Group representations
Exercise sheet 2

`https://www.maths.tcd.ie/~mascotn/teaching/2025/MAU34104/index.html`

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THIS ASSIGNMENT IS NOT MANDATORY.

You are still welcome to attempt it; your mark will be counted only if this results in an advantage to your overall mark for this class.

If you decide to attempt this assignment, you only need hand in Exercise 1; however this Exercise relies on some of the results of Exercise 2, so you should at least take a quick look at Exercise 2 as well (but you are allowed to admit the results of Exercise 2).

The other exercises are independent from each other (and again they are not required to get full marks; they are just here for your culture).

Submit your answers in class or to mismet@tcd.ie by Monday February 24, 9AM.

Exercise 1 *Submodules and short exact sequences (100 pts)*

In this exercise, all modules are over a fixed ring R. We write 0 for the 0 module $\{0\}$. We also denote by 0 the 0 morphism between any two submodules (i.e. the map taking all the elements of the source module to the 0 element of the target module).

Recall that we saw in class that the decomposability of a module M can be characterised[1] by the presence of non-trivial idempotents (meaning $T^2 = T$) elements of $\operatorname{End}(M)$. The goal of this exercise is to find a (vaguely) similar characterisation for reducibility of modules, and to use it to shed new light on the connection between reducibility and decomposability.

Define an *exact sequence* as a diagram

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

where the M_i are modules, the f_i are module morphisms, and such that $\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$ for all i.

1. Let $\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$ be an exact sequence. Prove that $f_{i+1} \circ f_i = 0$ for all i.
2. Let N, M, Q be modules, and let $f : M \longrightarrow N$ and $g : M \longrightarrow Q$ be module morphisms. Prove that $0 \xrightarrow{0} N \xrightarrow{f} M$ is an exact sequence if and only if f is injective. Also prove that $M \xrightarrow{g} Q \xrightarrow{0} 0$ is an exact sequence if and only if g is surjective.

[1] If you enjoyed this part of the class, you may also like exercise 3 of this other sheet, which shows that idempotents can also be used to decompose rings.

A *short exact sequence* is an exact sequence of the form

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0.$$

By the previous question, this implies that g is surjective and that f is injective.

In particular, it follows that any short exact sequence provides us with the submodule $\operatorname{Im} f \simeq N$ of the middle module M.

3. Prove that conversely, whenever M is a module and N is a submodule, there exists a short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

for some module Q, where f is the identity inclusion of N into M.

Hint: Use the exercise on quotient modules.

4. Provide a counterexample to show that however, the existence of a short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

does NOT imply that $M \simeq N \oplus Q$ as modules.

Hint: Take $R = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$.

From now on, we only consider short exact sequences. We say that a short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

is *left-split* if there exists a module morphism $f' : M \longrightarrow N$ such that $f' \circ f = \mathrm{Id}_M$. We also say that it is *right-split* if there exists a module morphism $g' : Q \longrightarrow M$ such that $g \circ g' = \mathrm{Id}_M$.

5. Let

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

be a short exact sequence, so that $\operatorname{Im} f$ is a submodule of M. Prove that if $\operatorname{Im} f$ admits a supplement submodule $M' \subset M$, then the short exact sequence above is both left-split and right-split.

Hint: Consider the restriction of g to M'.

6. Conversely, prove that if the short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

is left-split, then $\operatorname{Im} f$ admits a supplement in M.

Hint: Consider the map $T = f \circ f' \in \operatorname{End}(M)$.

7. Similarly, prove that if the short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

is right-split, then $\operatorname{Im} f$ admits a supplement in M.

In conclusion, submodules correspond to short exact sequences, and the submodule admits a complement iff. the sequence is left-split iff. it is right-split.

Solution 1

1. Let $m \in M_{i-1}$. Then $f_i(m) \in Imf_i = \operatorname{Ker} f_{i+1}$ since the sequence is exact, so $f_{i+1}(f_i(m)) = 0$.

2. The sequence $0 \xrightarrow{0} N \xrightarrow{f} M$ is exact iff. $\operatorname{Im} 0 = \operatorname{Ker} f$; but of course $\operatorname{Im} 0 = 0$.

   Similarly, $M \xrightarrow{g} Q \xrightarrow{0} 0$ is exact iff. $\operatorname{Im} g = \operatorname{Ker} 0$, but of course $\operatorname{Ker} 0$ is the whole of Q.

3. Take Q to be the quotient module M/N, and g the projection from M to M/N. Then $\operatorname{Ker} f = \operatorname{Im} 0 = 0$ as f, the identity inclusion, is injective; $\operatorname{Im} g = Q = \operatorname{Ker} 0$ as g, the projection to the quotient M/N, is surjective by construction; and finally $\operatorname{Im} f = N = \operatorname{Ker} g$ by definition of the projection to the quotient (an element $m \in M$ represents $0 \in M/N$ iff. $m \in N$).

4. Take $R = \mathbb{Z}$, $M = \mathbb{Z}/4\mathbb{Z}$, and N the submodule $2\mathbb{Z}/4\mathbb{Z}$. Then $M/N \simeq \mathbb{Z}/2\mathbb{Z}$ by the isomorphism theorem for modules; however, $N \simeq \mathbb{Z}/2\mathbb{Z}$, but $\mathbb{Z}/4\mathbb{Z}$ is NOT isomorphic to $N \oplus M/N = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}$-module (indeed, they are not even isomorphic as additive groups, since the former has elements of order 4 but not the latter).

   This should serve as a reminder that a sub-object is not, in general, the same thing as a quotient!

5. The assumption is that $M = \operatorname{Im} f \oplus M'$ for some submodule $M' \subset M$.

   Consider the restriction $g_{|M'}$ of g to M'. This is still a morphism (immediate). Its kernel is $M' \cap \operatorname{Ker} g = M' \cap \operatorname{Im} f = 0$ since the sequence is exact and the sum is direct. Furthermore, as the sequence is exact, g is surjective, so for all $q \in Q$ we can find $m \in M$ such that $q = g(m)$. As $M = \operatorname{Im} f \oplus M'$, we can decompose (uniquelely) $m = i + m'$ where $m' \in M'$ and $i \in \operatorname{Im} f = \operatorname{Ker} g$; thus $g(m') = g(m - i) = g(m) - g(i) = q - 0 = q$, which shows that $g_{|M'}$ is still surjective. Therefore $g_{|M'} : M' \longrightarrow Q$ is an isomorphism. Let $g' : Q \longrightarrow M' \subset M$ be its inverse; then for all $q \in Q$, $g'(q) = m' \in M'$ such that $g(m') = q$ by definition of g', so that $g(g'(q)) = q$. This shows that the sequence is right-split.

   Let us also show that it is left split. As $M = \operatorname{Im} f \oplus M'$, we can consider the projector T from M to $\operatorname{Im} f$, i.e. $T(m) = i$ where $m = i + m'$ is the unique decomposition of $m \in M$ with $i \in Imf$ and $m' \in M'$. Furthermore, f is injective by exactness of the sequence, so f corestricts into an isomorphism between N and $\operatorname{Im} f$. Post-composing T with the inverse of this isomorphism yields $f' : M \longrightarrow N$, which is thus defined by $f(m) = n$ where $m = i + m'$ is the unique decomposition of m as above and $n \in N$ is the unique element such that $f(n) = i$. This map f', being a composition of morphisms, is a morphism; and it satisfies that for all $n \in N$, $f'(f(n)) = f'(m)$ where $m = f(n) \in M$ decomposes as $m = i + m'$ where $i = f(n)$ and $m' = 0$, so $f'(f(n)) = n$ as required.

6. We are given a morphism $f' : M \longrightarrow N$ such that $f' \circ f = \mathrm{Id}_N$. Consider $T = f \circ f' : M \longrightarrow M$. This is a morphism (composition of morphisms). Furthermore $T \circ T = f \circ f' \circ f \circ f' = T$ since $f' \circ f = \mathrm{Id}_N$, so T is actually a projector, whence the decomposition $M = \operatorname{Im} T \oplus \operatorname{Ker} T$ of M.

To conclude, we are going to show that $\operatorname{Im} T = \operatorname{Im} f$ (so that $M' = \operatorname{Ker} T$ will serve as a supplement of $\operatorname{Im} f$). Indeed, first of all $\operatorname{Im} T = f \circ f' \subseteq \operatorname{Im} f$. Conversely, let $i \in \operatorname{Im} f$, so that $i = f(n)$ for some $n \in N$; then $i = f(n) = f(f'(f(n))) = T(f(n)) \in \operatorname{Im} T$ as $f' \circ f = \mathrm{Id}_N$, which shows the reverse inclusion $\operatorname{Im} f \subseteq \operatorname{Im} T$.

7. Same idea as the previous question: introduce $U = g' \circ g$, which is a morphism from M to itself. From $g \circ g' = \mathrm{Id}_Q$ we infer that $U^2 = U$, whence $M = \operatorname{Im} U \oplus \operatorname{Ker} U$. Next, if $m \in \operatorname{Im} f$, then $m \in \operatorname{Ker} g$ by exactness, so $U(m) = g'(g(m)) = g'(0) = 0$, which shows that $\operatorname{Im} f \subseteq \operatorname{Ker} U$; and conversely, if $m \in \operatorname{Im} f$, say $m = f(n)$ for some $n \in N$, then $U(m) = g'(g(f(n))) = g'(0) = 0$ as $g \circ f = 0$ by exactness (cf. first question of the exercise), which shows that $\operatorname{Im} f \subseteq \operatorname{Ker} U$. In conclusion, $\operatorname{Im} f = \operatorname{Ker} U$, so $M' = \operatorname{Im} U$ is a supplement to $\operatorname{Im} f$.

Exercise 2 *Quotient modules*

Let R be a ring, and let M be an R-module with addition $+ : \begin{array}{ccc} M \times M & \longrightarrow & M \\ (m_1, m_2) & \longmapsto & m_1 + m_2 \end{array}$ and scalar multiplication $\cdot : \begin{array}{ccc} R \times M & \longrightarrow & M \\ (\lambda, m) & \longmapsto & \lambda \cdot m \end{array}$.

Let $N \subset M$ be a submodule. Then in particular, N is a subgroup of the additive group $(M, +)$ which is Abelian, so we have the quotient group $M/N = \{m + N \mid m \in M\}$ on which $=$ is still well-defined. It comes with a projection map $\begin{array}{ccc} M & \longrightarrow & M/N \\ m & \longmapsto & m + N \end{array}$ which is surjective and whose kernel is N.

1. Prove that M/N is actually still an R-module, by proving that the scalar multiplication $\cdot$ descends to a well-defined map $\begin{array}{ccc} R \times M/N & \longrightarrow & M/N \\ (\lambda, m + N) & \longmapsto & (\lambda \cdot m) + N \end{array}$.

2. Prove the isomorphism theorem for modules: Any module morphism
$$f : M \longrightarrow M'$$
induces a module isomorphism $M/\operatorname{Ker} f \simeq \operatorname{Im} f$.

Solution 2

1. We must prove that if $\lambda \in R$ and if $m, m' \in M$ are such that $m + N = m' + N$, then $(\lambda m) + N = (\lambda m') + N$.

   First of all, observe that $m + N = m' + N$ iff. m and m' represent the same element of M/N iff. $m' = m + n$ for some $n \in N$.

   But then $\lambda m' = \lambda(m + n) = \lambda m + \lambda n$ is of the form $\lambda m + n'$ where $n' = \lambda n \in N$ since N is a submodule, so λm and $\lambda m'$ do represent the same element of M/N.

   To be complete, we should still check that the module axioms do hold for M/N, but they follow immediately from the fact that M itself satisfies them (they are equalities in M, whcih can be viewed as equalities in M/N).

2. Let $f : M \longrightarrow M'$ be a morphism. Then $\operatorname{Ker} f$ is a submodule of M, so the quotient module $M/\operatorname{Ker} f$ makes sense by the previous question. Define

$$f' : \begin{array}{rcl} M/\operatorname{Ker} f & \longrightarrow & \operatorname{Im} f \\ m + \operatorname{Ker} f & \longmapsto & f(m). \end{array}$$

This is well-defined, because $f(m) \in \operatorname{Im} M$ for all $m \in M$ by definition of the image, and because if $m + \operatorname{Ker} f = m' + \operatorname{Ker} f$, then $m' = m + k$ for some $k \in \operatorname{Ker} f$, so $f(m') = f(m + k) = f(m) + f(k) = f(m) + 0 = f(m)$. It is also clear that f' is a morphism (since f is), and that f' is surjective (again by definition of $\operatorname{Im} f$).

To conclude, we will prove that f' is injective, which will show that f' is an isomorphism whose existence we had to establish. Indeed, let $m + N \in M/N$; if $m + N \in \operatorname{Ker} f'$, then $f(m) = 0$ by definition of f', so $m \in \operatorname{Ker} f$. But then $m = 0 + m \in 0 + \operatorname{Ker} f$, so $m + \operatorname{Ker} f$ is actually 0 in M/N. This shows that $\operatorname{Ker} f' = \{0\}$, so f' is injective.

Exercise 3 *Preservation of semi-simplicity*

In this exercise, all modules are over a fixed ring R, and all modules are Artinian[2], meaning that there cannot exist an infinite descending chain of submodules.

1. Prove that a submodule of a semi-simple module is also semi-simple.

2. Prove that if $f : M \longrightarrow N$ is a module morphism, and if M is semi-simple, then $\operatorname{Im} f$ is also semi-simple.

3. Let now G be a group, K a field, and $f : V \longrightarrow W$ be a morphism between representations of G over K of finite degree. Prove that if V is semi-simple, then so are $\operatorname{Ker} f$ and $\operatorname{Im} f$.

Solution 3

We will constantly use the result that a module is semi-simple iff. every submodule admits a supplement.

1. Let M be a semi-simple module, and let $N \subseteq M$ be a submodule. If $P \subset N$ is a submodule of N, then it is also a submodule of M; as M is semi-simple, there exists a submodule $S \subseteq M$ such that $M = P \oplus S$, so that every $m \in M$ can be uniqueley decomposed as $m = p + s$ with $p \in P$ and $s \in S$. Let $S' = S \cap N$; this is a submodule of N. If $n \in N$, then also $n \in M$, so we can write $n = p + s$ with unique $p \in P$ and $s \in S$. Then $s = n - p \in N$ as $P \subseteq N$, so actually $s \in S \cap N = S'$. This shows that $N = P \oplus S'$, whence the result.

2. Since M is semi-simple, the submodule $\operatorname{Ker} f \subseteq M$ admits a supplement M'.

   Restricting f to M' and corestricting to $\operatorname{Im} f$ yields $f' = f_{|M'} : M' \longrightarrow \operatorname{Im} f$, which is injective since if $m \in \operatorname{Ker} f'$, then $m \in \operatorname{Ker} f \cap M'$ whence $m = 0$, and surjective, as if $n \in \operatorname{Im} f$, then $n = f(m)$ for some $m \in M$, which can be

[2] NB we make this assumption to make our lives easier, but it can be shown that the properties established in this exercise actually remain valid without this assumption.

(uniquely) decomposed as $m = k + m'$ with $k \in \operatorname{Ker} f$ and $m' \in M'$, but then $n = f(m) = f(k + m') = f(k) + f(m') = f(m')$ as $k \in \operatorname{Ker} f$.

Therefore f' is an isomorphism between M' and N. As M' is semi-simple by the previous question, so is N.

3. We can view V and W as R-modules where $R = K[G]$, which are Artinian since a descending chain of submodules would in particular be a descending chain of K-subspaces, and thus cannot be infinite (consider dimensions); and we can view f as a module morphism. Then $\operatorname{Ker} f$ is a submodule of V, and is therefore semi-simple by the first question, whereas $\operatorname{Im} f$ is a submodule of W, which is semi-simple by the second question.

Exercise 4 *A non-semi-simple ring*

Let K be a field, and let G be a finite group of order $n = \#G$. We will sometimes see the group ring $K[G] = \{\sum_{g \in G} \lambda_g e_g \mid \lambda_g \in K \text{ for all } g \in G\}$ as a module over itself.

1. Let $\Sigma = \sum_{g \in G} e_g \in K[G]$. Prove that $e_h \Sigma = \Sigma$ for all $h \in G$.
2. Prove that $S = \{\lambda\Sigma, \ \lambda \in K\}$ is a sub-$K[G]$-module of $K[G]$.
3. Identify S as a representation of G.

From now on, we assume that $n = 0$ in K.

4. Prove that $\Sigma^2 = 0$ in $K[G]$.
5. Deduce that $1 - \lambda\Sigma$ is invertible in $K[G]$ for all $\lambda \in K$, where $1 = e_{1_G}$ is the multiplicative identity of $K[G]$.

   Note that since $K[G]$ is not commutative in general, you must prove that your inverse works on both sides.

   Hint: For $x \in \mathbb{R}$ and $m \in \mathbb{N}$, what is the formula for the geometric series $1 + x + x^2 + \cdots + x^m$? How do you prove it?
6. Deduce that $K[G]$, viewed as a $K[G]$-module, is not semi-simple.

Solution 4

1. For all $h \in G$, we have

$$e_h \Sigma = e_h \sum_{g \in G} e_g = \sum_{g \in G} e_{hg} = \sum_{g \in G} e_g = \Sigma$$

   as $\begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & hg \end{array}$ is a bijection.

2. For all $\sum_{g \in G} \lambda_g e_g \in K[G]$ we have

$$\left(\sum_{g \in G} \lambda_g e_g\right) \Sigma = \sum_{g \in G} \lambda_g (e_g \Sigma) = \sum_{g \in G} \lambda_g \Sigma = \left(\sum_{g \in G} \lambda_g\right) \Sigma,$$

which proves that $M = \{\lambda\Sigma,\ \lambda \in K\} \subset K[G]$ is closed by multiplication by all "scalars" in $K[G]$. Since it is also an additive subgroup of $K[G]$ (and even a K-subspace), it is a submodule of $K[G]$.

3. We can therefore view M as a representation of G over K. Its degree is $\dim_K M = 1$, and since for all $h \in G$, $e_h\Sigma = \Sigma$, we see that G acts trivially on it. Thus this is the trivial representation $\mathbb{1}$.

4. We compute that
$$\Sigma^2 = \left(\sum_{g\in G} e_g\right)\Sigma = \sum_{g\in G} e_g\Sigma = \sum_{g\in G}\Sigma = n\Sigma$$
as we have proved that $e_g\Sigma = \Sigma$ for all $g \in G$ in the previous question. As $n = 0$ in K and therefore in $K[G]$, the result follows.

5. Expanding shows that $(1-x)(1+x+x^2+\cdots+x^n) = 1-x+x-x^2+\cdots-x^n+x^n-x^{n+1} = 1-x^{n+1}$ for all $x \in \mathbb{R}$, and in fact this identity remains valid in any ring, even if it is not commutative, since powers of x and 1 always commute with each other. In particular, for $n = 1$ we have that $(1-x)(1+x) = 1-x^2$, so $(1-\lambda\Sigma)(1+\lambda\Sigma) = 1-\lambda^2\Sigma^2 = 1$ as $\Sigma^2 = 0$, and similarly $(1+\lambda\Sigma)(1-\lambda\Sigma) = 1-\lambda^2\Sigma^2 = 1$. This shows that $1-\Sigma$ is invertible in $K[G]$, with inverse $1+\Sigma \in K[G]$.

6. Suppose by contradiction that $K[G]$ is semi-simple. An infinite descending chain of submodules of $K[G]$ would in particular be a descending chain of K-subspaces, which cannot exist (consider dimensions). Therefore, the submodule M of $K[G]$ would admit a supplement S, that is to say $K[G] = M \oplus S$. Then we could decompose (uniquely) $1 = m + s$ with $m \in M$ and $s \in S$, so we would have $m = \lambda\Sigma$ for some $\lambda \in K$ and $s = 1 - m = 1 - \lambda\Sigma$. As shown in the previous question, there exists $t \in K[G]$ such that $ts = 1$; but then $\Sigma = \Sigma 1 = (\Sigma t)s$ would lie in S since $\Sigma t \in K[G]$, $s \in S$, and S is a submodule; therefore $0 \neq \Sigma \in M \cap S$, which contradicts $K[G] = M \oplus S$ as we would have the two decompositions $\Sigma = \Sigma + 0 = 0 + \Sigma$.

Exercise 5 *Annihilators and simple modules*

Let R be a ring, which need not be commutative. We say that $I \subseteq R$ is a *left ideal* if it is an additive subgroup of $(R,+)$ and if $ri \in I$ for all $r \in R$ and $i \in I$. We define *right ideals* similarly. If I is both a left ideal and a right ideal, then we say that it is a *two-sided* ideal. A *maximal* left ideal is a left ideal $M \neq R$ such that there are no left ideals I such that $M \subsetneq I \subsetneq R$.

1. Let M be an R-module. Define its *annihilator* as
$$\operatorname{Ann} M = \{r \in R \mid rm = 0 \text{ for all } m \in M\} \subseteq R.$$
Prove that $\operatorname{Ann} M$ is a *two-sided* ideal of R.

2. The ring R can be viewed as a module over itself; we denote this module by ${}_RR$, so as to clearly distinguish between the ring R and the R-module ${}_RR$. Identify the submodules of ${}_RR$.

3. Let S be a module, and let $s \in S$.

   (a) Prove that the map
   $$\begin{array}{rccc} f_s : & {}_RR & \longrightarrow & S \\ & r & \longmapsto & rs \end{array}$$
   is a module morphism.

   (b) Prove that S is simple iff. f_s is surjective for all $s \neq 0$.

   (c) Deduce that if S is simple, then it is isomorphic to ${}_RR/M$, where M is a maximal left ideal of R.

   (d) Prove that conversely, if M is a maximal left ideal of R, then ${}_RR/M$ is a simple R-module.

4. Beware that in general, the annihilator of ${}_RR/M$, which is a two-sided ideal, does not agree with the left ideal M! Here is an example: Take $R = M_2(\mathbb{R})$, and $S = \mathbb{R}^2$, which is an R-module if we view its elements as column vectors. Prove that S is simple, determine $\operatorname{Ann} S$, and find a maximal left ideal M of R such that $S \simeq {}_RR/M$.

Solution 5

1. Let $m \in M$. Then $0m = 0$, so $0 \in \operatorname{Ann} M$. Besides, if $r, s \in \operatorname{Ann} M$, then $(r - s)m = rm - sm = 0 - 0 = 0$, so $r - s \in \operatorname{Ann} M$ as this holds for any $m \in M$. Finally, let $r \in \operatorname{Ann} M$ and $x \in R$. Then $(xr)m = x(rm) = x0 = 0$, and $(rx)m = r(xm) = 0$ as $xm \in M$ as M is a module; since these hold for any $m \in M$, $\operatorname{Ann} M$ is a two-sided ideal of R.

2. Let $M \subseteq {}_RR$ be a submodule. Then $M \subseteq (R, +)$ is an additive subgroup, and besides $xm \in M$ for all $x \in R$ and $m \in M$, so M is a left ideal. Conversely, we see that any left ideal of R is actually a submodule of ${}_RR$. So the submodules of ${}_RR$ are precisely the left ideals of R.

3. (a) f_s is additive since for all $r, r' \in {}_RR = R$, $f(r+r') = (r+r')s = rs+r's = f(r) + f(r')$, and linear because for all $\lambda \in R = {}_RR$ and $r \in {}_RR = R$, $f(\lambda r) = (\lambda r)s = \lambda(rs) = \lambda f(rs)$.

   (b) Note that $s = f_s(1_R) \in \operatorname{Im} f_s$. So if $s \neq 0$, then $\operatorname{Im} f$ is a nonzero submodule of S. So if S is simple, then this submodule must be all of S, so f_s is surjective.

   Conversely, suppose that S is not simple. Then it has a non-trivial submodule $\{0\} \subsetneq T \subsetneq S$. As $T \neq \{0\}$, we can find $0 \neq t \in T$; then $t \in S$, and $\operatorname{Im} f_t = \{rt, r \in R\} \subseteq T$ is strictly smaller that S as it is contained in T, so f_t is not surjective.

   (c) As S is simple, $S \neq \{0\}$, so let $0 \neq s \in S$. The previous question ensures that f_s is surjective, so the isomorphism theorem for modules yields $S \simeq_R R/M$ where $M = \operatorname{Ker} f_s$. In particular, M is a submodule of ${}_RR$, and therefore a left ideal by the first question.

If M were not maximal, we could find another ideal (=submodule) M' subst that $M \subsetneq M' \subsetneq {}_R R$, and then M'/M would be a non-trivial submodule of R/M, which is absurd as $R/M \simeq S$ is simple, so M must be maximal.

(d) The submodules of the quotient ${}_R R/M$ are precisely the M'/M, where M' is a submodule (= left ideal) of ${}_R R$ which contains M. As M is maximal, there are exactly two such submodules, namlely M and ${}_R R$, so correspondingly ${}_R R/M$ has only two submodules, which are $M/M = \{0\}$ and ${}_R R/M$ =itself. This shows that ${}_R R/M$ is a simple module.

4. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$, and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$; recall that for all $A \in M_2(\mathbb{R})$, Ae_1 is the left column of A, and Ae_2 is the right column of A.

Let now $0 \neq v \in \mathbb{R}^2$, and let $w \in \mathbb{R}^2$. There exists a matrix $B \in GL_2(\mathbb{R})$ whose first column is v, so that B^{-1} takes v to e_1, and a matrix $C \in M_2(\mathbb{R})$ whose first column is w, so that it takes e_1 to w; then $A = CB^{-1} \in M_2(\mathbb{R})$ satisfies $Av = w$. This shows that the map $f_v : A \mapsto Av$ is surjective for all $v \neq 0$, so by question 2, $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Still by question 2, for any $0 \neq v \in \mathbb{R}^2$, the $M_2(\mathbb{R})$-module $\mathbb{R}^2$ is isomorphic to $M_2(\mathbb{R})/N$, where N is the left ideal $\{A \in M_2(\mathbb{R}) \mid Av = 0\}$. For instance, if we take $v = e_1$, then

$$N = \{A \in M_2(\mathbb{R}) \mid Ae_1 = 0\} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}.$$

However, $\operatorname{Ann} \mathbb{R}^2$ is the set of matrices $A \in M_2(\mathbb{R})$ such that $Av = 0$ for all $v \in \mathbb{R}^2$; taking v to be e_1, we get that the left column Ae_1 of A must be 0, and then taking $v = e_2$, we see that the right column of A must also be 0. Thus

$$\operatorname{Ann} \mathbb{R}^2 = \{0\} \neq N.$$