Algebraic number theory — Exercise sheet 2

https://www.maths.tcd.ie/~mascotn/teaching/2022/MAU34109/index.html

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Email your answers to mascotn@tcd.ie by Wednesday October 19 noon.

Reminder: $\operatorname{disc}(x^n + bx + c) = (-1)^{n(n-1)/2}\big((1-n)^{n-1}b^n + n^n c^{n-1}\big)$.

Exercise 2.1: A cubic field (100pts)

Let $A(x) = x^3 - 5x + 1 \in \mathbb{Q}[x]$.

1. (10pts) Prove that $A(x)$ is irreducible over $\mathbb{Q}$.
2. (90pts) Let $K = \mathbb{Q}(\alpha)$, where $A(\alpha) = 0$. Determine the degree of K, the ring of integers of K, the discriminant of K, and the signature of K. Needless to say, justify your answers.

Solution 2.1:

1. The degree of $A(x)$ is only 3, so if it factored over $\mathbb{Q}$, it would have either one factor of degree 1 and one of degree 2, or 3 factors of degree 1 (possibly not all distinct); anyway, it would have at least one factor of degree 1, and therefore a root in $\mathbb{Q}$. But the rational root theorem shows that the only possible rational roots of $A(x)$ are ± 1 (NB such a root would be a rational number and also an algebraic integer since $A(x) \in \mathbb{Z}[x]$ is monic, and therefore a rational integer), and those are not actual roots. Therefore $A(x)$ is irreducible over $\mathbb{Q}$.

2. Since $A(x)$ is irreducible over $\mathbb{Q}$ and monic, it is the minimal polynomial of α over $\mathbb{Q}$, so
$$[K : \mathbb{Q}] = \deg A(x) = 3.$$
Next, since actually $A(x) \in \mathbb{Z}[x]$ and is monic, α is an algebraic integer, so $\mathbb{Z}[\alpha]$ is an order in K, and we have
$$\big[\mathbb{Z}_K : \mathbb{Z}[\alpha]\big]^2 \operatorname{disc} K = \operatorname{disc} \mathbb{Z}[\alpha] = \operatorname{disc} A = -4(-5)^3 - 27(-1)^2 = 500 - 27 = 473,$$
which factors as $473 = 11 \times 43$ and is thus squarefree. Therefore $\big[\mathbb{Z}_K : \mathbb{Z}[\alpha]\big] = 1$, so the ring of integers of K is
$$\mathbb{Z}_K = \mathbb{Z}[\alpha]$$
and its discriminant is
$$\operatorname{disc} K \stackrel{\text{def}}{=} \operatorname{disc} \mathbb{Z}_K = \operatorname{disc} \mathbb{Z}[\alpha] = 11 \times 43.$$

Finally, let (r_1, r_2) be the signature of K. Then $r_1 + 2r_2 = [K : \mathbb{Q}] = 3$, so this signature is either $(3, 0)$ or $(1, 1)$. But we know that $(-1)^{r_2}$ is the sign of $\operatorname{disc} K$, so r_2 is even, so the only possibility is that the signature of K is $(3, 0)$. This means that K is *totally real*; in other words, all 3 complex roots of $A(x)$ are actually real.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice (they may even give you inspiration to help you solve Exercise 1), and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2.2: To be or not to be integral

Is $\dfrac{3+2\sqrt{6}}{\sqrt{6}-2}$ an algebraic integer ?

Solution 2.2:

Let us first express this number as a polynomial in $\sqrt{6}$. In principle, we should use the $U(x)A(x) + B(x)V(x) = 1$ method as in exercise 3 of sheet 1 for that, but here we are in degree only 2, so we can just apply the good old "conjugate expression" trick:

$$\frac{3+2\sqrt{6}}{\sqrt{6}-2} = \frac{(3+2\sqrt{6})(\sqrt{6}+2)}{(\sqrt{6}-2)(\sqrt{6}+2)} = \frac{3\sqrt{6}+6+2(\sqrt{6})^2+4\sqrt{6}}{(\sqrt{6})^2-2^2} = 9+\frac{7}{2}\sqrt{6}.$$

Now, there are (at least) three ways to conclude:

- As $\sqrt{6}$ is algebraic of degree 2 over $\mathbb{Q}$, every element of $\mathbb{Q}(\sqrt{6})$ can be written *uniquely* as $a + b\sqrt{6}$ with $a, b \in \mathbb{Q}$; besides, 6 is squarefree and $6 \not\equiv 1 \bmod 4$ so the ring of integers is $\mathbb{Z}[\sqrt{6}]$, in other words, an element $a+b\sqrt{6}$ is an algebraic integer if and only if a and b are both integers. This is not the case for $9+\frac{7}{2}\sqrt{6}$, so it is NOT an algebraic integer.

- The characteristic polynomial of this number (with respect to the extension $\mathbb{Q}(\sqrt{6})/\mathbb{Q}$) is

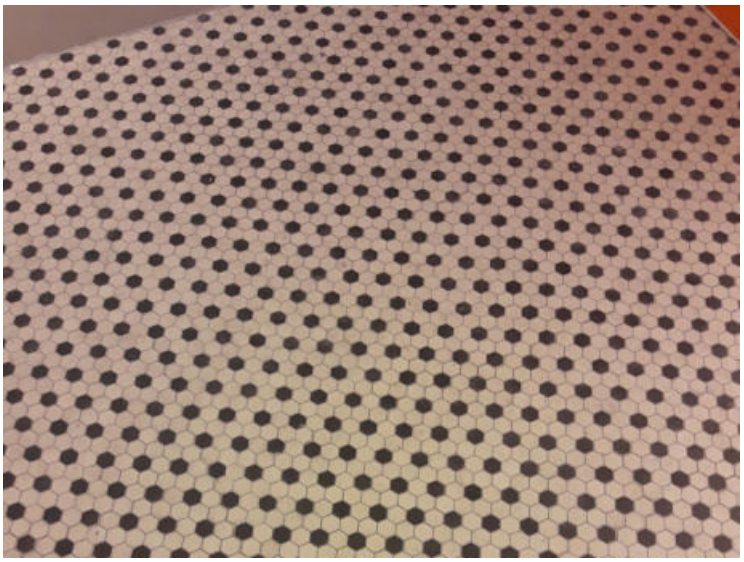
$$\left(x - \left(9+\frac{7}{2}\sqrt{6}\right)\right)\left(x - \left(9-\frac{7}{2}\sqrt{6}\right)\right) = x^2 - 18x + \frac{15}{2}$$

  (because of complex embeddings; we could also have written down the matrix of course). This does not lie in $\mathbb{Z}[x]$, so this number is NOT an algebraic integer.

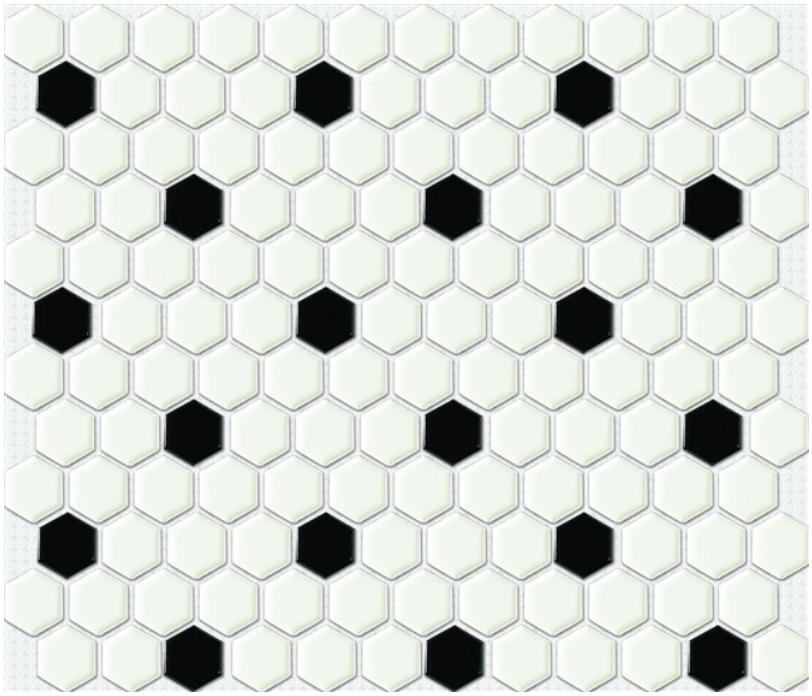
- If this number were an algebraic integer, then since algebraic integers form a ring containing $\mathbb{Z}$ and $\sqrt{6}$, we would have that $(9 + \frac{7}{2}\sqrt{6}) - 9 - 3\sqrt{6} = \frac{1}{2}\sqrt{6}$ is also an algebraic integer. However, this is not the case, since its minimal polynomial, which is $x^2 - 6/4$, does not lie in $\mathbb{Z}[x]$.

Exercise 2.3: Floor tilings

1. In the picture below, (the centre of) the hexagonal floor tiles (both the black ones and the white ones) form a lattice, and (the centre of) the black tiles form a sublattice. Compute the index of this sublattice by writing down a change-of-basis (= transition) matrix. What is the proportion of black tiles?

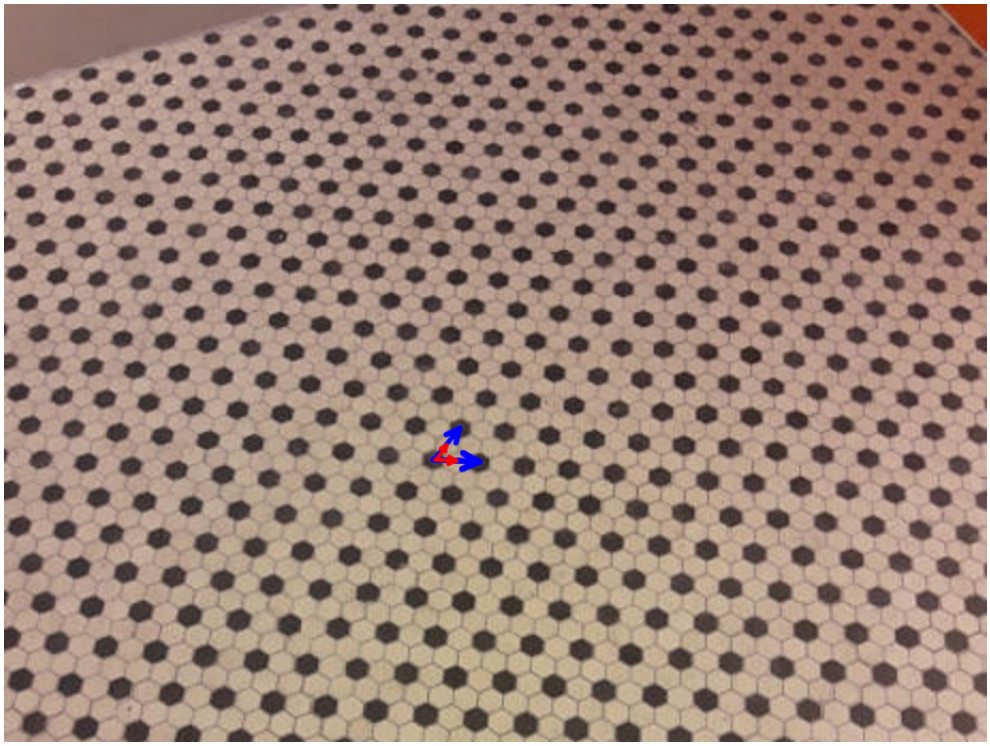


2. Same questions for this other tiling pattern.



Solution 2.3:

1. Let us choose a $\mathbb{Z}$-basis (in red) for the whole lattice, and another (in blue) for the black sublattice, as shown on the picture.

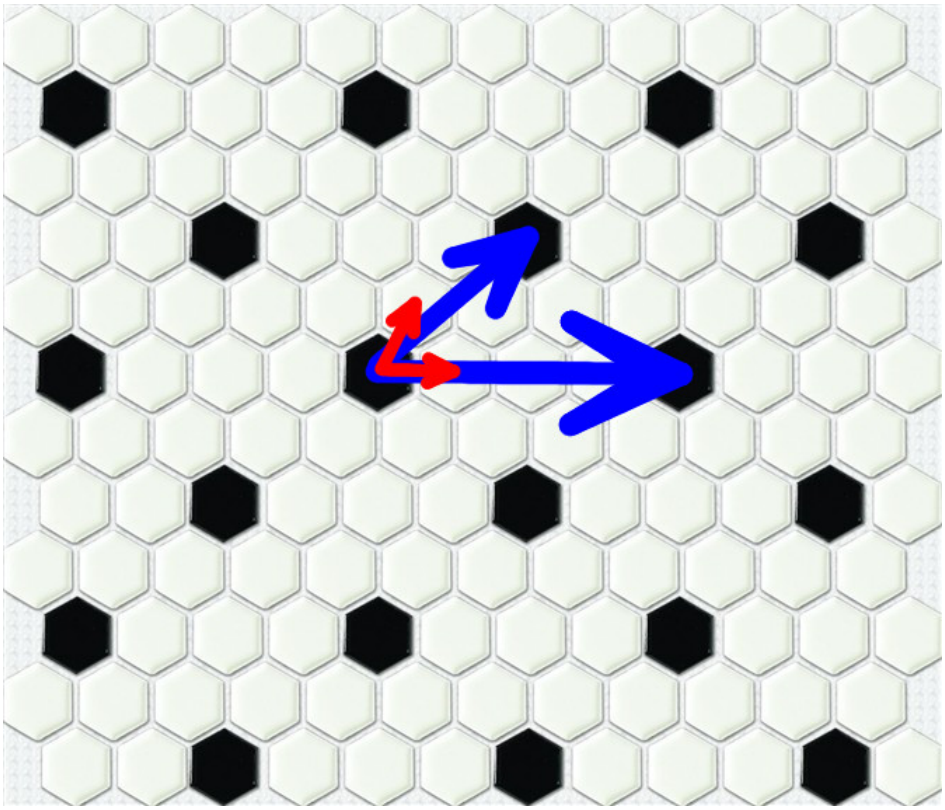


The change-of-basis matrix expressing the blue vectors in terms of the red ones is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

whose determinant is 4, so the index is 4. In other words, 1 out of 4 tiles is black.

2. Let us choose again $\mathbb{Z}$-basis (in red) for the whole lattice, and another (in blue) for the black sublattice.



This time, the change-of-basis matrix looks like

$$\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$$

(depending on how you order the bases). The determinant is 8, so the index is 8. In other words, 1 out of 8 tiles is black.

Exercise 2.4: A quartic field

Let $f(x) = x^4 - 2x + 4$, which you may assume without proof is irreducible over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha)$, where α satisfies $f(\alpha) = 0$.

1. Prove that $\mathbb{Z}[\alpha]$ is an order in K.
2. Compute and factor the discriminant of $\mathbb{Z}[\alpha]$.

   Hint: $2^{10} - 3^3 = 997$ *is prime.*
3. At this point, what are the possibilities for disc K, and the corresponding values of the index of $\mathbb{Z}[\alpha]$?
4. Let $\beta = \frac{\alpha^3}{2} \in K$, and consider the lattice $\mathcal{O} \subset K$ with $\mathbb{Z}$-basis

   $$1, \alpha, \alpha^2, \beta.$$

   Prove that $\mathcal{O}$ is stable under multiplication by β.

   Hint: what is $\beta \cdot \alpha$ *?.*

5. Deduce that β is an algebraic integer.

6. Which of the possibilities listed in question 2. remain?

7. Prove that $\mathcal{O}$ is actually an order in K.

   Hint: Prove that $\mathcal{O}$ is also stable under multiplication by α.

8. It turns out that $\mathbb{Z}_K = \mathcal{O}$. Give the discriminant of K in factored form.

Solution 2.4:

Let $f(x) = x^4 - 2x + 4$, which you may assume without proof is irreducible over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha)$, where α satisfies $f(\alpha) = 0$.

1. We know that $\operatorname{disc} \mathbb{Z}[\alpha] = \operatorname{disc} f$, and according to the formula

   $$\operatorname{disc}(x^n + bx + c) = (-1)^{n(n-1)/2}\big((1-n)^{n-1}b^n + n^n c^{n-1}\big),$$

   we have

   $$\operatorname{disc} \mathbb{Z}[\alpha] = +(-3^3 \cdot 2^4 + 4^4 \cdot 4^3) = 2^{14} - 3^3 \cdot 2^4 = 2^4 \cdot (2^{10} - 3^3) = 2^4 \cdot 997.$$

2. We know that

   $$\operatorname{disc} \mathbb{Z}[\alpha] = m^2 \operatorname{disc} K,$$

   where m is the index of $\mathbb{Z}[\alpha]$. Since 997 is prime, this leaves 3 possibilities:

   - Either $m = 1$ (which means that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$), and so $\operatorname{disc} K = 2^4 \cdot 997$,
   - or $m = 2$ (so $\mathbb{Z}[\alpha]$ is not 2-maximal, but is p-maximal for all primes $p \neq 2$), and $\operatorname{disc} K = 2^2 \cdot 997$,
   - or $m = 2^2$ (so again $\mathbb{Z}[\alpha]$ is not 2-maximal, but is p-maximal for all primes $p \neq 2$), and $\operatorname{disc} K = 997$.

   In the last two cases, the fact that $\mathbb{Z}[\alpha]$ is not 2-maximal would imply that the elements of $\mathbb{Z}_K$ would in general have denominators which are powers of 2 when we write then as polynomial of degree $< 4 = \deg f$ in α. In the first case, the elements of $\mathbb{Z}_K$ would have no denominators at all.

3. We have $\mathcal{O} = \{a + b\alpha + c\alpha^2 + d\beta,\ a, b, c, d \in \mathbb{Z}\}$.

   The relation $f(\alpha) = 0$ yields $\alpha^4 = 2\alpha - 4$, whence $\beta \cdot \alpha = \alpha - 2 \in \mathcal{O}$. Similarly, we find that $\beta \cdot \alpha^2 \in \mathcal{O}$, $\beta \cdot \beta \in \mathcal{O}$, and of course $\beta \cdot 1 \in \mathcal{O}$. Therefore,

   $$\beta \cdot (a + b\alpha + c\alpha^2 + d\beta) = a\beta \cdot 1 + b\beta \cdot \alpha + c\beta \cdot \alpha^2 + d\beta \cdot \beta \in \mathcal{O}$$

   as soon as $a, b, c, d \in \mathbb{Z}$, so $\beta \cdot \mathcal{O} \subset \mathcal{O}$.

4. Multiplication by β stabilises a lattice, so β is an algebraic integer (Recall the proof: we can write down the matrix of multiplication by β w.r.t. a $\mathbb{Z}$-basis of $\mathcal{O}$, and this matrix will have coefficients in $\mathbb{Z}$ since $\beta \cdot \mathcal{O} \subset \mathcal{O}$, so the characteristic polynomial of β lies in $\mathbb{Z}[x]$).

5. We have just seen that $\beta \in \mathbb{Z}_K$, but $\beta \notin \mathbb{Z}[\alpha]$ (because $\mathbb{Z}[\alpha] = \{a + b\alpha + c\alpha^2 + d\alpha^3,\ a, b, c, d \in \mathbb{Z}\}$ since α is an algebraic integer), so $\mathbb{Z}_K \neq \mathbb{Z}[\alpha]$. So only the second and third possibilities remain.

   Alternative, quicker proof : $\beta = \alpha^3/2 \in \mathbb{Z}_K$ has a denominator divisible by the prime 2, so $\mathbb{Z}[\alpha]$ is not 2-maximal.

6. We find that $\alpha \cdot 1$, $\alpha \cdot \alpha$, $\alpha \cdot \alpha^2$ and $\alpha \cdot \beta$ all lie in $\mathcal{O}$, which as in question 3. proves that $\mathcal{O}$ is stable under multiplication by α. It is therefore also stable under multiplication by α^2, and also by β by question 3., and of course also by 1. Therefore, $\mathcal{O}$ is stable under multiplication by any $\mathbb{Z}$-linear combination of $1, \alpha, \alpha^2$ and β, i.e. $\mathcal{O} \cdot \mathcal{O} \subset \mathcal{O}$. As $1 \in \mathcal{O}$, this proves that $\mathcal{O}$ is a subring of K. By construction, $\mathcal{O}$ is also a lattice in K, so it is an order in K.

7. Thanks to the relation
$$\operatorname{disc} \mathbb{Z}[\alpha] = m^2 \operatorname{disc} K,$$
   finding $\operatorname{disc} K$ amounts to computing the index m of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K = \mathcal{O}$. We do so by writing a change-of-basis matrix between a $\mathbb{Z}$-basis of $\mathcal{O}$ and a $\mathbb{Z}$-basis of $\mathbb{Z}[\alpha]$.

   A $\mathbb{Z}$-basis of $\mathcal{O}$ is $1, \alpha, \alpha^2, \beta = \alpha^3/2$, and a $\mathbb{Z}$-basis of $\mathbb{Z}[\alpha]$ is $1, \alpha, \alpha^2, \alpha^3$ since α is an algebraic integer of degree $4 = \deg f$. (Technically we could use any $\mathbb{Z}$-bases, but why make things complicated?). The matrix expressing the latter in terms of the former is
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$
   whose determinant is clearly 2, so the index is
$$m \overset{\text{def}}{=} [\mathcal{O} : \mathbb{Z}[\alpha]] = 2.$$
   Therefore, we are in the second of the three cases listed in question 2., so
$$\operatorname{disc} K = 2^2 \cdot 997.$$
   In conclusion, the factor 2^4 of $\operatorname{disc} \mathbb{Z}[\alpha]$ actually came *both* from $\operatorname{disc} K$ and from the index !

   Note that we could also have expressed $1, \alpha, \alpha^2, \beta = \alpha^3/2$ in terms of $1, \alpha, \alpha^2, \alpha^3$; this would lead us to the inverse change-of-basis matrix
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$
   whose determinant is (up to sign) the inverse of the index. Doing things this way is sometimes easier. Also, don't worry if you can't remember if you should

express this basis in terms of that basis, or the other way round: you'll either get a matrix with integer coefficients, whose determinant is (up to sign) the index, or a matrix with rational coefficients, whose determinant is (up to sign) the inverse of the index, and since the index is an integer, the result that you get will tell you which case you are in!