Algebraic number theory — Exercise sheet 1

`https://www.maths.tcd.ie/~mascotn/teaching/2022/MAU34109/index.html`

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Email your answers to mascotn@tcd.ie by Friday October 7 noon.

Exercise 1.1: Review of methods (26 pts)

Let $K = \mathbb{Q}(\sqrt{3})$, and let $\alpha = a + b\sqrt{3}$ ($a, b \in \mathbb{Q}$) be an element of K. Express the trace, norm, and characteristic polynomial of α (with respect to the extension $K/\mathbb{Q}$) in terms of a and b

1. (13 pts) by writing down the matrix of the multiplication-by-α map with respect to the $\mathbb{Q}$-basis of K of your choice,
2. (13 pts) by considering complex embeddings (*Hint: $K = \mathbb{Q}(\alpha)$ where $\alpha^2 = 3$*).

Remark: This exercise is meant as a warm-up for the next exercises, so as to remind you that some computations can be done in different ways. In the next exercises, remember that depending on the situation, some methods require less efforts than others!

Exercise 1.2: A biquadratic extension (74 pts)

1. (10 pts) Let $K = \mathbb{Q}(\sqrt{2})$. Prove that $\sqrt{5} \notin K$.
2. (10 pts) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. Determine $[L : \mathbb{Q}]$, and find a $\mathbb{Q}$-basis of L.
3. (10 pts) What is the signature of L?
4. (10 pts) Let $\alpha = \sqrt{2} + \sqrt{5} \in L$. Write down the matrix of
$$\begin{array}{rcl} \mu_\alpha : L & \longrightarrow & L \\ x & \longmapsto & \alpha x \end{array}$$
with respect to the $\mathbb{Q}$-basis of L that you found in question 2.
5. (4 pts) Deduce the value of $\mathrm{Tr}^L_{\mathbb{Q}}(\alpha)$.
6. (10 pts) Determine the norm $N^L_{\mathbb{Q}}(\alpha)$ of α.
Note: It is possible to do this without computing a big determinant.
7. (10 pts) The characteristic polynomial $\chi^L_{\mathbb{Q}}(\alpha)$ of α turns out to be $x^4 - 14x^2 + 9$ (we admit this without proof). Is it squarefree? What does this tell us about α?
8. (10 pts) Compute the characteristic polynomial $\chi^L_K(\alpha)$ of α with respect to the extension L/K.

These were the only mandatory exercises, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 1.3: Computations in $\mathbb{Q}(\sqrt[3]{2})$

1. Prove that $x^3 - 2$ is irreducible over $\mathbb{Q}$.
2. Let $K = \mathbb{Q}(\sqrt[3]{2})$, and let $\alpha = \frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1} \in K$. Find $a, b, c \in \mathbb{Q}$ such that $\alpha = a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$.
3. Are these rational numbers a, b, c unique ?
4. What is the degree of K over $\mathbb{Q}$?
5. Prove that $\sqrt{2} \notin K$.

   Hint: Think in terms of degrees.
6. Prove that $K = \mathbb{Q}(\alpha)$.
7. Compute the trace, norm, and characteristic polynomial of α.
8. What is the minimal polynomial of α over $\mathbb{Q}$?

Exercise 1.4: Resultant practice

1. Let $\gamma = \sqrt{3} - \sqrt[3]{2}$. Express a non-zero polynomial in $\mathbb{Q}[x]$ vanishing at γ as a resultant, and then express this resultant as a determinant.

   You are not required to compute this determinant explicitly. All this questions asks is to express this polynomial as a resultant, and then as a determinant, but you are not required to compute this polynomial explicitly.
2. Let $K = \mathbb{Q}(\alpha)$ be a number field, let $A(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α, and let $\beta = B(\alpha) \in K$, where $B(x) \in \mathbb{Q}[x]$ is some polynomial. Express the characteristic polynomial $\chi_{\mathbb{Q}}^{K}(\beta)$ of β in terms of a resultant involving A and B.

   Hint: Think in terms of complex embeddings.

Exercise 1.5: (Non)-repeated values of complex embeddings

Let K be a number field of degree $n = [K : \mathbb{Q}]$, let $\alpha \in K$, and let $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$.

Prove that α is a primitive element for $K/\mathbb{Q}$ if and only if the values $\{\sigma(\alpha), \sigma \in \mathrm{Hom}(K, \mathbb{C})\}$ are all distinct.

More generally (without assuming that α is a primitive element), relate the values $\{\sigma(\alpha), \sigma \in \mathrm{Hom}(K, \mathbb{C})\}$ to the roots of the characteristic polynomial and of the minimal polynomial of α, and explain what their multiplicities are. How many times is each value $\sigma(\alpha)$ repeated in $\{\sigma(\alpha), \sigma \in \mathrm{Hom}(K, \mathbb{C})\}$?