Algebraic number theory — Exercise sheet 5 (optional)

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Exercise 5.1: Units in a real quadratic field

Let $K = \mathbb{Q}(\sqrt{42})$, viewed as a subfield of $\mathbb{R}$.

1. Find a fundamental unit ε in K such that $\varepsilon > 1$.

2. Prove that the equation $x^2 - 42y^2 = -1$ has no solutions in integers.

   Hint: Write down generators for $\mathbb{Z}_K^\times$. What are their norms?

   We now wish to determine the class group $\mathrm{Cl}(K)$ of K.

3. First of all, prove that it is generated by the image of the prime $\mathfrak{p}_2$ above 2, and that this image has order at most 2 in $\mathrm{Cl}(K)$.

4. We want to prove that $\mathfrak{p}_2$ is not principal. Suppose by contradiction that it is, and let $\gamma = x + y\sqrt{42}$ be a generator. Explain why we may assume that $\frac{1}{\sqrt{\varepsilon}} \leq \gamma \leq \sqrt{\varepsilon}$, and deduce that $|y| < 2$.

5. Prove that $\mathrm{Cl}(K) \simeq \mathbb{Z}/2\mathbb{Z}$.

Solution 5.1:

1. We have $\mathbb{Z}_K = \mathbb{Z}[\sqrt{42}]$. The norm of the generic element $x + y\sqrt{42} \in \mathbb{Z}_K$ is $x^2 - 42y^2$, so units in K correspond to the pairs of integers (x, y) such that $x^2 - 42y^2 = \pm 1$, and we know that the pair with the smallest nonzero y corresponds to a fundamental unit. For $y = 1$, we get $x^2 = 41$ or 43, which has no solution in integers. But for $y = 2$, we find that $13^2 - 42 \cdot 2^2 = 1$, so we find the fundamental unit $\varepsilon = 13 + 2\sqrt{42}$, which clearly satisfies $\varepsilon > 1$.

2. This amounts to proving that there is no element of norm -1 in $\mathbb{Z}_K$. Such an element would be a unit, so let us take a look at $\mathbb{Z}_K^\times$.

   Since K is a subfield of $\mathbb{R}$, W_K is reduced to $\{\pm 1\}$, so $\mathbb{Z}_K^\times$ is generated by the root of unity -1 and by the fundamental unit ε.

   We observe that $N_{\mathbb{Q}}^K(-1) = (-1)^2 = +1$, and that $N_{\mathbb{Q}}^K(\varepsilon) = 13^2 - 42 \cdot 2^2 = +1$ as well. Since these two units generate $\mathbb{Z}_K^\times$, this means that every unit in K has norm $+1$, so there are indeed no units of norm -1 in K.

3. The discriminant of K is $2^2 \cdot 42$ and its signature is $(2, 0)$, so the Minkowski bound is
$$M_K = \frac{2!}{2^2}\sqrt{2^2 \cdot 42} = \sqrt{42} < 7,$$
   which means that $\mathrm{Cl}(K)$ is generated by the primes above 2, 3, and 5.

Now 2 and 3 both divide $\operatorname{disc} K$, so they ramify, say $(2) = \mathfrak{p}_2^2$ and $(3) = \mathfrak{p}_3^2$, whereas 5 is inert in K since 2 is a nonzero nonsquare mod 5. So $\mathrm{Cl}(K)$ is generated by $[\mathfrak{p}_2]$ and $[\mathfrak{p}_3]$, and we have the relations $[\mathfrak{p}_2]^2 = [\mathfrak{p}_3]^2 = 1$.

We now try to find a relation between $[\mathfrak{p}_2]$ and $[\mathfrak{p}_3]$. For this, we need to find an element of $\mathbb{Z}_K$ of norm $\pm 2 \cdot 3$, that is to say a pair of integers (x, y) such that $x^2 - 42y^2 = \pm 6$. We spot the solution $x = 6$, $y = 1$, which tells us that $6 + \sqrt{42}$ has norm -6, so that the ideal $(6 + \sqrt{42})$ factors as $\mathfrak{p}_2\mathfrak{p}_3$. This yields the relation $[\mathfrak{p}_2][\mathfrak{p}_3] = 1$ in $\mathrm{Cl}(K)$, which shows that $\mathrm{Cl}(K)$ is generated by $[\mathfrak{p}_2]$ alone. Besides, the relation $[\mathfrak{p}_2]^2 = 1$ tells us that the order of $[\mathfrak{p}_2]$ is at most 2.

4. After replacing γ with $\pm\varepsilon^n\gamma$ for some $n \in \mathbb{Z}$, which is legitimate as it is associate to γ and therefore generates the same ideal, we may assume that $\gamma > 0$ (thanks to the $\pm$) and that $\frac{1}{\sqrt{\varepsilon}} \leq \gamma \leq \sqrt{\varepsilon}$ (by adjusting n).

   Write $\gamma = x + y\sqrt{42}$. Since γ generates $\mathfrak{p}_2$, which has norm 2, γ must have norm ± 2, whence $\pm 2 = (x + y\sqrt{42})(x - y\sqrt{42})$. Thus

$$|2y\sqrt{42}| = |(x + y\sqrt{42}) - (x - y\sqrt{42})| = \left|\gamma \pm \frac{2}{\gamma}\right| \leq 3\sqrt{\varepsilon},$$

   so

$$|y| \leq \frac{3\sqrt{\varepsilon}}{2\sqrt{42}} < \frac{3\sqrt{13 + 2 \cdot 7}}{2\sqrt{42}} = \frac{9}{2\sqrt{14}} < \frac{9}{6} < 2.$$

5. Since $\gamma \in \mathbb{Z}_K = \mathbb{Z}[\sqrt{42}]$, y is an integer, so it must be 0 of ± 1. However, the equation $x^2 - 42y^2 = \pm 2$ has no integer solutions with such y, which contradicts our assumption that $\mathfrak{p}_2$ is principal. So $[\mathfrak{p}_2] \neq 1$, so $[\mathfrak{p}_2]$ has order exactly 2, and therefore

$$\mathrm{Cl}(K) = \langle [\mathfrak{p}_2] \rangle \simeq \mathbb{Z}/2\mathbb{Z}.$$

Exercise 5.2: Arbitrary unit groups

1. Prove that there is no number field K such that the unit group $\mathbb{Z}_K^\times$ is isomorphic to $\mathbb{Z}/50\mathbb{Z} \times \mathbb{Z}^{10}$.

2. Find a number field K such that $\mathbb{Z}_K \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}$.

Solution 5.2:

1. Let K be such a number field. Then K contains a primitive 50th root of unity ζ_{50}, so $\mathbb{Q}(\zeta_{50})$ is a subfield of K isomorphic to the 50th cyclotomic field. Therefore, the degree of K is a multiple of $\varphi(50) = \varphi(2 \cdot 5^2) = 20$, say $20k$ with $k \in \mathbb{Z}_{\geq 1}$. Moreover, K is totally complex since it contains non-trivial roots of unity. So the signature of K is $(0, 10k)$, and by Dirichlet's theorem, the rank of $\mathbb{Z}_K^\times$ is $10k - 1$, which cannot equal 10.

2. By the same analysis as above, such a number field must contain $\mathbb{Q}(\zeta_4) = \mathbb{Q}(i)$ and have degree 4 and signature $(0, 2)$. Moreover, such a field satisfies the required property, unless it has too many roots of unity. By listing the $n \in \mathbb{N}$ such that $\phi(4n) = 4$, we see that it is enough to find such a K that is not isomorphic to $\mathbb{Q}(\zeta_8)$ nor $\mathbb{Q}(\zeta_{12})$.

   Let $K = \mathbb{Q}(i, \sqrt{5}) \subset \mathbb{C}$. Then the inclusions

   $$\mathbb{Q} \subset \mathbb{Q}(\sqrt{5}) \subset K = \mathbb{Q}(\sqrt{5})(i)$$

   and the fact that $\mathbb{Q}(\sqrt{5})$ has not 4th root of unity since it admits real embeddings prove that K has degree 4. Besides, $\mathbb{Q}(\sqrt{5})$ is ramified at 5, so K itself is ramified at 5 and is therefore not isomorphic to $\mathbb{Q}(\zeta_8)$ or $\mathbb{Q}(\zeta_{12})$. So $\mathbb{Z}_K^\times \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}$.

Exercise 5.3: A lower bound on the regulator

Let K be a number field of degree 3 such that $\operatorname{disc} K < 0$.

1. Prove that the signature of K is $(1, 1)$.

2. From now on, we let σ be the unique real embedding of K. Prove that there exists $\varepsilon \in K$ such that $\mathbb{Z}_K^\times = \{\pm\varepsilon^n,\ n \in \mathbb{Z}\}$ and that $\sigma(\varepsilon) > 1$, and that such an ε is unique.

3. Prove that ε is a primitive element for K, and deduce that the minimal polynomial of ε factors over $\mathbb{R}$ as $(x - \sigma(\varepsilon))(x - u^{-1}e^{i\theta})(x - u^{-1}e^{-i\theta})$ for some $\theta \in \mathbb{R}$, where $u = \sqrt{\sigma(\varepsilon)}$.

4. Using without proof the fact that

$$\left(\frac{u^3 + u^{-3}}{2} - \cos\theta\right)^2 \sin^2\theta < \frac{u^6}{4} + \frac{3}{2}$$

   for all $\theta \in \mathbb{R}$ (you are **NOT** required to prove this), prove that

$$\sigma(\varepsilon) > \sqrt[3]{\frac{|\operatorname{disc} K|}{4} - 6}.$$

   Hint: Prove that

$$\operatorname{disc} \mathbb{Z}[\varepsilon] = -16\left(\frac{u^3 + u^{-3}}{2} - \cos\theta\right)^2 \sin^2\theta.$$

5. As an application, we want to find a fundamental unit in $K = \mathbb{Q}(\alpha)$ where α is a root of $f(x) = x^3 + 4x + 2$. We admit without proof that the only real root of $f(x)$ is approximately -0.473, and still denote by σ the corresponding embedding of K into $\mathbb{R}$.

   (a) By taking a look at the decomposition of $2\mathbb{Z}_K$, find a unit $u \in \mathbb{Z}_K^\times$ such that $\sigma(u) > 1$.

   (b) Prove that u is either a fundamental unit or the square of a fundamental unit.

   (c) By reducing u mod one of the primes above 3, prove that u is actually a fundamental unit.

   (d) What is the regulator of K?

Solution 5.3:

1. Let (r_1, r_2) be the signature of K, so that $r_1 + 2r_2 = 3$. Since $\operatorname{disc} K < 0$ has the same sign as $(-1)^{r_2}$, we see that r_2 is odd. This forces $r_1 = r_2 = 1$.

2. Since K has a real embedding, the group of roots of unity in K is $W_K = \{\pm 1\}$. By Dirichlet's theorem, the rank of the unit group is $r_1 + r_2 - 1 = 1$. Let ε be a fundamental unit of $\mathbb{Z}_K^\times$. Then we have

$$\mathbb{Z}_K^\times = \{\pm \varepsilon^n : n \in \mathbb{Z}\}.$$

   The other fundamental units are the $\pm\varepsilon^{\pm 1}$, and as $\sigma(\varepsilon) \neq \pm 1$ as $\varepsilon \neq \pm 1$, exactly one of these has it image by σ in the interval $(1, \infty)$.

3. Since $\mathbb{Z}$ does not have any unit of infinite order, $\varepsilon \notin \mathbb{Q}$. So the field $\mathbb{Q}(\varepsilon) \subset K$ is not $\mathbb{Q}$, but its degree over $\mathbb{Q}$ divides that of K, which is 3. So $[\mathbb{Q}(\varepsilon) : \mathbb{Q}] = 3$ and $\mathbb{Q}(\varepsilon) = K$, which means precisely that ε is a primitive element of K.

   Let P be the minimal polynomial of ε over $\mathbb{Q}$. Since ε is a primitive element of K, the polynomial P has degree 3, is irreducible over $\mathbb{Q}$, and has one real root and two conjugate complex roots since the signature of K is $(1, 1)$. The real root is $\sigma(\varepsilon)$, and let $z, \bar{z}$ be the complex roots. Since ε is a unit, it has norm ± 1. This norm is also the product of the complex embeddings of ε, so

$$\pm 1 = \sigma(\varepsilon) z \bar{z} = \sigma(\varepsilon)|z|^2 > 0, \text{ so } \sigma(\varepsilon)|z|^2 = 1.$$

   We get $|z|^2 = \sigma(\varepsilon)^{-1}$, so the polar decomposition of z is $z = u^{-1}e^{i\theta}$ with $u = \sqrt{\sigma(\varepsilon)}$ and $\theta \in \mathbb{R}$.

4. Since ε is a primitive element of K and is an algebraic integer, $\mathbb{Z}[\varepsilon]$ is an order in K, so $|\operatorname{disc} K| \leq |\operatorname{disc} \mathbb{Z}[\varepsilon]| = |\operatorname{disc} P|$. We compute

$$\begin{aligned}
\operatorname{disc} P &= \left[(\sigma(\varepsilon) - u^{-1}e^{i\theta})(\sigma(\varepsilon) - u^{-1}e^{-i\theta})(u^{-1}e^{i\theta} - u^{-1}e^{-i\theta})\right]^2 \\
&= \left[(\sigma(\varepsilon)^2 - 2\sigma(\varepsilon)u^{-1}\cos\theta + u^{-2})u^{-1}2i\sin\theta\right]^2 \\
&= -16\left(\frac{u^3 + u^{-3}}{2} - \cos\theta\right)^2 \sin^2\theta, \text{ since } u^2 = \sigma(\varepsilon).
\end{aligned}$$

   Taking absolute values, we have

$$\begin{aligned}
|\operatorname{disc} K| &\leq |\operatorname{disc} P| \\
&< 16\left(\frac{u^6}{4} + \frac{3}{2}\right) \\
&= 4(\sigma(\varepsilon)^3 + 6).
\end{aligned}$$

   Dividing by 4, subtracting 6 and taking cube roots, we obtain

$$\sigma(\varepsilon) > \sqrt[3]{\frac{|\operatorname{disc} K|}{4} - 6}.$$

5. (a) First of all, $f(x)$ is irreducible over $\mathbb{Q}$ since it is Eisenstein at $p=2$, so K is well-defined and has degree 3. Furthermore, since we are told that $f(x)$ has only one real root, the signature of K is $(1,1)$, so we can apply the previous questions to K.

Next, we compute that $\operatorname{disc} f = -4 \cdot 4^3 - 27 \cdot 2^2 = -364$ (which is negative, confirming that $\operatorname{sign} K = (1,1)$), which factors as $-2^2 \cdot 7 \cdot 13$. Since $f(x)$ is Eisenstein at $p=2$, the ring of integers of K is thus $\mathbb{Z}_K = \mathbb{Z}[\alpha]$.

Finally, since $f(x)$ is Eisenstein at $p=2$, 2 is totally ramified in K, and more specifically $(2) = \mathfrak{p}^3$ where $\mathfrak{p} = (2, \alpha)$ has norm 2^1. But $N_{\mathbb{Q}}^{K}(\alpha) = -2$ from the constant coefficient of $f(x)$, so actually $\mathfrak{p} = (\alpha)$.

It follows that $(2) = (\alpha)^3 = (\alpha^3)$, so $u_1 = \alpha^3/2 = -2\alpha + 1$ is a unit.

We have $\sigma(u_1) = -2y - 1 \approx -0.054$, so $u = -1/u_1$ is a unit with $\sigma(u) = -1/(-2y-1) \approx 18.5 > 1$.

(b) Since $\mathbb{Z}_K^\times = \{\pm\varepsilon^n, n \in \mathbb{Z}\}$ and $\sigma(u) > 1$, we must have $u = +\varepsilon^n$ for some integer $n \geqslant 1$. In particular, $\sigma(u) = \sigma(\varepsilon)^n$. But the lower bound from question 4. tells us that $\sigma(\varepsilon) > \sqrt[3]{\frac{364}{4} - 6} \approx 4.4$, whence $n \leqslant 2$ since $4.4^3 > 18.5$.

(c) We find that $x = -1$ is a root of $f(x)$ mod 3. As $\mathbb{Z}_K = \mathbb{Z}[\alpha]$, we deduce that $\mathfrak{q} = (3, \alpha+1)$ is a prime ideal above $p = 3$ such that $\mathbb{Z}_K/\mathfrak{q} \simeq \mathbb{Z}/3\mathbb{Z}$, the reduction being such that $\alpha \equiv -1 \bmod \mathfrak{q}$.

In particular, $u_1 \equiv -2 \times -1 - 1 = 1 \bmod \mathfrak{q}$, so $u = -1/u_1 \equiv -1/1 = -1 \bmod \mathfrak{q}$. But -1 is not a square in $\mathbb{Z}_K/\mathfrak{q} \simeq \mathbb{Z}/3\mathbb{Z}$, so u is anot a square mod $\mathfrak{q}$. Therefore u cannot be a square in $\mathbb{Z}_K$, so we cannot have $n = 2$ in $u = \varepsilon^n$. As $n \leqslant 2$, this forces $n = 1$, so $u = \varepsilon$ is a fundamental unit of K.

(d) Let $\tau, \bar{\tau}$ be the other complex embeddings of K. By definiton, the regulator R_K of K is the absolute value of the determinant of

$$\begin{pmatrix} \log|\sigma(\varepsilon)| \\ 2\log|\tau(\varepsilon)| \end{pmatrix}$$

with any one row deleted. Therefore

$$R_K = \big|\log|\sigma(\varepsilon)|\big| = \big|\log|1/(2y+1)|\big| = \big|\log|2y+1|\big| \approx 2.9.$$

Exercise 5.4: Units in a real cubic field

For this exercise, you will need a calculator so as to compute complex embeddings explicitly. Do not worry about accuracy issues.

Let $K = \mathbb{Q}(\alpha)$, where α is a root of $f(x) = x^3 - 12x + 6$. You will need to know the following:

- The roots of f are approximately -3.69, 0.511, and 3.18.
- f Assumes the following values:

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	-59	-10	15	22	17	6	-5	-10	-3	22	71

- The regulator of K is[1] approximately 21.

1. Prove that f is irreducible over $\mathbb{Q}$, and that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$.
2. Determine W_K and the rank of $\mathbb{Z}_K^\times$.
3. Determine explicitly the decomposition of 2, 3, and 5 in K.
4. Use the formula $N_{\mathbb{Q}}^K(\alpha + n) = -f(-n)$ to prove that $\alpha - 3$ generates the prime above 3. Explain how to use this to discover that $u = (\alpha - 3)^3/3$ is a unit in K.
5. Factor the ideals $(\alpha - 1)$ and $(\alpha + 4)$ into primes. Use this to find a generator γ for the prime above 2, and deduce that $v = \gamma^3/2$ is also a unit in K.

   *I recommend you **NOT** to try to express γ and v as polynomials in α.*
6. Compute approximately the regulator of $\{u, v\}$.
7. Let U be the subgroup of $\mathbb{Z}_K^\times$ generated by W_K, u, and v. Is U equal to $\mathbb{Z}_K^\times$? What is the (possibly infinite) index of U in $\mathbb{Z}_K^\times$?
8. Compute the factorisation of the ideal (α) into primes, and use it to find a third unit $w \in \mathbb{Z}_K^\times$.
9. Prove that $\{u, w\}$ is a system of fundamental units for K.
10. Use the logarithmic embedding to conjecture a simple expression for v in terms of u and w (you do not have to prove that your conjecture is correct). Is your guess compatible with question 7?

[1] I determined this using a computer and methods beyond the scope of this class.

Solution 5.4:

1. f is Eisenstein at 2 (and also at 3), so it is irreducible over $\mathbb{Q}$ and K is well-defined up to isomorphism. Besides, f is monic and in $\mathbb{Z}[x]$, so $\alpha \in \mathbb{Z}_K$ and therefore $\mathbb{Z}[\alpha]$ is an order of discriminant

   $$\operatorname{disc} f = -4\cdot(-12)^3 - 27\cdot(-6)^2 = 2^8\cdot 3^3 - 2^2\cdot 3^5 = 2^2\cdot 3^3\cdot(2^6-3^2) = 2^2\cdot 3^3\cdot 5\cdot 11.$$

   We see from this factorisation that the index of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$ must divide 6. However, it is also coprime with 2 and 3 since f is Eisenstein at these primes, so this index is 1 and $\mathbb{Z}_K = \mathbb{Z}[\alpha]$.

2. Since $f(x)$ has three real roots, K has signature $(3,0)$ (one could also see this from the fact that $\operatorname{disc} K > 0$). In particular, K is embeddable into $\mathbb{R}$ (in three different ways), so W_K is reduced to $\{\pm 1\}$. As for the rank of $\mathbb{Z}_K^\times$, it is $3+0-1=2$ by Dirichlet's theorem.

3. As $\mathbb{Z}_K = \mathbb{Z}[\alpha]$, we can determine the decomposition of any prime $p \in \mathbb{N}$ by factoring $f \bmod p$. Now f is Eisenstein at 2, so 2 is totally ramified in K, namely $(2) = \mathfrak{p}_2^3$ where $\mathfrak{p}_2 = (2,\alpha)$ has norm 2. For the same reason, $(3) = \mathfrak{p}_3^3$, where $\mathfrak{p}_3 = (3,\alpha)^3$ has norm 3. Finally, we know that 5 ramifies in K since it divides $\operatorname{disc} K = \operatorname{disc} \mathbb{Z}[\alpha]$. From the table of values of f, we spot that 1 and 2 are the only roots of f mod 5, so $f \equiv (x-1)^2(x-2)$ or $(x-1)(x-2)^2$ mod 5. Since $f' = 3(x^2-4)$ vanishes mod 5 at 2 but not at 1, we have in fact $f \equiv (x-1)(x-2)^2 \mod 5$, whence $(5) = \mathfrak{p}_5\mathfrak{q}_5^2$, where $\mathfrak{p}_5 = (5, \alpha-1)$ and $\mathfrak{q}_5 = (5,\alpha-2)$ both have norm 5.

4. We get from the table that $N_{\mathbb{Q}}^K(\alpha-3) = -f(3) = 3$, so $(\alpha-3)$ is an ideal of norm 3, which can only be $\mathfrak{p}_3$ given the decomposition of 3 in K. Therefore $(3) = \mathfrak{p}_3^3 = (\alpha-3)^3 = \big((\alpha-3)^3\big)$, so 3 and $(\alpha-3)^3$ are associate in $\mathbb{Z}_K$.

5. From the table, $N_{\mathbb{Q}}^K(\alpha-1) = -f(1) = -5$, so $(\alpha+1)$ is an ideal of norm 5 and is thus either $\mathfrak{p}_5$ or $\mathfrak{q}_5$. But $\alpha - 1 \in \mathfrak{p}_5$, so $\mathfrak{p}_5 \mid (\alpha-1)$ and so $(\alpha-1) = \mathfrak{p}_5$. Similarly, $N_{\mathbb{Q}}^K(\alpha+4) = -f(-4) = 10$, so $(\alpha+4)$ has norm 10 and must thus factor either as $\mathfrak{p}_2\mathfrak{p}_5$ or as $\mathfrak{p}_2\mathfrak{q}_5$, but $\alpha+4 = \alpha-1+5 \in \mathfrak{p}_5$ so $\mathfrak{p}_5 \mid (\alpha+4)$ so actually $(\alpha+4) = \mathfrak{p}_2\mathfrak{p}_5$.

   We deduce that $\gamma = \frac{\alpha+4}{\alpha-1}$ satisfies $(\gamma) = \mathfrak{p}_2$ (as fractional ideals, so also as integral ideals; in particular $\gamma \in \mathbb{Z}_K$).

   Then as in the previous question we get $(2) = \mathfrak{p}_2^3 = (\gamma)^3 = (\gamma^3)$, so 2 and γ^3 are associate in $\mathbb{Z}_K$.

6. The regulator of $\{u,v\}$ is well-defined since we have seen that $\mathbb{Z}_K^\times$ has rank 2.

   In order to evaluate it, we need to approximate $\mathcal{L}(u)$ and $\mathcal{L}(v)$, where $\mathcal{L}$ is the logarithmic embedding. Let $\alpha_1 \approx -3.69$, $\alpha_2 \approx 0.511$, and $\alpha_3 \approx 3.28$ be the roots of f, and let $\sigma_1, \sigma_2, \sigma_3$ be the corresponding embeddings of K into $\mathbb{R}$; then $\mathcal{L}(\varepsilon) = \big(\log|\sigma_k(\varepsilon)|\big)_{k=1,2,3}$ for all $\varepsilon \in \mathbb{Z}_K^\times$. Therefore

   $$\mathcal{L}(u) = \left(\log\left|\sigma_k\left(\frac{(\alpha-3)^3}{3}\right)\right|\right)_{k=1,2,3} = \left(\log\left|\frac{(\alpha_k-3)^3}{3}\right|\right)_{k=1,2,3} \approx (4.60, 1.64, -6.24)$$

and

$$\mathcal{L}(v) = \left(\log\left|\sigma_k\left(\frac{\gamma^3}{2}\right)\right|\right)_{k=1,2,3} = \left(\log\left|\frac{\left(\frac{\alpha_k+4}{\alpha_k-1}\right)^3}{2}\right|\right)_{k=1,2,3} \approx (-8.84, 5.97, 2.88).$$

Deleting any row from

$$R = \begin{pmatrix} 4.60 & -8.84 \\ 1.64 & 5.97 \\ -6.24 & 2.88 \end{pmatrix}$$

and taking the absolute value of the determinant of the resulting matrix yields $\mathrm{Reg}(\{u, v\}) \approx 42$.

7. The regulator of $\{u, v\}$ is approximately twice that of K, so U is a strict subgroup of $\mathbb{Z}_K^\times$ of index 2.

8. $N_{\mathbb{Q}}^K(\alpha) = -f(0) = -6$, so (α) has norm $6 = 2 \cdot 3$. Therefore $(\alpha) = \mathfrak{p}_2\mathfrak{p}_3$.

   Hence $(6) = (2)(3) = \mathfrak{p}_2^3\mathfrak{p}_3^3 = (\alpha)^3 = (\alpha^3)$, so 6 and α^3 are associate in $\mathbb{Z}_K$ and we may take $w = \alpha^3/6 = 2\alpha - 1$.

   Alternatively, we can say that $(\alpha) = \mathfrak{p}_2\mathfrak{p}_3 = (\gamma)(\alpha - 3)$ and take $w' = \frac{\alpha}{\gamma(\alpha-3)} = \frac{\alpha(\alpha-1)}{(\alpha-3)(\alpha+4)}$; this does not matter since it turns out that $w' = w$.

9. We evaluate

   $$\mathcal{L}(w) = (\log|\sigma_k(2\alpha - 1)|)_{k=1,2,3} = (\log|2\alpha_k - 1|)_{k=1,2,3} \approx (2.12, -3.82, 1.68).$$

   Thus the regulator of $\{u, w\}$ is approximately equal to the absolute value of the determinant of

   $$R' = \begin{pmatrix} 4.60 & 2.12 \\ 1.64 & -3.82 \\ -6.24 & 1.68 \end{pmatrix}$$

   with any row deleted, namely ≈ 21. This time, this is the same as the regulator of K, so $\{u, w\}$ is a system of fundamental units for K.

10. We notice that $\mathcal{L}(u) + \mathcal{L}(v) + 2\mathcal{L}(w) \approx 0$. Assuming that this is in fact an exact 0, this would mean that uvw^2 is trivial in $\mathbb{Z}_K^\times/W_K$. As $W_K = \{\pm 1\}$, this would mean that $v = \pm u^{-1}w^{-2}$.

    Comparing $\sigma_k(v)$ with $\sigma_k(u^{-1}w^{-2}) = \sigma_k(u)^{-1}\sigma_k(w)^{-2}$ for $k = 1, 2, 3$, we guess that $v = +u^{-1}w^{-2}$.

    This is consistent with question 7: the transition matrix between $\{u, w\}$ and $\{u, v\}$ would be $\begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}$, whose determinant has absolute value 2.

    Remark: In fact, we can of course check that we indeed have $v = u^{-1}w^{-2}$ by expressing everything as polynomials in α, but this is not in the spirit of this exercise.