Group representations
Exercise sheet 1

https://www.maths.tcd.ie/~mascotn/teaching/2025/MAU34104/index.html

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Submit your answers in class or to mismet@tcd.ie by Wednesday February 12 noon.

Exercise 1 *(Sub)representations of degree 1 (60 pts)*

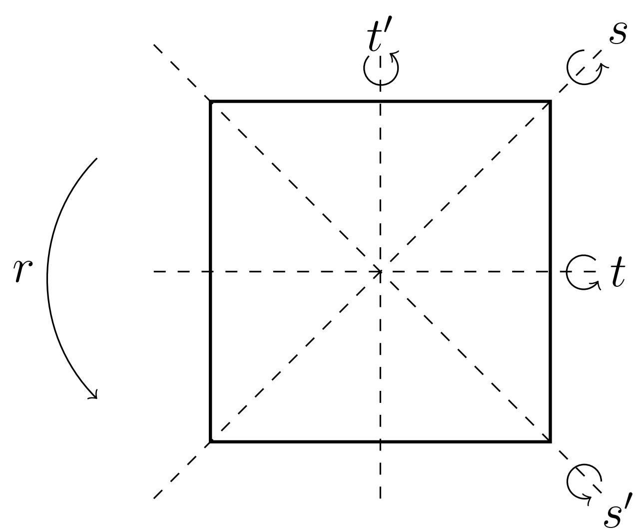
Let G be a group, K a field, and $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation of G over K.

1. (10 pts) Prove that if ρ is of degree 1, then ρ is irreducible and indecomposable.
2. (20 pts) We no longer assume that $\deg \rho = 1$. Let $W \subseteq V$ be a subspace of dimension 1, so that $W = Kv = \{\lambda v \mid \lambda \in K\}$ for some $0 \neq v \in V$. Prove that W is a subrepresentation of V if and only if v is an eigenvector of $\rho(g)$ for all $g \in G$.
3. (30 pts) Suppose finally that $K = \mathbb{C}$, that G is Abelian, and that $\dim V$ is finite and nonzero (but not necessarily 1). Prove that V admits at least one subrepresentation of degree 1.

   Hint: Prove that if $T, U : V \longrightarrow V$ are linear transformations which commute with each other, then for each eigenvalue λ of T, the λ-eigensubspace of T is stable by U. Then induct on $\dim V$. You may want to treat separately the case where all the $\rho(g)$ act as scalar matrices.

Exercise 2 *The dihedral group D_8 (40 pts)*

Let $G = D_8$ be the group of transformations of the plane $\mathbb{R}^2$ that preserve a square. It consists of 8 elements: Id, the rotations r (by $\pi/2$), r^2, r^3, (note $r^4 = \mathrm{Id}$), and the reflections s, s' and t, t', as shown on the figure below:



It thus comes naturally with a representation $\rho : D_8 \longrightarrow \mathrm{GL}(\mathbb{R}^2)$ of D_8 of degree 2 over $\mathbb{R}$.

1. (5 pts) For each of the 8 elements $g \in D_8$, write down the matrix of $\rho(g)$ with respect to the (fixed) basis of $\mathbb{R}^2$ of your choice (specify which basis you pick).
2. (10 pts) Deduce that ρ is faithful.
3. (15 pts) Use the results of Exercise 1 to prove that ρ is irreducible.
4. (10 pts) We can also view ρ as a representation $\rho_{\mathbb{C}}$ over $\mathbb{C}$ instead of $\mathbb{R}$ (keep the same matrices, but view them as complex matrices rather than real ones). Is $\rho_{\mathbb{C}}$ irreducible, i.e. is ρ still irreducible over $\mathbb{C}$?

   Hint: Use the results of Exercise 1 again, but work slightly harder.

These were the only mandatory exercises, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I *strongly* recommend you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 3 *Representation morphisms form a subspace*

Let G be a group, K a field, and let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$ and $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$ be two representations of G over K.

Recall that the set $\mathrm{Hom}(V_1, V_2)$ of all linear transformations from V_1 to V_2 has a vector space structure, with addition and scalar multiplications defined pointwise (that is to say if $T, U \in \mathrm{Hom}_(V_1, V_2)$ and $\lambda \in K$, then $T + U$ is defined as $(T + U)(v_1) = T(v_1) + U(v_1)$ for all $v_1 \in V_1$, and λT is defined as $(\lambda T)(v_1) = \lambda\big(T(v_1)\big)$ for all $v_1 \in V_1$).

Prove that the subset $\mathrm{Hom}_G(V_1, V_2)$ consisting of linear transformations which are representation morphisms is a subspace of $\mathrm{Hom}(V_1, V_2)$.

Exercise 4 *Decomposition over* $\mathbb{R}$

In this exercise, we consider over $K = \mathbb{R}$ two representations of $G = S_3$ constructed in the lectures, namely the permutation representation

$$\mathrm{Perm} : S_3 \longrightarrow \mathrm{GL}_3(K)$$

induced by $S_3 \circlearrowright \{1, 2, 3\}$, and

$$\triangleleft : S_3 \longrightarrow \mathrm{GL}_2(K), \quad (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad (12) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

obtained by labelling the vertices of an equilateral triangle by $\{1, 2, 3\}$.

1. Prove that $\triangleleft$ is irreducible. Is $\triangleleft$ indecomposable?
2. Prove that $\mathrm{Perm} \simeq \mathbb{1} \oplus \triangleleft$ as representations of S_3.

The following exercise has been included for those of you who wish to try their skills in more "exotic" situations. It is not meant to be as profitable for your understanding of the material of this module as the previous exercises. You are still welcome to try to solve it for practice, and to email me if you have questions about it. The solution will be also made available with the solution to the other exercises.

Exercise 5 *Decomposition over* $\mathbb{Z}/p\mathbb{Z}$

Redo Exercise 4, but with $K = \mathbb{Z}/p\mathbb{Z}$ instead of $\mathbb{R}$, where $p \in \mathbb{N}$ is prime. Is $\triangleleft$ still irreducible? Indecomposable? Do we still have $\mathrm{Perm} \simeq \mathbb{1} \oplus \triangleleft$?
Your answers may depend on the value of p; explore all cases!