

Algebraic number theory — Exercise sheet 2

<https://www.maths.tcd.ie/~mascotn/teaching/2026/MAU34109/index.html>

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Submit your answers in class or to neumayec@tcd.ie by Thursday October 22, 1pm.

Exercise 2.1: A cubic field with an elusive maximal order (100pts)

Let $A(x) = x^3 + 2x^2 + 5x - 1 \in \mathbb{Q}[x]$.

We admit without proof that $\text{disc } A = -575 = -5^2 \cdot 23$.

We also observe that $A(0) = -1$ whereas $A(1) = 7$.

1. (12 pts) Prove that $A(x)$ has no repeated root in $\mathbb{C}$, and determine how many roots it has in $\mathbb{R}$.
2. (12 pts) Use the notion of algebraic integer to prove that if $A(x)$ were reducible over $\mathbb{Q}$, then it would have a root in $\mathbb{Z}$. *You are NOT allowed to invoke the rational root theorem for this question.*
3. (4 pts) Hence deduce that $A(x)$ is irreducible over $\mathbb{Q}$.

In what follows, we let $K = \mathbb{Q}(\alpha)$, where α is a root of $A(x)$.

4. (8 pts) Prove that $\mathbb{Z}[\alpha]$ is an order in K , and specify a $\mathbb{Z}$ -basis of it.
5. (16 pts) Find an upper bound on the index of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$.

Let $\beta = 1 + \frac{\alpha^2 - 6\alpha - 2}{5} \in K$.

We admit without proof that the characteristic polynomial of β is $x^3 + 11x + 7$.

6. (10 pts) Prove that $\mathbb{Z}_K \neq \mathbb{Z}[\alpha]$.
7. (10 pts) Determine $\text{disc } K$.
8. (16 pts) Prove that $1, \alpha, \beta$ is a $\mathbb{Z}$ -basis of $\mathbb{Z}_K$.
9. (12 pts) Is it true that $\mathbb{Z}_K = \mathbb{Z}[\beta]$? If not, what is the index of $\mathbb{Z}[\beta]$ in $\mathbb{Z}_K$?

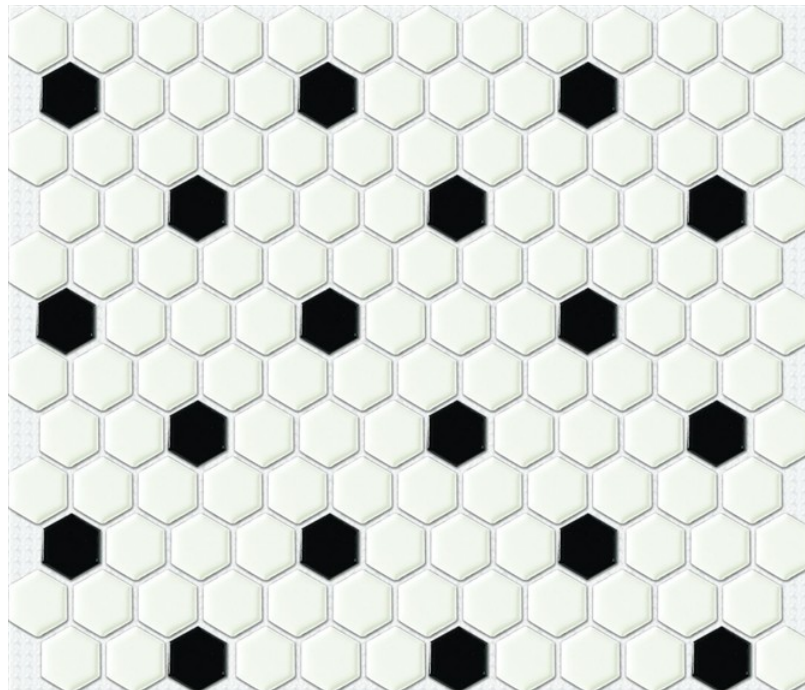
This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice (they may even give you inspiration to help you solve Exercise 1), and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2.2: Floor tilings

1. In the picture below¹, (the centre of) the hexagonal floor tiles (both the black ones and the white ones) form a lattice, and (the centre of) the black tiles form a sublattice. Compute the index of this sublattice by writing down a change-of-basis (= transition) matrix. What is the proportion of black tiles?



2. Same questions for this other tiling pattern.



¹In case you were wondering, no, that's not my home; I just googled "floor tiling".

Exercise 2.3: To be or not to be integral

Is $\frac{3 + 2\sqrt{6}}{\sqrt{6} - 2}$ an algebraic integer ?

Exercise 2.4: Another cubic field

Let $A(x) = x^3 - 5x + 1 \in \mathbb{Q}[x]$.

1. Prove that $A(x)$ is irreducible over $\mathbb{Q}$.
2. Let $K = \mathbb{Q}(\alpha)$, where $A(\alpha) = 0$. Determine the degree of K , the ring of integers of K , the discriminant of K , and the signature of K . Needless to say, justify your answers.

Exercise 2.5: And finally, a quartic field

Reminder: $\text{disc}(x^n + bx + c) = (-1)^{n(n-1)/2}((1-n)^{n-1}b^n + n^n c^{n-1})$.

Let $f(x) = x^4 - 2x + 4$, which you may assume without proof is irreducible over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha)$, where α satisfies $f(\alpha) = 0$.

1. Prove that $\mathbb{Z}[\alpha]$ is an order in K .
2. Compute and factor the discriminant of $\mathbb{Z}[\alpha]$.
Hint: $2^{10} - 3^3 = 997$ is prime.
3. At this point, what are the possibilities for $\text{disc } K$, and the corresponding values of the index of $\mathbb{Z}[\alpha]$?
4. Let $\beta = \frac{\alpha^3}{2} \in K$, and consider the lattice $\mathcal{O} \subset K$ with $\mathbb{Z}$ -basis

$$1, \alpha, \alpha^2, \beta.$$

Prove that $\mathcal{O}$ is stable under multiplication by β .

Hint: what is $\beta \cdot \alpha$?

5. Deduce that β is an algebraic integer.
6. Which of the possibilities listed in question 2. remain?
7. Prove that $\mathcal{O}$ is actually an order in K .
Hint: Prove that $\mathcal{O}$ is also stable under multiplication by α .
8. It turns out that $\mathbb{Z}_K = \mathcal{O}$. Give the discriminant of K in factored form.