BIT Problem Sheet II

Question 1
If the demand function is given by $P = -2.5Q_D + 47.5$ and the supply function is given by $P = Q_S + 23$
(i)Determine the equilibrium price and quantity by Gaussian Elimination.
(ii) Verify the answer in (i) using the graphical method.

Question 2
If the demand function is given by $P = -4Q_D + 36$ and the supply function is given by $P = 2Q_S + 12$
(i)Determine the equilibrium price and quantity by Gaussian Elimination.
(ii) Verify the answer in (i) using the graphical method.

Question 3
Find the roots of the following quadractic functions:

$$\text{(i) } f(x) = x^2 - x - 6 \qquad \text{(ii) } y = x^2 - 1\frac{1}{2}x - 4\frac{1}{2}$$
$$\text{(iii) } y = x^2 - 10x + 21 \qquad \text{(iv) } y = x^2 - 6x + 8$$
$$\text{(v) } f(x) = 12x^2 - 11x + 1 \qquad \text{(vi) } f(x) = 3x^2 + 9x$$

Question 4
Sketch the following quadratic functions and use symmetry to locate the maximum or minimum.

(i)	$f(x) = x^2 + 2x - 99$	(ii)	$f(x) = x^2 + 5x + 6$
(iii)	$y = -x^2 + 5x - 6$	(iv)	$y = 10x^2 + 6x - 4$
(v)	$f(x) = 4x^2 - 11x + 7$	(vi)	$y = x^2 + 11x - 26$

Question 5
Given the demand function $P = 175 - 3Q$
(i) Express the total revenue (TR) as a function of Q and sketch its graph.
(ii) For what values of Q is TR zero?
(iii) What is the maximum value of TR?

Question 6
Given the demand function $P = 270 - 9Q$
(i) Express the total revenue (TR) as a function of Q and sketch its graph.
(ii) For what values of Q is TR zero?
(iii) What is the maximum value of TR?

Question 7
If fixed costs are 12, variable costs per unit are 2 and the demand function is $P = 27-3Q$.
(i) Obtain an expression for Π, the profit function, in terms of Q and hence sketch the graph of Π against Q.

(ii) For what value of Q does the firm break even?
(iii) What is the maximum profit?

Question 8
If fixed costs are 10, variable costs per unit are 3 and the demand function is $P = 20-4Q$.
(i) Obtain an expression for Π, the profit function, in terms of Q and hence sketch the graph of Π against Q.
(ii) For what value of Q does the firm break even?
(iii) What is the maximum profit?

Question 9
Differentiate the following functions and find their value at the points $x = 0$, $x = -3$ and $x = 2$.

$$\begin{array}{rr} \text{(i) } f(x) = x^9 & \text{(ii) } f(x) = x^{-99} \\ \text{(iii) } y = x^{-1\frac{1}{2}} & \text{(iv) } y = x^{-\frac{11}{2}} \\ \text{(v) } y = x^{+7} & \text{(vi) } f(x) = x^{-2.8} \end{array}$$

Question 10
Differentiate the following functions

$$\begin{array}{rr} \text{(i) } y = 7x^2 & \text{(ii) } y = 3x^4 \\ \text{(iii) } f(x) = 2x^2 - 4x^3 & \text{(iv) } f(x) = -9x^7 - 11x^5 \\ \text{(v) } y = 2x^{\frac{1}{2}} + 8 & \text{(vi) } y = 11.214x \\ \text{(vii) } f(x) = 7x^{\frac{1}{3}} + 3x^{-\frac{1}{3}} & \text{(viii) } f(x) = 27x^{\frac{1}{9}} + 27 \\ \text{(ix) } y = 2x^2 - 3x^3 + 4x^4 - 5x^{-5} & \text{(x) } f(x) = -2x^{-2.5} - 3x^{-3.25} \end{array}$$

Question 11
Find the second derivative of the following functions

$$\begin{array}{rr} \text{(i) } y = 3x^2 + 7x^3 & \text{(ii) } y = 3x^{-2} + 7x^{-3} \\ \text{(iii) } f(x) = 2x^4 + 5x^9 - 4x^{-\frac{1}{2}} & \text{(iv) } f(x) = x^{-8} + 2x^{-6} + 12 \end{array}$$

Question 12
Use the chain rule to differentiate the following

$$\begin{array}{rr} \text{(i) } y = (x-5)^7 & \text{(ii) } y = (4-5x)^3 \end{array}$$

(iii) $y = (1 - 0.8x)^{-5}$ (iv) $f(x) = (3x^2 - \frac{1}{2}x^3)^2$

(v) $f(x) = (x^2 + 12)^{\frac{1}{2}}$ (vi) $y = \sqrt{x^5 + 3x^2}$

(vii) $y = (x^3 + 2x^2)^{-\frac{1}{5}}$ (viii) $f(x) = (5x + 7)^{-4}$

(ix) $f(x) = (4x^2 + 3x^{\frac{1}{3}})^{-1}$ (x) $y = (2x + 2)^{-\frac{1}{9}}$

Question 13

Use the product rule to differentiate the following

(i) $y = (3x - 1)(2x + 5)$ (ii) $y = (5 - x)(4 - 2x)$

(iii) $y = (x^2)(x^3 - 1)$ (iv) $y = (x - 1)(x^2 + x + 1)$

(v) $y = (2x - 5)(x^2 + 4)$ (vi) $y = (x + 1)(x^3 + x + 3)$

(vii) $y = (x^2 + x + 1)(x + 1)$ (viii) $y = (2x - x^3 - x^4)(3x - 7)$

(ix) $y = (x + 1)(3x - 5)(x^2 + 4)$ (x) $y = (x^2 + 1)(x^2 + 2)(x^2 + 3)$

Question 14

Use the quotient rule to differentiate the following

(i) $f(x) = \dfrac{(2x - 3)}{(5 - x)}$ (ii) $y = \dfrac{(x^2)}{(x - 1)}$

(iii) $f(x) = \dfrac{(3 - 2x^2)}{(x^3 + 7)}$ (iv) $f(x) = \dfrac{(4x - x^2)}{(x^2 + 1)}$

(v) $y = \dfrac{(1 + x^2)}{(1 - x^2)}$ (vi) $y = \dfrac{(x^3 - 1)}{(x^3 + 1)}$

(vii) $f(x) = \dfrac{(x^2 + x + 1)}{(x^2 + 1)}$ (viii) $y = \dfrac{(x^3 - x - 1)}{(x^3 + x + 1)}$

(ix) $y = \dfrac{(3x - 7)}{(2x - 1)}$ (x) $f(x) = \dfrac{(2x^2 + x - 1)}{(x - 3)}$

Question 15

Find and classify the turning points of the following functions:

(i) $f(x) = x^3 - 3x - 2$

(ii) $f(x) = x^3 - 3x^2 - x + 3$

(iii) $f(x) = x^3 - 6x^2 + 3x + 10$

(iv) $f(x) = 2x^3 + 3x^2 - 12x + 8$

(v) $f(x) = 4x^3 - 8x^2 - 11x + 15$

Question 16
Let P be the price and Q the quantity of a good produced. If the demand equation of the good is $P + Q = 38$ and the total cost function is $TC = Q^2 + 8Q + 27$
(i) Find the level of output which maximises total revenue and state this maximum revenue.
(ii) Find the level of output which maximises profit and give the maximum profit.

Question 17
Let P be the price and Q the quantity of a good produced. If the demand equation of the good is $P + Q = 36$ and the total cost function is $TC = (0.5)Q^2 + 27Q + 12$
(i) Find the level of output which maximises total revenue and state this maximum revenue.
(ii) Find the level of output which maximises profit and give the maximum profit.

Question 18
Let $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 3 & -2 \end{pmatrix}, C = \begin{pmatrix} 1 & -4 \\ 2 & -1 \end{pmatrix} D = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} E = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$
Evaluate the following products (i) AB (ii) BA (iii) CB (iv) BC (v) DE (vi) ED.

Question 19
Invert the following matrices

$$\begin{pmatrix} 2 & 2 \\ 2 & 6 \end{pmatrix}, \quad \begin{pmatrix} 3 & 4 \\ 12 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 2 & 1 \\ 1 & -5 & -1 \\ 2 & -1 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 2 \\ 6 & 3 & 6 \end{pmatrix}.$$

Question 20
For each of the following systems of linear equations express the problem in matrix form and solve the system:

$$\begin{aligned} x + 2y &= -1 \\ 2x - y &= -3 \end{aligned} \qquad \begin{aligned} 2x + y - 2x &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 4y + 3z &= 4 \end{aligned}$$

$$\begin{aligned} 2x - 3y + z &= -5 \\ x + 4y + 2z &= 13 \\ x - 2y + z &= -2 \end{aligned} \qquad \begin{aligned} 2x + y - 3z &= 4 \\ x + 2y - 2z &= 6 \\ 3x - y - z &= 2 \end{aligned}$$

Question 21
Given the matrix of technical coefficients, A, and final demand B for industries 1 and 2.

$$A = \begin{pmatrix} 0.5 & 0.4 \\ 0.25 & 0.4 \end{pmatrix} \quad B = \begin{pmatrix} 40 \\ 100 \end{pmatrix}$$

(a) Determine $(I - A)^{-1}$.
(b) Calculate the total output required from each sector.

Question 22
Given the matrix of technical coefficients, A, and final demand B for industries 1,2 and 3.

$$A = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.25 & 0.1 \\ 0.4 & 0.2 & 0.2 \end{pmatrix} \quad B = \begin{pmatrix} 1020 \\ 1200 \\ 700 \end{pmatrix}$$

(a) Determine $(I - A)^{-1}$.
(b) Calculate the total output required from each sector.

Answers

Question 1 $Q = 7,\ P = 30$

Question 2 $Q = 4,\ P = 20$

Question 3
(i) $3,\ -2$ (ii) $-1.5,\ 3$ (iii)$7,\ 3$ (iv) $4,\ 2$ (v)$-\frac{1}{12},\ 1$ (vi) $0,\ -3$

Question 4
(i) The curve is "U-shaped" as the coefficient of x^2 is positive (+1). The y intercept is at $y = -99$. The roots are at $x = 9$ and $x = -11$

(ii) The curve is "U-shaped" as the coefficient of x^2 is positive (+1). The y intercept is at $y = 6$. The roots are at $x = -3$ and $x = -2$

(iii) The curve has an inverted "U-shape" as the coefficient of x^2 is negative (-1). The y intercept is at $y = -6$. The roots are at $x = 3$ and $x = 2$

(iv) The curve is "U-shaped" as the coefficient of x^2 is positive (+10). The y intercept is at $y = -4$. The roots are at $x = \frac{2}{5}$ and $x = -1$

(v) The curve is "U-shaped" as the coefficient of x^2 is positive (+4). The y intercept is at $y = 7$. The roots are at $x = 1$ and $x = -1\frac{3}{4}$

(vi) The curve is "U-shaped" as the coefficient of x^2 is positive (+1). The y intercept is at $y = -26$. The roots are at $x = 2$ and $x = -13$

Question 5
(i) $TR = 175Q - 3Q^2$
(ii) TR is zero for $Q = 0$ and $Q = 58\frac{1}{3}$
(iii) Maximum total revenue is $2552\frac{1}{12} = 2552.083$. This occurs at $Q = 29\frac{1}{6} = 29.167$.

Question 6
(i) $TR = 270Q - 9Q^2$
(ii) TR is zero for $Q = 0$ and $Q = 30$

(iii) Maximum total revenue is 2025. This occurs at $Q = 15$.

Question 7

(i) $\Pi = -3Q^2 + 25Q - 12$, this has an inverted "U-shape". It cuts the vertical axis at $\Pi = -12$ and it has roots 0.51 and 7.82.
(ii) The firm breaks even when $Q = 0.51$ and $Q = 7.82$.
(iii) The maximum profit is 40.08, which occurs at $Q = 4.165$.

Question 8

(i) $\Pi = -4Q^2 + 17Q - 10$, this has an inverted "U-shape". It cuts the vertical axis at $\Pi = -10$ and it has roots 0.705 and 3.545.
(ii) The firm breaks even when $Q = 0.705$ and $Q = 3.545$.
(iii) The maximum profit is 8.0625, which occurs at 2.125.

Question 9

(i) $f'(x) = 9x^8$ (ii) $f'(x) = -99x^{-100}$ (iii) $\frac{dy}{dx} = -1\frac{1}{2}x^{-2\frac{1}{2}}$ (IV) $\frac{dy}{dx} = -\frac{11}{2}X^{-\frac{13}{2}}$ (v) $\frac{dy}{dx} = 7x^6$ (vi) $f'(x) = -2.8x^{-3.8}$

Question 10

(i) $\frac{dy}{dx} = 14x$ (ii) $\frac{dy}{dx} = 12x^3$ (iii) $f'(x) = 4x - 12x^2$ (iV) $f'(X) = -63x^6 - 55x^4$ (v) $\frac{dy}{dx} = x^{-\frac{1}{2}}$ (vi) $\frac{dy}{dx} = 11.214$ (vii) $f'(x) = \frac{7}{3}x^{-\frac{2}{3}} - x^{-1\frac{1}{3}}$ (viii) $f'(x) = 3x^{-\frac{8}{9}}$
(ix) $\frac{dy}{dx} = 4x - 9x^2 + 16x^3 + 25x^{-6}$ (x) $f'(x) = 5x^{-3.5} + 9.75x^{-4.25}$

Question 11

(i) $\frac{d^2y}{dx^2} = 6 + 42x$
(ii) $\frac{d^2y}{dx^2} = 18x^{-4} + 84x^{-5}$
(iii)$f''(x) = 24x^2 + 360x^7 - 3x^{2\frac{1}{2}}$
(iv)$f''(x) = 72x^{-10} + 84x^{-8}$

Question 12

(i) $\frac{dy}{dx} = 7(x-5)^6$ (ii) $\frac{dy}{dx} = -15(4-5x)^2$

(iii) $\frac{dy}{dx} = 4(1-0.8x)^{-6}$ (iv) $f'(x) = 2(3x^2 - \frac{1}{2}x^3)(6x - \frac{3}{2}x^2)$

(v) $f'(x) = (x^2+12)^{-\frac{1}{2}}$ (vi) $\frac{dy}{dx} = \frac{1}{2}(x^5 + 3x^2)^{-\frac{1}{2}}(5x^4 + 6x)$

(vii) $\frac{dy}{dx} = -\frac{1}{5}(x^3 + 2x^2)^{-1\frac{1}{5}}(3x^2 + 4x)$ (viii) $f'(x) = -20(5x+7)^{-5}$

(ix) $f'(x) = -(4x^2 + 3x^{\frac{1}{3}})^{-2}(8x + x^{-\frac{2}{3}})$ (x) $\frac{dy}{dx} = -\frac{2}{9}(2x+2)^{-1\frac{1}{9}}$

Question 13

(i)$\frac{dy}{dx} = 12x + 13$ (ii)$\frac{dy}{dx} = 4x - 14$

$$\text{(iii)}\frac{dy}{dx} = 5x^4 - 2x \qquad \text{(iv)}\frac{dy}{dx} = 3x^2$$
$$\text{(v)}\frac{dy}{dx} = 6x^2 - 10x + 8 \qquad \text{(vi)}\frac{dy}{dx} = 4x^3 + 3x^2 + 2x + 4$$
$$\text{(vii)}\frac{dy}{dx} = 3x^2 + 4x + 2 \qquad \text{(viii)}\frac{dy}{dx} = -15x^4 + 16x^3 + 21x^2 + 12x - 14$$
$$\text{(ix)}\frac{dy}{dx} = 12x^3 - 6x^2 + 14x - 8 \qquad \text{(x)}\frac{dy}{dx} = 6x^5 + 24x^3 + 24x$$

Question 14

$$f'(x) = \frac{7}{(5-x)^2} \qquad \frac{dy}{dx} = \frac{x(x-2)}{(x-1)^2}$$
$$f'(x) = \frac{2x^4 - 9x^2 - 28x}{(x^3+7)^2} \qquad f'(x) = \frac{4 - 2x - 4x^2}{(x^2+1)^2}$$
$$\frac{dy}{dx} = \frac{4x}{(1-x^2)^2} \qquad \frac{dy}{dx} = \frac{6x^2}{(x^3+1)^2}$$
$$f'(x) = \frac{1-x^2}{(1+x^2)^2} \qquad \frac{dy}{dx} = \frac{2x^2(2x+3)}{(x^3+x+1)^2}$$
$$\frac{dy}{dx} = \frac{11}{(2x-1)^2} \qquad f'(x) = \frac{2x^2 - 12x - 2}{(x-3)^2}$$

Question 15
(i) Max $(-1, 0)$, min $(1, -4)$
(ii) max $(2.15, -3.08)$ max $(-0.15, 3.08)$ (iii) max $(0.26, 10.39)$ min $(3.73, -10.39)$ (iv) max $(-2, 28)$, min $(1, 1)$ (iv) max $(-\frac{1}{2}, 18)$, min $(\frac{11}{6}, -\frac{200}{7})$,

Question 16
(i) $Q = 19,\ TR_{max} = 361$
(ii) $Q = 7.5, \Pi = 85.5$

Question 17
(i) $Q = 18,\ TR_{max} = 324$
(ii) $Q = 3, \Pi = 1.5$

Question 18
(i) $AB = \begin{pmatrix} 11 & -6 \\ 1 & 2 \end{pmatrix}$
(ii) $BA = \begin{pmatrix} 2 & 6 \\ -1 & 11 \end{pmatrix}$
(iii) $CB = \begin{pmatrix} -10 & 8 \\ 1 & 2 \end{pmatrix}$
(iv) $BC = \begin{pmatrix} 2 & -8 \\ -1 & -10 \end{pmatrix}$
(v) $DE = \begin{pmatrix} 4 & 9 & 4 \\ 5 & 5 & 1 \\ 9 & 7 & 0 \end{pmatrix}$ (vi) $ED = \begin{pmatrix} -7 & -4 & 1 \\ 18 & 8 & 11 \\ 15 & 7 & 8 \end{pmatrix}$

Question 19

(i) $\frac{1}{8}\begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$

(ii) $-\frac{1}{48}\begin{pmatrix} 0 & -4 \\ -12 & 3 \end{pmatrix}$

(iii) The matrix has no inverse because the determinant is zero.

(iv) $\frac{1}{55}\begin{pmatrix} -31 & -13 & 3 \\ -8 & -14 & -1 \\ 8 & 2 & 8 \end{pmatrix}$

(v) $-\frac{1}{13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}$

(vi) $\begin{pmatrix} 1 & -\frac{1}{4} & \frac{1}{12} \\ 0 & \frac{1}{4} & -\frac{1}{12} \\ -1 & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$

Question 20

(a) $x = -1.4,\ y = 0.2,\ A^{-1} = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{pmatrix}$

(b) $x = 1,\ y = 2,\ x = -3,\ A^{-1} = -\frac{1}{7}\begin{pmatrix} 2 & 11 & -6 \\ -1 & -16 & 10 \\ -2 & 3 & -1 \end{pmatrix}$

(c) $x = -1,\ y = 2,\ z = 3\ A^{-1} = \frac{1}{7}\begin{pmatrix} 8 & 1 & -10 \\ 1 & 1 & -1 \\ -6 & 1 & 11 \end{pmatrix}$

(d) $x = 2,\ y = 3,\ z = 1,\ A^{-1} = \frac{1}{16}\begin{pmatrix} -8 & 8 & 8 \\ -10 & 14 & 2 \\ -14 & 10 & 6 \end{pmatrix}$

Question 21

(i) $(I - A)^{-1} = \begin{pmatrix} 3 & 2 \\ 1.25 & 2.5 \end{pmatrix}$

(ii) Total Output $= \begin{pmatrix} 320 \\ 300 \end{pmatrix}$

Question 22

(i) $(I - A)^{-1} = \begin{pmatrix} 2.9 & 1.8 & 0.95 \\ 1 & 2 & 0.5 \\ 1.7 & 1.4 & 1.85 \end{pmatrix}$

(ii) Total Output $= \begin{pmatrix} 5783 \\ 3770 \\ 4709 \end{pmatrix}$