Group representations
Exercise sheet 4

https://www.maths.tcd.ie/~mascotn/teaching/2025/MAU34104/index.html

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Submit your answers in class or to mismet@tcd.ie by Monday 24 March, 09:00.

Exercise 1 *Group theory thanks to representation theory (40 pts)*

Let G be a non-Abelian group of order 8. Prove that G has exactly 5 conjugacy classes, and that the Abelianisation $G^{\mathrm{ab}} = G/D(G)$ of G must be of order 4.

Hint: This exercise is part of an assignment on group representations!

Exercise 2 *The character table of A_4 (60 pts)*

Let $G = A_4$ be the alternating group of even permutations on 4 objects.

1. (12 pts) Let V_4 be the Klein subgroup of A_4 consisting of the double transpositions and of the identity. Prove that V_4 is normal in A_4, and that A_4/V_4 is cyclic.
2. (6 pts) Prove that (123) and (132) are **not** conjugate in A_4.
3. (30 pts) Determine the character table of A_4.
   You may want to define $\omega = e^{2\pi i/3}$; note that $\omega^2 = \overline{\omega} = -\omega - 1$.
4. (6 pts) Deduce that V_4 is the derived subgroup of A_4.
5. (6 pts) Determine the decomposition into irreducible representations of the restriction to A_4 of each the five irreducible representations of S_4.
6. (Bonus question, 0 pts) We admit that the group of rotations of $\mathbb{R}^3$ that preserve a regular tetrahedron invariant is isomorphic to A_4 via the permutations induced on the 4 vertices, whence a representation of A_4 of degree 3. Write down the decomposition (over $\mathbb{C}$) of this representation into irreducible representations.

These were the only mandatory exercises, that you must submit before the deadline. The following exercises are not mandatory; they is not worth any points, and you do not have to submit them. However, I *strongly* suggest you can try to solve them, as this is *excellent practice for the exam.* You are welcome to email me if you have questions about them. The solution will be made available with the solution to the mandatory exercises.

Exercise 3 *The character table of* $C_2 \times C_2$

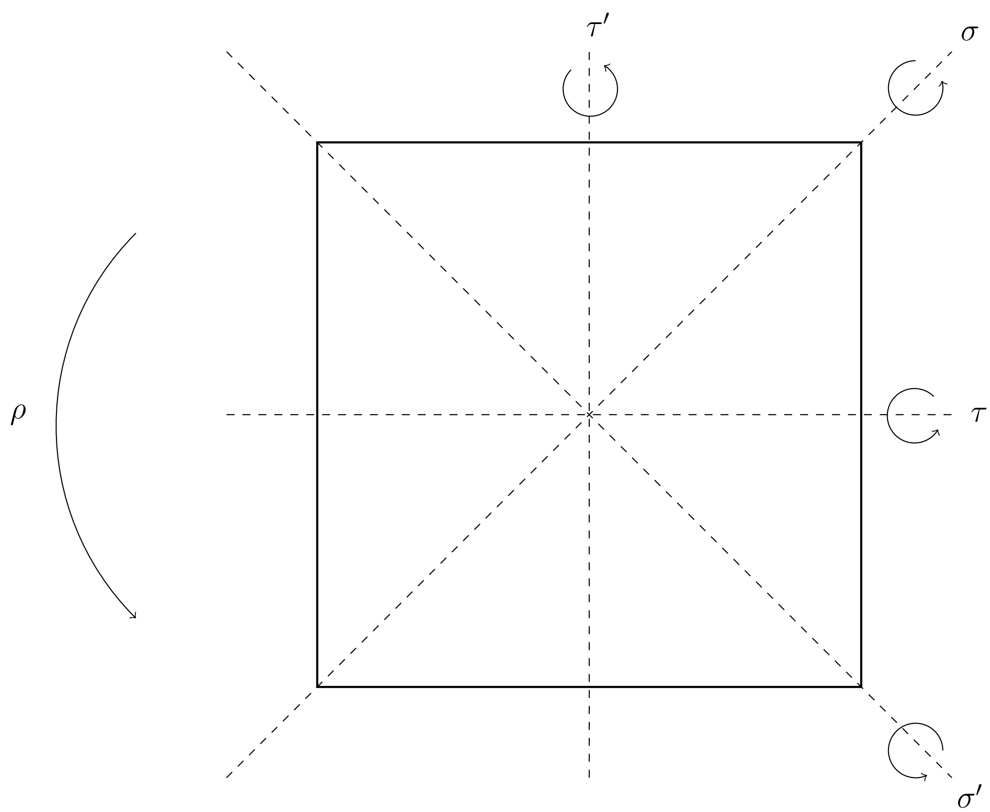
In this exercise, we write[1] C_2 for the group $\mathbb{Z}/2\mathbb{Z}$, and we consider representations of the group

$$G = C_2 \times C_2 = \{(x, y) \,|\, x \in C_2, y \in C_2\}.$$

1. Let χ be an irreducible character of G. Prove that $\chi(g) \in \{+1, -1\}$ for all $g \in G$.
2. Write down the character table of G.
3. Is any of the irreducible represenations of G faithful?
4. Does there exist faithful representations of G? If so, what is the smallest possible degree of such a representation?

Exercise 4 *The character table of* D_8

1. Let $G = D_8$ be the group of symmetries of the square, most of whose elements we name as follows:



   Determine the derived subgroup $D(G)$ and the structure of the quotient $G/D(G)$.

2. Determine the character table of D_8.

[1]For clarity, I personally prefer to write C_n for the group, $\mathbb{Z}/n\mathbb{Z}$ for the ring, and, if n is prime, $\mathbb{F}_n$ for the field.

Exercise 5 *The character table of Q_8*

Let $Q_8 = \{1, -1, I, -I, J, -J, K, -K\}$ be the (Hamiltonian) quaternionic group, whose multiplication is defined by the rules

$$\text{For all } x, y \in Q_8, (-x)y = x(-y) = -(xy), \quad x1 = 1x = x,$$

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = K = -JI, \quad JK = I = -KJ, \quad KI = J = -IK.$$

1. By Exercise 1, Q_8 has exactly 5 conjugacy classes. Check that these classes are $\{1\}$, $\{-1\}$, $\{I, -I\}$, $\{J, -J\}$, and $\{K, -K\}$.
2. Determine the centre Z of Q_8.
3. Prove that Q_8/Z is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.
4. Determine the character table of Q_8. Any comments?
5. It is standard to realise Q_8 as a group of 2×2 complex matrices by identifying I with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, J with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and K with $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ (since these matrices satisfy the relations defining the group law of Q_8, as you may check if you wish). How would you interpret this in terms of the character table of Q_8?

Exercise 6 *To be or not to be real*

Nicolas M., lecturer at a college somewhere in Europe, has been trying to hammer into his students that they must not write things such as

$$(\chi \mid \chi) = \frac{1}{\#G} \sum_{g \in G} \chi(g)^2$$

as this is incorrect since characters are complex-valued, the correct formula being

$$(\chi \mid \chi) = \frac{1}{\#G} \sum_{g \in G} |\chi(g)|^2.$$

However, he admits that it may be misleading that most of the characters in his lectures were real-valued. But how come?

1. Let G be a finite group. Prove that every character of G is real-valued if and only if every $g \in G$ is conjugate to its inverse.

   Hint: Recall that $\chi(g^{-1}) = \overline{\chi(g)}$.
2. Let $n \in \mathbb{N}$. Prove that every character of S_n is real-valued. What about A_4?