Algebraic number theory — Exercise sheet 2

`https://www.maths.tcd.ie/~mascotn/teaching/2022/MAU34109/index.html`

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Email your answers to mascotn@tcd.ie by Wednesday October 19 noon.

Reminder: $\operatorname{disc}(x^n + bx + c) = (-1)^{n(n-1)/2}\big((1-n)^{n-1}b^n + n^n c^{n-1}\big)$.

Exercise 2.1: A cubic field (100pts)

Let $A(x) = x^3 - 5x + 1 \in \mathbb{Q}[x]$.

1. (10pts) Prove that $A(x)$ is irreducible over $\mathbb{Q}$.
2. (90pts) Let $K = \mathbb{Q}(\alpha)$, where $A(\alpha) = 0$. Determine the degree of K, the ring of integers of K, the discriminant of K, and the signature of K. Needless to say, justify your answers.

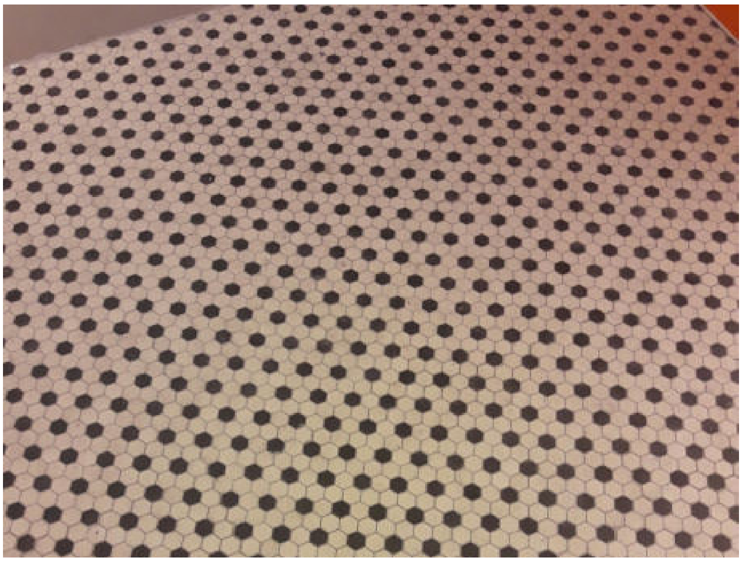
This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice (they may even give you inspiration to help you solve Exercise 1), and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2.2: To be or not to be integral

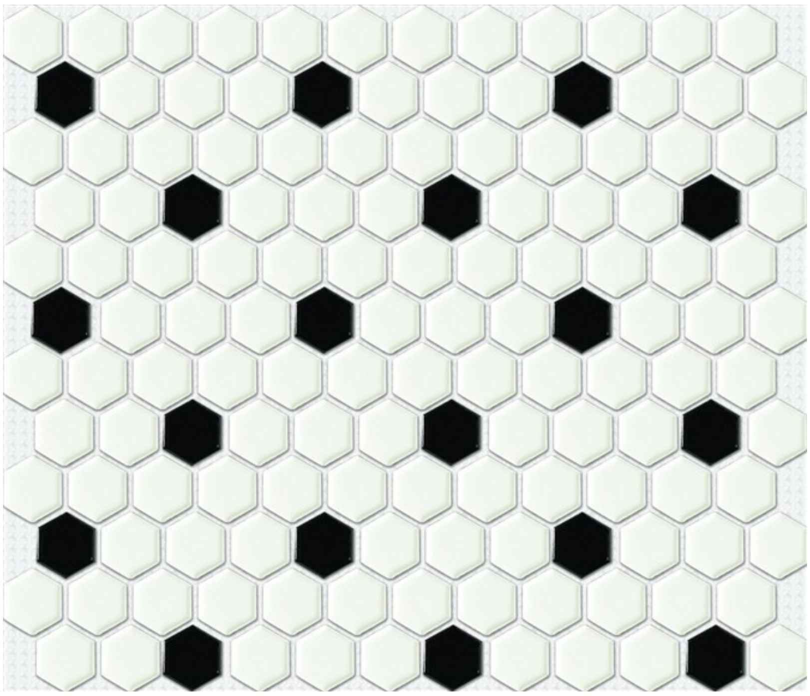
Is $\dfrac{3+2\sqrt{6}}{\sqrt{6}-2}$ an algebraic integer ?

Exercise 2.3: Floor tilings

1. In the picture below, (the centre of) the hexagonal floor tiles (both the black ones and the white ones) form a lattice, and (the centre of) the black tiles form a sublattice. Compute the index of this sublattice by writing down a change-of-basis (= transition) matrix. What is the proportion of black tiles?



2. Same questions for this other tiling pattern.



Exercise 2.4: A quartic field

Let $f(x) = x^4 - 2x + 4$, which you may assume without proof is irreducible over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha)$, where α satisfies $f(\alpha) = 0$.

1. Prove that $\mathbb{Z}[\alpha]$ is an order in K.
2. Compute and factor the discriminant of $\mathbb{Z}[\alpha]$.

   Hint: $2^{10} - 3^3 = 997$ *is prime.*
3. At this point, what are the possibilities for $\operatorname{disc} K$, and the corresponding values of the index of $\mathbb{Z}[\alpha]$?
4. Let $\beta = \frac{\alpha^3}{2} \in K$, and consider the lattice $\mathcal{O} \subset K$ with $\mathbb{Z}$-basis

   $$1, \alpha, \alpha^2, \beta.$$

   Prove that $\mathcal{O}$ is stable under multiplication by β.

   Hint: what is $\beta \cdot \alpha$ *?.*
5. Deduce that β is an algebraic integer.
6. Which of the possibilities listed in question 2. remain?
7. Prove that $\mathcal{O}$ is actually an order in K.

   Hint: Prove that $\mathcal{O}$ *is also stable under multiplication by* α.
8. It turns out that $\mathbb{Z}_K = \mathcal{O}$. Give the discriminant of K in factored form.