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JS/SS Maths/TP/TJH **Semester 2, 2024–2025**

MAU34104 Group representations

PRACTICE PAPER

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Instructions to candidates:

All the representations considered in this exam are over the field $\mathbb{C}$ of complex numbers.
This is a mock exam for revisions.
When solving a question in this exercise, you are allowed to admit the result of the previous questions (except those of question 7., because they rely on an absurd hypothesis), but *not* the results of the *next* questions.
Even if you were not able to solve previous questions, you are encouraged to explain how you would use their results if you had solved them.
For example, when you solve question 5., you are allowed to use the results of questions 3. and 4. even if you were unable to solve these questions (in which case you should try to explain how you could solve question 5. if you had solved questions 3. and 4.), but you are not allowed to use the results of question 6.

The use of non-programmable calculators is allowed.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1 *Bookwork*

1. Define a group representation, and the character of this representation.
2. Define the inner product of two characters of a group G.
3. State the second orthogonality relations.
4. State the second formula for index characters.
5. State the Frobenius reciprocity theorem.

Question 2 *The mysterious group*

As he was exploring a Frobenian temple lost deep inside of the German jungle, Indiana Jones has stumbled upon mysterious markings on a wall:

	C_1	C_2	C_3	C_4	C_5	C_6
			$\vdots$			
α	3	-1	0	1	z	$\bar{z}$
β	3	-1	0	1	$\bar{z}$	z
γ	6	2	0	0	-1	-1
δ	7	-1	1	-1	0	0
ϵ	8	0	-1	0	1	1

where z and $\bar{z}$ are the complex-conjugate roots of $x^2 + x + 2$.

Having taking MAU34104 in his youth, Indiana quickly realises that this is the character table of a group G; unfortunately, the top of the table is damaged, so that the information about the conjugacy classes of G is unreadable and some of the characters of G may be missing. Fortunately, there is no more damage (no column has been erased), so Indiana can clearly see that there are 6 conjugacy classes, which he has taken upon himself to denote by C_1, ..., C_6. He has also named α, ..., ϵ the characters that are still readable.

1. How many irreducible characters are missing? Write them down.
2. Prove that the conjugacy class of the identity 1_G of G is C_1.

3. Prove that G has 168 elements.

4. Prove that G is a simple group.

5. Determine the size of the conjugacy classes.

6. An inscription next to the table claims that the elements of C_2 have order 2 (we admit this without proof so as not to anger the Frobenian gods). Let $h \in C_2$ be such an element, so that $H = \{1_G, h\}$ is a subgroup of G isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and let $\psi : H \longrightarrow \mathbb{C}$ be the character of H defined by

$$\phi(1_G) = 1, \quad \psi(h) = -1.$$

   (a) Write dow the induced character $\operatorname{Ind}_H^G \psi$ of G.

   (b) Determine the decomposition of $\operatorname{Ind}_H^G \psi$ into irreducible characters of G.

Solution 2

1. We know there are always as many irreducible characters as conjugacy classes (the character table is square). As we are told there are 6 conjugacy classes, this means exactly 1 irreducible character is missing.

   Furthermore, every group admits the trivial character as an irreducible character. The trivial character does not appear on the partial character table, so it must be the missing one.

   This means that the complete character table is actually

	C_1	C_2	C_3	C_4	C_5	C_6
			$\vdots$			
$\mathbb{1}$	1	1	1	1	1	1
α	3	-1	0	1	z	$\bar{z}$
β	3	-1	0	1	$\bar{z}$	z
γ	6	2	0	0	-1	-1
δ	7	-1	1	-1	0	0
ϵ	8	0	-1	0	1	1

2. We have $\chi(1_G) = \deg \chi \in \mathbb{N}$ for every character χ, and the first column of the character table is the only one that only contains positive integers.

3. We have

$$\#G = \sum_{\chi \in \mathrm{Irr}(G)} (\deg \chi)^2$$

for any group G.

For this particular group G, this yields

$$\#G = 1^2 + 3^2 + 3^3 + 6^2 + 7^2 + 8^2 = 168$$

(read the column of C_1, as we have established in the previous question that $1_G \in C_1$.)

4. We know that every normal subgroup of G is an intersection of $\mathrm{Ker}\,\chi$ for some of the irreducible characters χ of G. But we see that while $\mathrm{Ker}\,\mathbb{1} = G$ as usual, all the other irreducible characters have trivial kernel; therefore $\{1_G\}$ and G are the only normal subgroups of G, which shows that G is a simple group.

5. By the second orthogonality formula, we have

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(g)} = \#G / \# \operatorname{cc}_G(g).$$

As we know that $\#G = 168$, we deduce that $\#C_1 = 1$ (which we already knew), $\#C_2 = 21$, $\#C_3 = 56$, $\#C_4 = 42$, and $\#C_5 = \#C_6 = 24$ (for C_5 and C_6, we use the fact that since z and $\bar{z}$ are the two complex roots of $x^2 + x + 2$, we have $z\bar{z} =$ product of the roots $= 2$).

6. (a) The second formula for induced characters informs us that for all $g \in G$,

$$(\operatorname{Ind}_H^G \psi)(g) = \frac{[G:H]}{\# \operatorname{cc}_G(g)} \sum_j \# cc_H(h_j)\psi(h_j)$$

where $\operatorname{cc}_G(g) \cap H$ is the (automatically disjoint) union of the $cc_H(h_j)$.

Furthermore, $[G:H] = 168/2 = 84$.

For $g = 1_G$, we have $\operatorname{cc}_G(1_G) = \{1_G\}$ so $\operatorname{cc}_G(1_G) \cap H = \{1_G\} = \operatorname{cc}_H(1_G)$, whence

$$(\operatorname{Ind}_H^G \psi)(1_G) = \frac{84}{1} 1\psi(1_G) = 84$$

(which we already knew since $\deg \operatorname{Ind}_H^G \psi = [G:H] \deg \psi = 84 \cdot 1$).

For $g \in C_2$, we have $\operatorname{cc}_G(g) = C_2$ so $\operatorname{cc}_G(1_G) \cap H = \{h\} = \operatorname{cc}_H(h)$ since $h \in C_2$ by assumption, so

$$(\operatorname{Ind}_H^G \psi)(g) = \frac{84}{21} 1\psi(h) = -4$$

as we have determined that $\# C_2 = 21$ in the previous question.

For $g \in C_j$ with $j = 3, 4, 5,$ or 6, we have $\operatorname{cc}_G(g) \cap H = C_j \cap H = \emptyset$ as $H = \{1, h\}$, so

$$(\operatorname{Ind}_H^G \psi)(g) = 0.$$

(b) We *could* calculate the inner product of $\operatorname{Ind}_H^G \psi$ with the irreducible characters $\mathbb{1}, \alpha, \cdots, \epsilon$ of G.

But there is a much better way: Frobenius reciprocicity tells us that if χ is any character of G, we have

$$(\operatorname{Ind}_H^G \psi | \chi)_G = (\psi | \operatorname{Res}_H^G \chi)_H = \frac{1}{\# H} \sum_{H \in H} \psi(h)\overline{\chi(h)} = \frac{1}{2}(\overline{\chi(1_G) - \chi(h)}).$$

We thus find $(\operatorname{Ind}_H^G \psi | \mathbb{1}) = 0$, $(\operatorname{Ind}_H^G \psi | \alpha) = 2$, $(\operatorname{Ind}_H^G \psi | \beta) = 2$, $(\operatorname{Ind}_H^G \psi | \gamma) = 2$, $(\operatorname{Ind}_H^G \psi | \delta) = 4$, and $(\operatorname{Ind}_H^G \psi | \epsilon) = 4$, so we conclude that $\operatorname{Ind}_H^G \psi$ decomposes as

$$\operatorname{Ind}_H^G \psi = 2\alpha + 2\beta + 2\gamma + 4\delta + 4\epsilon.$$

Note: thanks to Frobenius reciprocity, we get all of this without even having to determine the values of $\operatorname{Ind}_H^G \psi$ as in the previous question.

Question 3 *The character table of a group of order* 12

Let G be the group generated by two elements $s, t \in G$ with s of order 3 and t of order 4 (so $s^3 = t^4 = 1_G$, where $1_G \in G$ is the identity element) such that

$$tst^{-1} = s^{-1}. \qquad (\star)$$

We *admit* the following facts *without proof* (they are easily proved, but this is not the point of this exercise):

- The relation $(\star)$ implies $st = ts^{-1}$ and $ts = s^{-1}t$,
- Each element of G can be expressed as $s^m t^n$ for some *unique* integers $0 \leqslant m < 3$ and $0 \leqslant n < 4$,
- In particular, $\#G = 12$,
- $H = \{1_G, t, t^2, t^3\}$ is a subgroup of G isomorphic to $\mathbb{Z}/4\mathbb{Z}$,
- $N = \{1_G, s, s^2\}$ is a *normal* subgroup of G, with quotient G/N isomorphic to $\mathbb{Z}/4\mathbb{Z}$ via $s^m t^n \mapsto n \bmod 4$.

1. Write down the character table of $\mathbb{Z}/4\mathbb{Z}$.
2. Deduce that G admits at least four irreducible representations of degree 1, and describe them. Do not forget to explain why they are irreducible!
3. Prove that not every irreducible representation of G is of degree 1.
4. We admit that in G,
    - s and s^2 are conjugate,
    - t, st, and s^2t are conjugate,
    - t^3, st^3, and s^2t^3 are conjugate,
    - and that st^2 and s^2t^2 are conjugate.

Prove that up to isomorphism, G admits six irreducible representations, of respective degrees $1, 1, 1, 1, 2, 2$, and that the conjugacy classes of G are as follows:

$$\{1_G\}, \{t^2\}, \{s, s^2\}, \{t, st, s^2t\}, \{t^3, st^3, s^2t^3\}, \{st^2, s^2t^2\}.$$

From now on, let ϕ be the character of one of the irreducible representations of G of degree 1 constructed in question 2. such that the other irreducible characters of G of degree 1 are the powers of ϕ, and let ψ be either of the irreducible characters of degree 2 of G. You may find it useful to write down a partially completed character table of G, and to update it after each question.

5. (a) Prove that $\phi\psi$ is also a character of G.

   (b) Prove that $\phi\psi$ is also of degree 2.

   (c) Prove that $\phi\psi$ is also irreducible.

6. In this question, we suppose **by contradiction** that $\phi\psi = \psi$.

   (a) Prove that ψ vanishes at t, t^2, t^3, and st^2.

   (b) Use an orthogonality relation to deduce the value of $\psi(s)$.

   (c) Prove that this value of $\psi(s)$ is incompatible with the fact that ψ is an irreducible character.

It follows that $\phi\psi \neq \psi$, so the irreducible characters of degree 2 of G are ψ and $\phi\psi$.

7. (a) Prove that $\phi^2\psi = \psi$.

   (b) Deduce the values of ψ at t and at t^3.

8. Let χ be an irreducible character of H (pick any of them, it does not matter, but clearly state which one you have picked), and let $\eta = \operatorname{Ind}_H^G \chi$.

   (a) Prove that $\deg \eta = 3$.

   (b) Prove that η decomposes as the sum of an irreducible character of degree 1 and of an irreducible character of degree 2 of G. Identify this character of degree 1.

From now on, we assume without loss of generality that the irreducible character of degree 2 which occurs in the decomposition of η is ψ.

(c) Compute $\eta(t^2)$, and deduce the value of $\psi(t^2)$. Determine $\psi(s)$ similarly.

9. Write down the complete character table of G.

10. Determine the centre of G and the derived subgroup of G.

Solution 3

1. *(The character table of $\mathbb{Z}/n\mathbb{Z}$ has been seen in class.)*

   The fourth roots of 1 are the powers of $e^{2\pi i/4} = i$, whence

	0	1	2	3
$\mathbb{1} = \chi^0$	1	1	1	1
χ	1	i	-1	$-i$
χ^2	1	-1	1	-1
χ^3	1	$-i$	-1	i.

2. If $\rho' : \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathrm{GL}(V)$ is a representation, then we can inflate it into the representation

$$\rho : G \xrightarrow{\pi} G/N \simeq \mathbb{Z}/4\mathbb{Z} \xrightarrow{\rho} \mathrm{GL}(V).$$

   This representation has the same space V, so $\deg \rho = \dim V = \deg \rho'$.

   Apply the previous question to the four representations of degree 1 of $\mathbb{Z}/4\mathbb{Z}$ mentioned in question 1. The resulting representations still have degree 1, and are therefore irreducible.

3. We have seen in class that the only groups whose irreducible representations have all degree 1 are the Abelian groups. But in G, we have $tst^{-1} = s^{-1} = s^2 \neq t$ (because each element of G can be uniqueley expressed as $s^m t^n$ as mentioned in the preamble), so G is not Abelian.

4. We are given that each conjugacy class of G is a union of some of the following six subsets:

$$\{1_G\}, \{t^2\}, \{s, s^2\}, \{t, st, s^2t\}, \{t^3, st^3, s^2t^3\}, \{st^2, s^2t^2\}.$$

Therefore, G has at most 6 conjugacy classes, and thus at most 6 irreducible representations. We also know that G has 4 irreducible representations of degree 1 (from question 3.), and at least one of degree at least 2 (from question 4.), so G has either 5 or 6 irreducible representations. Finally, we know that the squares of the degrees of these representations sum to $\#G = 12$.

If G had 5 irreducible representations, their degrees would be $1, 1, 1, 1, d$ for some integer $d \geqslant 2$ such that $1^2 + 1^2 + 1^2 + 1^2 + d^2 = 12$. But this implies $d^2 = 8$, which is absurd.

Therefore, G has 6 irreducible representations, whose degrees are $1, 1, 1, 1, d, d'$ for some integers $d \geqslant 1$ and $d' \geqslant 2$ such that $1^2+1^2+1^2+1^2+d^2+d'^2 = 12$, whence $d^2+d'^2 = 8$. The only possibility is that $d = d' = 2$.

Finally, this shows that G has exactly 6 conjugacy classes, which must be the subset displayed above.

We can start to fill in the character table of G, using the fact that since $N = \langle s \rangle$, the projection $G \to G/N$ is simply given by $s^m t^n \mapsto t^n$:

	$\{1_G\}$	$\{t^2\}$	$\{s, s^2\}$	$\{t, st, s^2t\}$	$\{t^3, st^3, s^2t^3\}$	$\{st^2, s^2t^2\}$
$\mathbb{1}$	1	1	1	1	1	1
ϕ	1	-1	1	i	$-i$	-1
ϕ^2	1	1	1	-1	-1	-1
ϕ^3	1	-1	1	$-i$	i	-1
ψ						
ψ'						

where ψ' is the other irreducible character of degree 2.

5. (a) We know from class that $\bar{\phi}$ is also a character of G (alternatively, we can observe that on the character table above); therefore, so is $\phi\psi$, which is the character of $\operatorname{Hom}(\overline{\phi}, \psi)$ by the formula seen in class.

(b) $\deg \phi\psi = \dim \operatorname{Hom}(\overline{\phi}, \psi) = \dim \overline{\phi} \times \dim \psi = 1 \times 2.$

(c) We compute

$$(\phi\psi, \phi\psi) = \frac{1}{\#G} \sum_{g \in G} \phi\psi(g)\overline{\phi\psi(g)} = \frac{1}{\#G} \sum_{g \in G} |\phi(g)|^2 \psi(g)\overline{\psi(g)} = (\psi, \psi) = 1,$$

where we have used the facts that $|\phi(g)|^2 = 1$ for all g and that $(\psi, \psi) = 1$ as ψ is irreducible.

6. In this question, we suppose by contradiction that $\phi\psi = \psi$, and we let $x = \psi(s)$.

(a) The relation $\phi(g)\psi(g) = \psi(g)$ implies that $\psi(g) = 0$ whenever $\phi(g) \neq 1$, which is the case for $g = t, t^2, t^3, st^2$.

(b) By the previous question and the fact that $\phi(1_G) = \deg \psi$, our character table would look like

	$\{1_G\}$	$\{t^2\}$	$\{s, s^2\}$	$\{t, st, s^2t\}$	$\{t^3, st^3, s^2t^3\}$	$\{st^2, s^2t^2\}$
$\mathbb{1}$	1	1	1	1	1	1
ϕ	1	-1	1	i	$-i$	-1
ϕ^2	1	1	1	-1	-1	-1
ϕ^3	1	-1	1	$-i$	i	-1
ψ	2	0	$\psi(s)$	0	0	0
ψ'						.

As irreducible characters are orthogonal, we must have $(\psi, \mathbb{1}) = 0$, whence $2 + 2\psi(s) = 0$. This shows that $\psi(s) = -1$.

(c) It follows that $(\psi, \psi) = \frac{1}{2}$; however we should have $(\psi, \psi) = 1$ since ψ is an irreducible character.

7. (a) By the same logic as in 7., we see that $\phi^2\psi$ is an irreducible character of degree 2, so either $\phi^2\psi = \psi$ or $\phi\psi$. The latter would imply $\phi\psi = \psi$, but we have just proved that this is not possible.

(b) We deduce that ψ vanishes at t and t^3 by the same logic as in 7a.

Summary of the situation:

	$\{1_G\}$	$\{t^2\}$	$\{s, s^2\}$	$\{t, st, s^2t\}$	$\{t^3, st^3, s^2t^3\}$	$\{st^2, s^2t^2\}$
$\mathbb{1}$	1	1	1	1	1	1
ϕ	1	-1	1	i	$-i$	-1
ϕ^2	1	1	1	-1	-1	-1
ϕ^3	1	-1	1	$-i$	i	-1
$\psi = \phi^2\psi$	2	?	?	0	0	?
$\phi\psi$	2	?	?	0	0	?.

8. *The irreducible characters of $H = \{1_G, t, t^2, t^3\}$ are precisely the restrictions of the powers of ϕ. In what follows, we let $n \in \mathbb{Z}/4\mathbb{Z}$ be such that we have chose $\chi = \phi^n$. I do expect that most students will take $\chi = \mathbb{1}$, which corresponds to $n = 0$.*

   (a) $\deg \eta = [G : H] \deg \chi = 3 \times 1$.

   (b) By using the character table of $\mathbb{Z}/4\mathbb{Z}$ established in question 1, we find that $(\eta, \phi^m) = 1$ if $m = n$ and 0 if $m \neq n$. This shows that η contains exactly one copy of exactly one irreducible character of G.

   (c) By the second formula for induced characters,

$$\eta(g) = \operatorname{Ind}_H^G \chi(g) = \frac{[G : H]}{\# cc_G(g)} \sum_j \# cc_H(h_j) \chi(h_j)$$

   whenever $cc_G(g) \cap H = \coprod_j cc_H(h_j)$. We can deduce $\psi(g)$ as $\eta = \phi^n + \psi$.

   For $g = t^2$, we have $cc_G(g) \cap H = \{t^2\} \cap H = \{t^2\}$, so $\eta(t^2) = \frac{3}{1} 1 \times (-1)^n$, whence $\psi(g) = 2(-1)^n$.

   For $g = s$, we have $cc_G(g) \cap H = \{s, s^2\} \cap H = \emptyset$, so $\eta(t^2) = 0$, whence $\psi(g) = -1$.

9. We find the value of $\psi(st^2)$ by orthogonality as in 7b. We get

	$\{1_G\}$	$\{t^2\}$	$\{s, s^2\}$	$\{t, st, s^2t\}$	$\{t^3, st^3, s^2t^3\}$	$\{st^2, s^2t^2\}$
$\mathbb{1}$	1	1	1	1	1	1
ϕ	1	-1	1	i	$-i$	-1
ϕ^2	1	1	1	-1	-1	-1
ϕ^3	1	-1	1	$-i$	i	-1
$\psi = \phi^2\psi$	2	± 2	-1	0	0	∓ 2
$\phi\psi$	2	∓ 2	-1	0	0	± 2

where $\pm = (-1)^n$ *(Again, most students will have taken* $n = 0$*).*

10. As seen in class:

    The centre of G is made up of the elements that sit alone in their conjugacy class, namely $Z(G) = \{1_G, t^2\}$.

    The derived subgroup of G is formed by the elements where the irreducible characters of degree 1 assume the value 1, namely $D(G) = \{1, s, s^2\} = N$.

Question 4 *The affine group* $\mathrm{AGL}(1,5)$

Let $\mathbb{F}_5$ be the field $\mathbb{Z}/5\mathbb{Z} = \{0,1,2,3=-2,4=-1\}$, and let G be the set of polynomials $f_{a,b} = ax+b$ with $a,b \in \mathbb{F}_5$, $a \neq 0$. We equip G with the law given by composition:

$$(f_{a,b} \cdot f_{a',b'})(x) = f_{a,b}\big(f_{a',b'}(x)\big).$$

We admit that G is a group, whose identity is $f_{1,0}$ and whose conjugacy classes are:

$$\{f_{1,0}\},\ \{f_{1,b} \mid b \neq 0\},\ \{f_{2,b} \mid b \in \mathbb{F}_5\},\ \{f_{-2,b} \mid b \in \mathbb{F}_5\},\ \{f_{-1,b} \mid b \in \mathbb{F}_5\}.$$

1. Prove that the map

$$\begin{aligned} \lambda : G &\longrightarrow \mathbb{F}_5^\times \\ f_{a,b} &\longmapsto a \end{aligned}$$

is a group morphism.

2. Deduce the existence of four pairwise non-isomorphic irreducible representations of G of degree 1, and write down their characters.

   Hint: $\mathbb{F}_5^\times$ is cyclic and generated by 2.

3. Determine the number of irreducible representations of G, and their degrees.

4. Determine the character table of G.

5. We have a natural action of G on $\mathbb{F}_5$ defined by $f_{a,b} \cdot x = f_{a,b}(x)$. Determine up to isomorphism the decomposition into irreducible representations of the permutation representation $\mathbb{C}[\mathbb{F}_5]$ attached to this action of G on $\mathbb{F}_5$.

6. The element $f_{-1,0}$ of G has order 2, and thus generates a subgroup $H = \{f_{1,0}, f_{-1,0}\}$ of G isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let $\varepsilon : H \longrightarrow \mathrm{GL}_1(\mathbb{C})$ be the degree 1 representation of H such that $\varepsilon(f_{-1,0}) = -\mathrm{Id}_{\mathbb{C}}$. Determine up to isomorphism the decomposition into irreducible representations of the induced representation $\mathrm{Ind}_H^G(\varepsilon)$.

Solution 4

1. We have

$$f_{a,b}\big(f_{a',b'}(x)\big) = a(a'x+b') + b = aa'x + (ab'+b)$$

so $\lambda(f_{a,b} \circ f_{a'b'}) = aa' = \lambda(f_{a,b})\lambda(f_{a',b'})$; besides $\lambda(\mathrm{Id}) = \lambda(f_{1,0}) = 1$.

2. We can use the morphism λ to view any representation of $\mathbb{F}_5^\times$ as a representation of G. Moreover, since λ is clearly surjective, irreducible representations of $\mathbb{F}_5^\times$ yield irreducible representations of G.

   As mentioned in the hint, $\mathbb{F}_5^\times = \langle 2 \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ is cyclic, so it has four irreducible representations of degree one, whose characters are the $\psi_j : 2^x \mapsto i^{jx}$ for $j \in \mathbb{Z}/4\mathbb{Z}$. The corresponding irreducible representations of G are clearly still of degree 1, and their characters are the $\chi_j : f_{a,b} \mapsto \psi_j\big((\lambda(f_{a,b})\big) = i^{ja}$ for $j \in \mathbb{Z}/4\mathbb{Z}$. These characters are pairwise distinct, so these representations are pairwise non-isomorphic.

3. The number of irreducible representations of G is the number of conjugacy classes of G, which is five. Let $n_1 \leq \cdots \leq n_5$ be their degrees; by the previous question, $n_1 = \cdots = n_4 = 1$. We also know that $\sum_{j=1}^5 n_j^2 = \#G$, which is 20 as can be seen form the explicit decomposition of G into conjugacy classes, or directly on the definition of G. It follows that $n_5 = 4$.

4. Let ϕ be the irreducible character of G of degree 4 whose existence was established at the previous question. We know that the character of the regular representation of G, which is $\sum_{\psi \in \mathrm{Irr}(G)} (deg\psi)\psi = \chi_0 + \chi_1 + \chi_2 + \chi_3 + 4\phi$, assumes the value $\#G = 20$ at $f_{1,0}$ and 0 everywhere else; since the χ_j are known, this allows us to determine ϕ. We find the following character table:

Class	$f_{1,0}$	$f_{1,*}$	$f_{2,*}$	$f_{2^2,*}$	$f_{2^3,*}$
$\#$	1	4	5	5	5
$\mathbb{1} = \chi_0$	1	1	1	1	1
χ_1	1	1	i	-1	$-i$
χ_2	1	1	-1	1	-1
χ_3	1	1	$-i$	-1	i
ϕ	4	-1	0	0	0

5. Let P be the character of this permutation representation. We know that for each $g \in G$, $P(g)$ is the number of fixed points of g. $f_{1,0}$ is the identity and therefore fixes

all five elements of $\mathbb{F}_5$, $f_{1,*}$ is a translation and fixes none, and finally, if $a \neq 1$, then $f_{a,b}(x) = ax + b$ has a unique fixed point, namely $b/(1-a) \in \mathbb{F}_5$. In conclusion, the values of P are $5, 0, 1, 1, 1$.

We have $P = \mathbb{1} + \phi$, either by dot products or by direct inspection; therefore this permutation representation decomposes into the direct sum of one copy of the trivial representation and of one copy of the irreducible representation of character ϕ.

NB this question can also be solved by invoking the double transitivity of this action of G, which implies that $\mathbb{C}[\mathbb{F}_5]$ decomposes into the direct sum of $\mathbb{1}$ and of an irreducible representation of degree $5 - 1 = 4$, as seen in one of the homework assignments.

6. Note that we can identify the degree 1 representation ε with it s character.

   Let $\psi \in \operatorname{Irr}(G)$. The number of copies of the representation of character ψ in the decomposition of $\operatorname{Ind}_H^G(\varepsilon)$ is $(\operatorname{Ind}_H^G(\varepsilon) \mid \psi)_G$, which agrees with

$$(\varepsilon \mid \operatorname{Res}_H^G \psi)_H = \frac{1}{\#H} \sum_{h \in H} \varepsilon(h)\overline{\psi(h)} = \frac{\overline{\psi(f_{1,0})} - \overline{\psi(f_{-1,0})}}{2}$$

   by Frobenius reciprocity. As the conjugacy class of $f_{-1,0}$ is $f_{2^2,*}$, this yields

$$\operatorname{Ind}_H^G(\varepsilon) = \chi_1 + \chi_3 + 2\phi$$

   which means that $\operatorname{Ind}_H^G(\varepsilon)$ decomposes as the direct sum of one copy of the representation of character χ_1, one copy of the representation of character χ_3, and two copies of the representation of character ϕ.

Question 5 *The determinant of the character table*

Let G be a finite group of order $n \in \mathbb{N}$. We arrange the conjugacy classes $C_1, \cdots, C_r$ and the irreducible characters $\chi_1, \cdots, \chi_r$ of G arbitrarily, and we view the character table of G as an $r \times r$ complex matrix X. In other words, for all $1 \leq i, j \leq r$, the i,j-coefficient $X_{i,j}$ of X is $\chi_i(g_j)$, where g_j is any element of C_j.

1. (a) Prove that the complex conjugate of a character is a character.

(b) Prove that the complex conjugate of an **irreducible** character is an **irreducible** character.

(c) Deduce that $\overline{\det X} = \pm \det X$. What does this imply about the complex number $\det X$?

2. (a) Let D be the diagonal matrix whose j,j-coefficient is $\#C_j$ for all $1 \le j \le r$. Express the orthogonality relations in terms of the matrices X and D.

(b) Deduce the value of $|\det X|$ in terms of n and of the $\#C_j$.

(c) (Trick question) What is the exact value of $\det X$?

Solution 5

1. (a) If χ is the character of a representation ρ, then $\overline{\chi}$ is the character of the dual representation $\operatorname{Hom}(\rho, \mathbb{1})$.

(b) Let $\chi \in \operatorname{Irr}(G)$. As χ is irreducible, we have $(\chi \,|\, \chi) = 1$; therefore

$$(\overline{\chi} \,|\, \overline{\chi}) = \frac{1}{n} \sum_{g \in G} \bar{\chi}(g)\overline{\overline{\chi}(g)} = \frac{1}{n} \sum_{g \in G} \chi(g)\overline{\chi(g)} = (\chi \,|\, \chi) = 1,$$

which shows that $\overline{\chi}$ is also irreducible.

(c) The previous question shows that the complex-conjugate $\overline{X}$ of X can be obtained from X by permuting the rows of X in some way. The properties of the determinant thus ensure that

$$\overline{\det X} = \det \overline{X} = \pm \det X,$$

which means that $\det X$ is either real (if the sign is $+$) or purely imaginary (if the sign is $-$).

2. (a) The orthogonality relations state that for all $\chi, \psi \in \operatorname{Irr}(G)$,

$$(\chi \,|\, \psi) = \frac{1}{n} \sum_{g \in G} \chi(g)\overline{\psi(g)}$$

is 1 if $\psi = \chi$ and 0 else. For each $j \le r$, pick $g_j \in C_j$. As character are class functions, we have that

$$(\chi \,|\, \psi) = \frac{1}{n} \sum_{g \in G} \chi(g)\overline{\psi(g)} = \frac{1}{n} \sum_{j=1}^{r} \#C_j \chi(g_j)\overline{\psi(g_j)} = \frac{1}{n} \sum_{j=1}^{r} X_{\chi,j} D_{j,j} \overline{{}^t X_{j,\psi}}$$

is the χ, ψ coefficient of the matrix

$$\frac{1}{n} X D \overline{^t X}.$$

Therefore, the orthogonality relations are equivalent to the statement that $\frac{1}{n} X D \overline{^t X}$ is the identity matrix.

(b) Taking the determinant of

$$n I_r = X D \overline{^t X},$$

where I_r is the identity matrix of size $r \times r$, we get that

$$n^r = \det(X D \overline{^t X}) = \det(X) \det(D) \overline{\det(X)}.$$

But as D is diagonal, we have $\det(D) = \prod_{j=1}^{r} \# C_j$, whence

$$|\det X|^2 = \det(X) \overline{\det(X)} = n^r / \prod_{j=1}^{r} \# C_j,$$

so finally

$$|\det X| = \sqrt{n^r / \prod_{j=1}^{r} \# C_j} = \sqrt{\prod_{j=1}^{r} \frac{n}{\# C_j}}.$$

NB $n / \# C_j$ is the order of the centraliser of g_j in G; however mentioning this fact is not required to get full marks to this question.