee r)$

Question 2

p	q	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Table 1: Truth Table for $(\neg p) \vee (\neg q)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg(q \rightarrow p)$	$(p \rightarrow q) \rightarrow (\neg(q \rightarrow p))$
T	T	T	T	F	F
T	F	F	T	F	T
F	T	T	F	T	T
F	F	T	T	F	F

Table 2: Truth Table for $((p \rightarrow q) \rightarrow (\neg(q \rightarrow p)))$

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Table 3: Truth Table for $(p \wedge q) \rightarrow r$

p	q	$\neg q$	$p \leftrightarrow (\neg q)$	$(p \leftrightarrow (\neg q)) \vee q$
T	T	F	F	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	F

Table 4: Truth Table for $((p \leftrightarrow (\neg q)) \vee q)$

p	q	r	$\neg p$	$(\neg p) \wedge q$	$\neg q$	$(\neg q) \wedge r)$	$(((\neg p) \wedge q) \rightarrow ((\neg q) \wedge r))$
T	T	T	F	F	F	F	T
T	T	F	F	F	F	F	T
T	F	T	F	F	T	T	T
T	F	F	F	F	T	F	T
F	T	T	T	T	F	F	F
F	T	F	T	T	F	F	F
F	F	T	T	F	T	T	T
F	F	F	T	F	T	F	T

Table 5: Truth Table for $(((\neg p) \wedge q) \rightarrow ((\neg q) \wedge r))$

Question 3

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

Table 6: Truth Table for $(p \rightarrow (q \rightarrow p))$, this is a tautology.

q	r	$q \vee r$	$\neg r$	$\neg r \rightarrow q$	$(q \vee r) \rightarrow ((\neg r) \rightarrow q)$
T	T	T	F	T	T
T	F	T	T	T	T
F	T	T	F	T	T
F	F	F	T	F	T

Table 7: Truth Table for $((q \vee r) \rightarrow ((\neg r) \rightarrow q))$, this is a tautology.

p	q	r	$\neg q$	$p \wedge (\neg q)$	$\neg r$	$q \wedge (\neg r)$	$\neg p$	$r \wedge (\neg p)$
T	T	T	F	F	F	F	F	F
T	T	F	F	F	T	T	F	F
T	F	T	T	T	F	F	F	F
T	F	F	T	T	T	F	F	F
F	T	T	F	F	F	F	T	T
F	T	F	F	F	T	T	T	F
F	F	T	T	F	F	F	T	T
F	F	F	T	F	T	F	T	F

$(q \wedge (\neg r)) \vee (r \wedge (\neg p))$	$(p \wedge (\neg q)) \vee ((q \wedge (\neg r)) \vee (r \wedge (\neg p)))$
F	F
T	T
F	T
F	T
T	T
T	T
T	T
F	F

Table 8: Truth Table for $((p \wedge (\neg q)) \vee ((q \wedge (\neg r)) \vee (r \wedge (\neg p))))$, this is not a tautology.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$\neg q$	$p \wedge (\neg q)$	$(p \wedge (\neg q)) \vee r$	$(p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge (\neg q) \vee r)$
T	T	T	T	T	F	F	T	T
T	T	F	F	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	T	T	F	F	T	T
F	T	F	F	T	F	F	F	F
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	F	F

Table 9: Truth Table for $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge (\neg q)) \vee r))$, this is not a tautology.

Question 4

$p =$ " Sandra finishes her work"
$q =$" Sandra goes to the cinema"
Statement: $p \rightarrow q$
Converse $(q \rightarrow p)$: "If Sandra goes to the cinema then she will finish her work"
Inverse: $((\neg p) \rightarrow (\neg q)) =$ "If Sandra does not finish her work then she will not go to the cinema"
Contrapositive: $((\neg q) \rightarrow (\neg p)) =$ "If Sandra does not go to the cinema then she will not finish her work"
Negation: $(\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge (\neg q)) =$ " Sandra finishes her work and does not go to the cinema"

Question 5

(a)

$$\begin{aligned} p \to q &\equiv \neg p \vee q \\ &\equiv q \vee \neg p \\ &\equiv (\neg q) \to (\neg p) \end{aligned}$$

(b)

$$\begin{aligned} ((\neg p) \vee (\neg q)) \to (\neg r) &\equiv (\neg(p \wedge q)) \to (\neg r) \\ &\equiv (\neg(\neg(p \wedge q))) \vee (\neg r) \\ &\equiv (p \wedge q) \vee (\neg r) \\ &\equiv (\neg r) \vee (q \wedge p) \\ &\equiv r \to (q \wedge p) \end{aligned}$$

(c)

$$\begin{aligned} ((\neg p) \vee q) \to r &\equiv (\neg((\neg p) \vee q)) \vee r \\ &\equiv (p \vee (\neg q)) \vee r \end{aligned}$$

Question 6
(i) $a \vee (\neg b)$
(ii) $(\neg a) \wedge b$
(iii) $(\neg a) \wedge ((\neg b) \vee (\neg c))$
(iv) $(\neg a) \wedge (b \vee (\neg c))$
(v) $a \wedge ((\neg b) \vee (\neg c))$
(vi) $(a \vee (\neg b)) \wedge (d \vee (\neg c))$
(vii) $((\neg a) \vee (\neg b)) \wedge (d \wedge c)$

Question 7
(a)
p = "Function f is continuous."
q = "Function g is differentiable".
$(\neg p) \to (\neg q)$. q . Therefore p.
$(((\neg p) \to (\neg q)) \vee q) \to p$ is a tautology, therefore the argument is valid.

(b)
p = "There is oil in Polygonia"
q="The experts are right"
r="The government is lying"

$p \to (q \vee r)$. $(\neg p) \vee (\neg q)$. Therefore $(\neg r)$
$((p \to (q \vee r)) \wedge ((\neg p) \vee (\neg q))) \to (\neg r)$ is not a tautology, therefore the argument is not valid.

(c)
p = "U is a subspace of V."
q = "U is a subset of V."
r = "U contains the zero vector."
s = "U is closed."
$p \to (q \wedge r \wedge s)$. q. $(s \to r)$ Therefore$(s \to p)$.
$((p \to (q \wedge r \wedge s)) \wedge (q) \wedge (s \to r))) \to (s \to p)$ is not a tautology, therefore the argument is not valid.

(d)
p= "Rachel gets the supervisor's position"
q = "Rachel works hard"
r = "Rachel gets a raise"
s= "Rachel buys a new car"
$(p \wedge q) \to r$. $r \to s.(\neg s)$. Therefore$(\neg p) \vee (\neg q)$.
$(((p \wedge q) \to r) \wedge (r \to s) \wedge (\neg s)) \to ((\neg p) \vee (\neg q))$ is a tautology, therefore the argument is valid.

Question 8
(i) $\exists x\ [q(x)]$
(ii) $\forall x\ [q(x) \to (\neg t(x))]$
(iii) $\forall x\ [q(x) \to (\neg t(x))]$
(iv) $\exists x\ [q(x) \wedge t(x)]$
(v) $\forall x\ [(q(x) \wedge r(x)) \to s(x)]$

Question 9
(i) $\forall x\ [(\neg p(x)) \wedge (\neg q(x))]$
(ii) $\exists x\ [(\neg p(x)) \vee q(x)]$
(iii) $\exists x\ [p(x) \wedge (\neg q(x))]$
(iv) $\forall x\ [(p(x) \vee q(x)) \wedge (\neg r(x))]$
(v) $\forall x\ [p(x) \wedge ((\neg q(x)) \vee (\neg r(x)))]$