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JS/SS Maths/TP/TJH **Semester 2, 2024–2025**

MAU34104 Group representations

PRACTICE PAPER

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Instructions to candidates:

All the representations considered in this exam are over the field $\mathbb{C}$ of complex numbers.
This is a mock exam for revisions.
When solving a question in this exercise, you are allowed to admit the result of the previous questions (except those of question 7., because they rely on an absurd hypothesis), but *not* the results of the *next* questions.
Even if you were not able to solve previous questions, you are encouraged to explain how you would use their results if you had solved them.
For example, when you solve question 5., you are allowed to use the results of questions 3. and 4. even if you were unable to solve these questions (in which case you should try to explain how you could solve question 5. if you had solved questions 3. and 4.), but you are not allowed to use the results of question 6.

The use of non-programmable calculators is allowed.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1 *Bookwork*

1. Define a group representation, and the character of this representation.

2. Define the inner product of two characters of a group G.

3. State the second orthogonality relations.

4. State the second formula for index characters.

5. State the Frobenius reciprocity theorem.

Question 2 *The mysterious group*

As he was exploring a Frobenian temple lost deep inside of the German jungle, Indiana Jones has stumbled upon mysterious markings on a wall:

	C_1	C_2	C_3	C_4	C_5	C_6
			$\vdots$			
α	3	-1	0	1	z	$\bar{z}$
β	3	-1	0	1	$\bar{z}$	z
γ	6	2	0	0	-1	-1
δ	7	-1	1	-1	0	0
ϵ	8	0	-1	0	1	1

where z and $\bar{z}$ are the complex-conjugate roots of $x^2 + x + 2$.

Having taking MAU34104 in his youth, Indiana quickly realises that this is the character table of a group G; unfortunately, the top of the table is damaged, so that the information about the conjugacy classes of G is unreadable and some of the characters of G may be missing. Fortunately, there is no more damage (no column has been erased), so Indiana can clearly see that there are 6 conjugacy classes, which he has taken upon himself to denote by $C_1, \ldots, C_6$. He has also named $\alpha, \ldots, \epsilon$ the characters that are still readable.

1. How many irreducible characters are missing? Write them down.

2. Prove that the conjugacy class of the identity 1_G of G is C_1.

3. Prove that G has 168 elements.

4. Prove that G is a simple group.

5. Determine the size of the conjugacy classes.

6. An inscription next to the table claims that the elements of C_2 have order 2 (we admit this without prooof so as not to anger the Frobenian gods). Let $h \in C_2$ be such an element, so that $H = \{1_G, h\}$ is a subgroup of G isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and let $\psi : H \longrightarrow \mathbb{C}$ be the character of H defined by

$$\phi(1_G) = 1, \quad \psi(h) = -1.$$

   (a) Write dow the induced character $\operatorname{Ind}_H^G \psi$ of G.

   (b) Determine the decompositions of $\operatorname{Ind}_H^G \psi$ into irreducible characters of G.

Question 3 *The character table of a group of order* 12

Let G be the group generated by two elements $s, t \in G$ with s of order 3 and t of order 4 (so $s^3 = t^4 = 1_G$, where $1_G \in G$ is the identity element) such that

$$tst^{-1} = s^{-1}. \tag{$\star$}$$

We *admit* the following facts *without proof* (they are easily proved, but this is not the point of this exercise):

- The relation $(\star)$ implies $st = ts^{-1}$ and $ts = s^{-1}t$,
- Each element of G can be expressed as $s^m t^n$ for some *unique* integers $0 \leqslant m < 3$ and $0 \leqslant n < 4$,
- In particular, $\#G = 12$,
- $H = \{1_G, t, t^2, t^3\}$ is a subgroup of G isomorphic to $\mathbb{Z}/4\mathbb{Z}$,
- $N = \{1_G, s, s^2\}$ is a *normal* subgroup of G, with quotient G/N isomorphic to $\mathbb{Z}/4\mathbb{Z}$ via $s^m t^n \mapsto n \bmod 4$.

1. Write down the character table of $\mathbb{Z}/4\mathbb{Z}$.

2. Deduce that G admits at least four irreducible representations of degree 1, and describe them. Do not forget to explain why they are irreducible!

3. Prove that not every irreducible representation of G is of degree 1.

4. We admit that in G,

   - s and s^2 are conjugate,
   - t, st, and s^2t are conjugate,
   - t^3, st^3, and s^2t^3 are conjugate,
   - and that st^2 and s^2t^2 are conjugate.

   Prove that up to isomorphism, G admits six irreducible representations, of respective degrees $1, 1, 1, 1, 2, 2$, and that the conjugacy classes of G are as follows:

   $$\{1_G\}, \{t^2\}, \{s, s^2\}, \{t, st, s^2t\}, \{t^3, st^3, s^2t^3\}, \{st^2, s^2t^2\}.$$

   From now on, let ϕ be the character of one of the irreducible representations of G of degree 1 constructed in question 2. such that the other irreducible characters of G of degree 1 are the powers of ϕ, and let ψ be either of the irreducible characters of degree 2 of G. You may find it useful to write down a partially completed character table of G, and to update it after each question.

5. (a) Prove that $\phi\psi$ is also a character of G.

   (b) Prove that $\phi\psi$ is also of degree 2.

   (c) Prove that $\phi\psi$ is also irreducible.

6. In this question, we suppose **by contradiction** that $\phi\psi = \psi$.

   (a) Prove that ψ vanishes at t, t^2, t^3, and st^2.

   (b) Use an orthogonality relation to deduce the value of $\psi(s)$.

   (c) Prove that this value of $\psi(s)$ is incompatible with the fact that ψ is an irreducible character.

It follows that $\phi\psi \neq \psi$, so the irreducible characters of degree 2 of G are ψ and $\phi\psi$.

7. (a) Prove that $\phi^2\psi = \psi$.

   (b) Deduce the values of ψ at t and at t^3.

8. Let χ be an irreducible character of H (pick any of them, it does not matter, but clearly state which one you have picked), and let $\eta = \operatorname{Ind}_H^G \chi$.

   (a) Prove that $\deg \eta = 3$.

   (b) Prove that η decomposes as the sum of an irreducible character of degree 1 and of an irreducible character of degree 2 of G. Identify this character of degree 1.

   From now on, we assume without loss of generality that the irreducible character of degree 2 which occurs in the decomposition of η is ψ.

   (c) Compute $\eta(t^2)$, and deduce the value of $\psi(t^2)$. Determine $\psi(s)$ similarly.

9. Write down the complete character table of G.

10. Determine the centre of G and the derived subgroup of G.

Question 4 *The affine group* $\mathrm{AGL}(1,5)$

Let $\mathbb{F}_5$ be the field $\mathbb{Z}/5\mathbb{Z} = \{0,1,2,3=-2,4=-1\}$, and let G be the set of polynomials $f_{a,b} = ax+b$ with $a,b \in \mathbb{F}_5$, $a \neq 0$. We equip G with the law given by composition:

$$(f_{a,b} \cdot f_{a',b'})(x) = f_{a,b}\big(f_{a',b'}(x)\big).$$

We admit that G is a group, whose identity is $f_{1,0}$ and whose conjugacy classes are:

$\{f_{1,0}\}$, $\{f_{1,b} \mid b \neq 0\}$, $\{f_{2,b} \mid b \in \mathbb{F}_5\}$, $\{f_{-2,b} \mid b \in \mathbb{F}_5\}$, $\{f_{-1,b} \mid b \in \mathbb{F}_5\}$.

1. Prove that the map

$$\begin{aligned} \lambda : G &\longrightarrow \mathbb{F}_5^\times \\ f_{a,b} &\longmapsto a \end{aligned}$$

is a group morphism.

2. Deduce the existence of four pairwise non-isomorphic irreducible representations of G of degree 1, and write down their characters.

   Hint: $\mathbb{F}_5^\times$ is cyclic and generated by 2.

3. Determine the number of irreducible representations of G, and their degrees.

4. Determine the character table of G.

5. We have a natural action of G on $\mathbb{F}_5$ defined by $f_{a,b} \cdot x = f_{a,b}(x)$. Determine up to isomorphism the decomposition into irreducible representations of the permutation representation $\mathbb{C}[\mathbb{F}_5]$ attached to this action of G on $\mathbb{F}_5$.

6. The element $f_{-1,0}$ of G has order 2, and thus generates a subgroup $H = \{f_{1,0}, f_{-1,0}\}$ of G isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let $\varepsilon : H \longrightarrow \mathrm{GL}_1(\mathbb{C})$ be the degree 1 representation of H such that $\varepsilon(f_{-1,0}) = -\mathrm{Id}_{\mathbb{C}}$. Determine up to isomorphism the decomposition into irreducible representations of the induced representation $\mathrm{Ind}_H^G(\varepsilon)$.

Question 5 *The determinant of the character table*

Let G be a finite group of order $n \in \mathbb{N}$. We arrange the conjugacy classes $C_1, \cdots, C_r$ and the irreducible characters $\chi_1, \cdots, \chi_r$ of G arbitrarily, and we view the character table of G as an $r \times r$ complex matrix X. In other words, for all $1 \leq i, j \leq r$, the i, j-coefficient $X_{i,j}$ of X is $\chi_i(g_j)$, where g_j is any element of C_j.

1. (a) Prove that the complex conjugate of a character is a character.

   (b) Prove that the complex conjugate of an **irreducible** character is an **irreducible** character.

   (c) Deduce that $\overline{\det X} = \pm \det X$. What does this imply about the complex number $\det X$?

2. (a) Let D be the diagonal matrix whose j, j-coefficient is $\#C_j$ for all $1 \leq j \leq r$. Express the orthogonality relations in terms of the matrices X and D.

   (b) Deduce the value of $|\det X|$ in terms of n and of the $\#C_j$.

   (c) (Trick question) What is the exact value of $\det X$?