Group representations
Exercise sheet 2

https://www.maths.tcd.ie/~mascotn/teaching/2025/MAU34104/index.html

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THIS ASSIGNMENT IS NOT MANDATORY.

You are still welcome to attempt it; your mark will be counted only if this results in an advantage to your overall mark for this class.

If you decide to attempt this assignment, you only need hand in Exercise 1; however this Exercise relies on some of the results of Exercise 2, so you should at least take a quick look at Exercise 2 as well (but you are allowed to admit the results of Exercise 2).

The other exercises are independent from each other (and again they are not required to get full marks; they are just here for your culture).

Submit your answers in class or to mismet@tcd.ie by Monday February 24, 9AM.

Exercise 1 *Submodules and short exact sequences (100 pts)*

In this exercise, all modules are over a fixed ring R. We write 0 for the 0 module $\{0\}$. We also denote by 0 the 0 morphism between any two submodules (i.e. the map taking all the elements of the source module to the 0 element of the target module).

Recall that we saw in class that the decomposability of a module M can be characterised[1] by the presence of non-trivial idempotents (meaning $T^2 = T$) elements of $\operatorname{End}(M)$. The goal of this exercise is to find a (vaguely) similar characterisation for reducibility of modules, and to use it to shed new light on the connection between reducibility and decomposability.

Define an *exact sequence* as a diagram

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

where the M_i are modules, the f_i are module morphisms, and such that $\operatorname{Im} f_i = \operatorname{Ker} f_{i+1}$ for all i.

1. Let $\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$ be an exact sequence. Prove that $f_{i+1} \circ f_i = 0$ for all i.

2. Let N, M, Q be modules, and let $f : N \longrightarrow M$ and $g : M \longrightarrow Q$ be module morphisms. Prove that $0 \xrightarrow{0} N \xrightarrow{f} M$ is an exact sequence if and only if f is injective. Also prove that $M \xrightarrow{g} Q \xrightarrow{0} 0$ is an exact sequence if and only if g is surjective.

[1] If you enjoyed this part of the class, you may also like exercise 3 of this other sheet, which shows that idempotents can also be used to decompose rings.

A *short exact sequence* is an exact sequence of the form

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0.$$

By the previous question, this implies that g is surjective and that f is injective.

In particular, it follows that any short exact sequence provides us with the submodule $\operatorname{Im} f \simeq N$ of the middle module M.

3. Prove that conversely, whenever M is a module and N is a submodule, there exists a short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

for some module Q, where f is the identity inclusion of N into M.

Hint: Use the exercise on quotient modules.

4. Provide a counterexample to show that however, the existence of a short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

does NOT imply that $M \simeq N \oplus Q$ as modules.

Hint: Take $R = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$.

From now on, we only consider short exact sequences. We say that a short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

is *left-split* if there exists a module morphism $f' : M \longrightarrow N$ such that $f' \circ f = \mathrm{Id}_N$. We also say that it is *right-split* if there exists a module morphism $g' : Q \longrightarrow M$ such that $g \circ g' = \mathrm{Id}_Q$.

5. Let

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

be a short exact sequence, so that $\operatorname{Im} f$ is a submodule of M. Prove that if $\operatorname{Im} f$ admits a supplement submodule $M' \subset M$, then the short exact sequence above is both left-split and right-split.

Hint: Consider the restriction of g to M'.

6. Conversely, prove that if the short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

is left-split, then $\operatorname{Im} f$ admits a supplement in M.

Hint: Consider the map $T = f \circ f' \in \mathrm{End}(M)$.

7. Similarly, prove that if the short exact sequence

$$0 \xrightarrow{0} N \xrightarrow{f} M \xrightarrow{g} Q \xrightarrow{0} 0$$

is right-split, then $\operatorname{Im} f$ admits a supplement in M.

In conclusion, submodules correspond to short exact sequences, and the submodule admits a complement iff. the sequence is left-split iff. it is right-split.

Exercise 2 *Quotient modules*

Let R be a ring, and let M be an R-module with addition $+ : \begin{array}{ccc} M \times M & \longrightarrow & M \\ (m_1, m_2) & \longmapsto & m_1 + m_2 \end{array}$ and scalar multiplication $\cdot : \begin{array}{ccc} R \times M & \longrightarrow & M \\ (\lambda, m) & \longmapsto & \lambda \cdot m \end{array}$.

Let $N \subset M$ be a submodule. Then in particular, N is a subgroup of the additive group $(M, +)$ which is Abelian, so we have the quotient group $M/N = \{m + N \mid m \in M\}$ on which $=$ is still well-defined. It comes with a projection map $\begin{array}{ccc} M & \longrightarrow & M/N \\ m & \longmapsto & m + N \end{array}$ which is surjective and whose kernel is N.

1. Prove that M/N is actually still an R-module, by proving that the scalar multiplication $\cdot$ descends to a well-defined map $\begin{array}{ccc} R \times M/N & \longrightarrow & M/N \\ (\lambda, m + N) & \longmapsto & (\lambda \cdot m) + N \end{array}$.

2. Prove the isomorphism theorem for modules: Any module morphism

$$f : M \longrightarrow M'$$

induces a module isomorphism $M/\operatorname{Ker} f \simeq \operatorname{Im} f$.

Exercise 3 *Preservation of semi-simplicity*

In this exercise, all modules are over a fixed ring R, and all modules are Artinian[2], meaning that there cannot exist an infinite descending chain of submodules.

1. Prove that a submodule of a semi-simple module is also semi-simple.

2. Prove that if $f : M \longrightarrow N$ is a module morphism, and if M is semi-simple, then $\operatorname{Im} f$ is also semi-simple.

3. Let now G be a group, K a field, and $f : V \longrightarrow W$ be a morphism between representations of G over K of finite degree. Prove that if V is semi-simple, then so are $\operatorname{Ker} f$ and $\operatorname{Im} f$.

Exercise 4 *A non-semi-simple ring*

Let K be a field, and let G be a finite group of order $n = \#G$. We will sometimes see the group ring $K[G] = \{\sum_{g \in G} \lambda_g e_g \mid \lambda_g \in K \text{ for all } g \in G\}$ as a module over itself.

1. Let $\Sigma = \sum_{g \in G} e_g \in K[G]$. Prove that $e_h \Sigma = \Sigma$ for all $h \in G$.

2. Prove that $S = \{\lambda\Sigma,\ \lambda \in K\}$ is a sub-$K[G]$-module of $K[G]$.

3. Identify S as a representation of G.

From now on, we assume that $n = 0$ in K.

4. Prove that $\Sigma^2 = 0$ in $K[G]$.

[2] NB we make this assumption to make our lives easier, but it can be shown that the properties established in this exercise actually remain valid without this assumption.

5. Deduce that $1 - \lambda\Sigma$ is invertible in $K[G]$ for all $\lambda \in K$, where $1 = e_{1_G}$ is the multiplicative identity of $K[G]$.

   Note that since $K[G]$ is not commutative in general, you must prove that your inverse works on both sides.

   Hint: For $x \in \mathbb{R}$ and $m \in \mathbb{N}$, what is the formula for the geometric series $1 + x + x^2 + \cdots + x^m$? How do you prove it?

6. Deduce that $K[G]$, viewed as a $K[G]$-module, is not semi-simple.

Exercise 5 *Annihilators and simple modules*

Let R be a ring, which need not be commutative. We say that $I \subseteq R$ is a *left ideal* if it is an additive subgroup of $(R, +)$ and if $ri \in I$ for all $r \in R$ and $i \in I$. We define *right ideals* similarly. If I is both a left ideal and a right ideal, then we say that it is a *two-sided* ideal. A *maximal* left ideal is a left ideal $M \neq R$ such that there are no left ideals I such that $M \subsetneq I \subsetneq R$.

1. Let M be an R-module. Define its *annihilator* as

   $$\operatorname{Ann} M = \{r \in R \mid rm = 0 \text{ for all } m \in M\} \subseteq R.$$

   Prove that $\operatorname{Ann} M$ is a *two-sided* ideal of R.

2. The ring R can be viewed as a module over itself; we denote this module by ${}_RR$, so as to clearly distinguish between the ring R and the R-module ${}_RR$. Identify the submodules of ${}_RR$.

3. Let S be a module, and let $s \in S$.

   (a) Prove that the map

   $$\begin{array}{rcl} f_s : {}_RR & \longrightarrow & S \\ r & \longmapsto & rs \end{array}$$

   is a module morphism.

   (b) Prove that S is simple iff. f_s is surjective for all $s \neq 0$.

   (c) Deduce that if S is simple, then it is isomorphic to ${}_RR/M$, where M is a maximal left ideal of R.

   (d) Prove that conversely, if M is a maximal left ideal of R, then ${}_RR/M$ is a simple R-module.

4. Beware that in general, the annihilator of ${}_RR/M$, which is a two-sided ideal, does not agree with the left ideal M! Here is an example: Take $R = M_2(\mathbb{R})$, and $S = \mathbb{R}^2$, which is an R-module if we view its elements as column vectors. Prove that S is simple, determine $\operatorname{Ann} S$, and find a maximal left ideal M of R such that $S \simeq {}_RR/M$.