Algebraic number theory — Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2022/MAU34109/index.html

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Email your answers to mascotn@tcd.ie by Wednesday November 02 noon.

Exercise 3.1: A cubic field (100 pts)

Let $f(x) = x^3 - 5x + 5$, and let $K = \mathbb{Q}(\alpha)$, where α is a root of $f(x)$.

1. (15 pts) Determine the degree $[K : \mathbb{Q}]$ and the ring of integers $\mathbb{Z}_K$ of K.
2. (10 pts) Which primes $p \in \mathbb{N}$ ramify in K ?
3. (25 pts) For each $n \in \mathbb{N}$, $n \leqslant 7$, compute explicitly the decomposition of $n\mathbb{Z}_K$ as a product of prime ideals.
4. (15 pts) Prove that the prime(s) above 5 are principal, and find an explicit generator for them.
5. (15pts) List all the ideals $\mathfrak{a}$ of $\mathbb{Z}_K$ such that $N(\mathfrak{a}) \leqslant 7$.
6. (20 pts) Factor the ideals $(\alpha - 2)$ and $(\alpha + 1)$ into primes.

   You may use without proof the result of Exercise 3.2 below (which BTW I really encourage you to solve).

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice (they may even give you inspiration to help you solve Exercise 1), and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 3.2: A useful formula

Let $A(x) \in \mathbb{Q}[x]$ be monic and irreducible, and consider the number field $K = \mathbb{Q}(\alpha)$ where $A(\alpha) = 0$. Prove that for all $n \in \mathbb{Z}$,

$$N_{\mathbb{Q}}^{K}(\alpha + n) = (-1)^{\deg A} A(-n).$$

Exercise 3.3: Ideals of fixed norm

1. How many ideals of norm 900 are there in the ring of integers of $\mathbb{Q}(\sqrt{7})$?

   Hint: Compute the decomposition in $\mathbb{Q}(\sqrt{7})$ *of the primes* $p \in \mathbb{N}$ *that divide 900.*

2. How many ideals of norm 80 are there in the ring of integers of $\mathbb{Q}(\zeta)$, where ζ is a primitive 60th root of unity?

Exercise 3.4: Similar-looking yet non-isomorphic number fields

The goal of this exercise is to prove that the number fields $\mathbb{Q}(\sqrt[3]{6})$ and $\mathbb{Q}(\sqrt[3]{12})$ have the same degree and discriminant, but are not isomorphic.

To ease notation, we let $\alpha = \sqrt[3]{6}$, $\beta = \sqrt[3]{12}$, $K = \mathbb{Q}(\alpha)$ and $L = \mathbb{Q}(\beta)$.

1. Prove that $[K : \mathbb{Q}] = 3$.

2. Prove that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ and compute $\operatorname{disc} K$.

3. Prove that $[L : \mathbb{Q}] = 3$ and that $\operatorname{disc} L$ is of the form $-2^a 3^5$ for some integer $a \geqslant 0$. What are the possible values of a ?

4. Prove that $L \simeq \mathbb{Q}(\sqrt[3]{18})$.

   Hint: Take a look at $\gamma = \beta^2/2$.

5. Deduce that $\operatorname{disc} L = \operatorname{disc} K$.

6. Which primes $p \in \mathbb{N}$ ramify in K? What about L?

7. Compute explicitly the decomposition of 7 in K and in L.

8. Deduce that K and L are not isomorphic.

9. Compute explicitly the decomposition of 2 and 3 in K and in L.

10. Deduce the factorisation of the ideals $\alpha\mathbb{Z}_K$, $\beta\mathbb{Z}_L$ and $\gamma\mathbb{Z}_L$ into primes.