Algebraic number theory — Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2026/MAU34109/index.html

Version: November 2, 2026

Submit your answers in class or to neumayec@tcd.ie by Thursday November 12, 1pm.

Exercise 3.1: Ideals in a cubic field (100 pts)

Let $A(x) = x^3 - 8x + 2$, and let $K = \mathbb{Q}(\alpha)$, where α is a root of $A(x)$.
Here is a table of values of $A(x)$ which you may find useful (even though you may have to compute other values of $A(x)$ at some point...):

x	-3	-2	-1	0	1	2	3
$A(x)$	-1	10	9	2	-5	-6	5

1. (2 pts) Determine the degree $[K : \mathbb{Q}]$ of K.
2. (10 pts) Prove that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$.
3. (10 pts) Which primes $p \in \mathbb{N}$ ramify in K?
4. (6 pts each) For each prime $p \in \mathbb{N}$, $p \leqslant 7$, compute explicitly the decomposition of p in K (in other words, the factorisation of (p) as a product of prime ideals). Choose a name, determine the norm, and specify generators for each of the prime ideals occurring in these decompositions.
5. (4 pts each) For each $n \in \{1, 2, 3, 4, 5, 6, 7\}$, list all the ideals $\mathfrak{a}$ of $\mathbb{Z}_K$ such that $N(\mathfrak{a}) = n$.
6. Factor the following ideals into prime ideals:
   (a) (5 pts) (α),
   (b) (7 pts) $(\alpha - 3)$,
   (c) (7 pts) $(\alpha + 2)$,
   (d) (7 pts) $(\alpha + 1)$.

   Hint: Remember that $N_{\mathbb{Q}}^K(\alpha + m) = -A(-m)$ for all $m \in \mathbb{Z}$.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice (they may even give you inspiration to help you solve Exercise 1), and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 3.2: Ideals of fixed norm

How many ideals of norm 900 are there in the ring of integers of $\mathbb{Q}(\sqrt{7})$?
Hint: Compute the decomposition in $\mathbb{Q}(\sqrt{7})$ of the prime factors of 900.

Exercise 3.3: Similar-looking yet non-isomorphic number fields

The goal of this exercise is to prove that the number fields $\mathbb{Q}(\sqrt[3]{6})$ and $\mathbb{Q}(\sqrt[3]{12})$ have the same degree and discriminant, but are not isomorphic.

To ease notation, we let $\alpha = \sqrt[3]{6}$, $\beta = \sqrt[3]{12}$, $K = \mathbb{Q}(\alpha)$ and $L = \mathbb{Q}(\beta)$.

1. Prove that $[K : \mathbb{Q}] = 3$.
2. Prove that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ and compute $\operatorname{disc} K$.
3. Prove that $[L : \mathbb{Q}] = 3$ and that $\operatorname{disc} L$ is of the form $-2^a 3^5$ for some integer $a \geqslant 0$. What are the possible values of a ?
4. Prove that $L \simeq \mathbb{Q}(\sqrt[3]{18})$.
   Hint: Take a look at $\gamma = \beta^2/2$.
5. Deduce that $\operatorname{disc} L = \operatorname{disc} K$.
6. Which primes $p \in \mathbb{N}$ ramify in K? What about L?
7. Compute explicitly the decomposition of 7 in K and in L.
8. Deduce that K and L are not isomorphic.
9. Compute explicitly the decomposition of 2 and 3 in K and in L.
10. Deduce the factorisation of the ideals $\alpha\mathbb{Z}_K$, $\beta\mathbb{Z}_L$ and $\gamma\mathbb{Z}_L$ into primes.

Exercise 3.4: A necessarily complicated ring of integers

Let K be a number field such that $[K : \mathbb{Q}] = 3$, and suppose[1] that $p = 2$ is totally split in K. Prove that $\mathbb{Z}_K$ is *not* of the form $\mathbb{Z}[\alpha]$ for any $\alpha \in K$.

[1] This can really happen, e.g. $K = \mathbb{Q}(\alpha)$ where $\alpha^3 - \alpha^2 - 2\alpha - 8 = 0$.