MAU34104
Group representations
2 - The module point of view

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Modules over a ring

Modules

Definition (Module, morphism of modules)

Let R be a ring. An <u>R-module</u> is a set M with two laws

$$\begin{array}{ccc} M \times M & \longrightarrow & M \\ (m, n) & \longmapsto & m + n, \end{array} \qquad \begin{array}{ccc} R \times M & \longrightarrow & M \\ (\lambda, m) & \longmapsto & \lambda m, \end{array}$$

such that $(M, +)$ is an Abelian group, and that for all $\lambda, \mu \in R$ and $m, n \in M$, we have

$$\begin{array}{rcl} \lambda(\mu m) = (\lambda\mu)m, & & 1m = m, \\ (\lambda + \mu)m = (\lambda m) + (\mu m), & & \lambda(m + n) = (\lambda m) + (\lambda n). \end{array}$$

A morphism $f : M \longrightarrow N$ between R-modules is an <u>R-linear</u> map, meaning

$$f(m + m') = f(m) + f(m') \text{ and } f(\lambda m) = \lambda f(m)$$

for all $m, m' \in M$ and $\lambda \in R$.

Modules: examples

Example

If R is actually a field, then R-module $=$ R-vector space, and module morphism $=$ linear transformation.

Example

Let R be a ring, and let $n \in \mathbb{N}$. Then

$$R^n = \{(x_1, \cdots, x_n) \mid x_i \in R\}$$

is an R-module.

Example

For $R = \mathbb{Z}$, $\mathbb{Z}$-module $=$ Abelian group:

$$ng = \underbrace{g + \cdots + g}_{n \text{ times}} \qquad (n \in \mathbb{Z}, g \in G)$$

and module morphism $=$ group morphism.

Modules: notation

Let R be a ring. Given two R-modules M, N, we denote the set of morphisms from M to N by $\mathrm{Hom}_R(M, N)$. It is actually an Abelian group for pointwise addition.

In the case $M = N$, we write $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$. It is actually a ring, where multiplication is given by composition.

Example

If R is actually a field, and if M is an R-vector space of dimension n, then $\mathrm{End}_R(M) \simeq \mathcal{M}_n(R)$.

Submodules and direct sums

Definition (Submodule)

Let M be an R-module. A submodule of M is a subset of M which is nonempty and closed under $+$ and under multiplication by R.

Example

Let $M = R$, viewed as an R-module. Then the submodules of M are the ideals of R.

Proposition

Let $f : M \longrightarrow N$ be a morphism of modules. Then $\operatorname{Ker} f$ is a submodule of M, and $\operatorname{Im} f$ is a submodule of N.

Direct sums

Definition (Direct sum)

Let M be an R-module, and $(M_i)_{i \in I}$ a collection of submodules of M. We say that $M = \bigoplus_{i \in I} M_i$ if every element $m \in M$ can be expressed as $m = \sum_{i \in I}^{\text{finite}} m_i$ with $m_i \in M_i$ in a unique way.

Example

$$R[x] = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} R x^n.$$

Remark

In the case of 2 submodules, $M = M_1 \oplus M_2$ iff. every $m \in M$ is of the form $m = m_1 + m_2$, $m_1 \in M_1$, $m_2 \in M_2$, and if $M_1 \cap M_2 = \{0\}$. Indeed, if $m = m_1 + m_2 = m_1' + m_2'$, then $m_1' - m_1 = m_2 - m_2' \in M_1 \cap M_2$.

Representations as modules

The group ring

Let K be a field, and X be a set. Recall that

$$K[X] = \left\{ \sum_{x\in X}^{\text{finite}} \lambda_x e_x \;\middle|\; \lambda_x \in K \right\} = \bigoplus_{x\in X} K e_x$$

is the K-vector space with basis $\{e_x\}_{x\in X}$ indexed by X.

If $X = G$ is actually a group, then $K[G]$ is actually a ring by the rule $e_g e_h = e_{gh}$ for all $g, h \in G$; thus

$$\left(\sum_{g\in G} \lambda_g e_g\right)\left(\sum_{g\in G} \mu_g e_g\right) = \sum_{g\in G} \left(\sum_{\substack{g_1,g_2\in G\\ g_1 g_2 = g}} \lambda_{g_1}\mu_{g_2} \right) e_g .$$

Remark

The ring $K[G]$ is commutative iff. G is Abelian.
It contains subring $\{\lambda e_{1_G},\ \lambda \in K\}$ which is a copy of K.

G-modules

A $K[G]$-module = a representation of G over K.
Indeed, a $K[G]$-module M is a K-vector space by

$$\lambda m = (\lambda e_{1_G})m \quad (\lambda \in K, m \in M),$$

and a representation by

$$\begin{array}{rcl} \rho : G & \longrightarrow & \mathrm{GL}(M) \\ g & \longmapsto & (m \mapsto e_g m). \end{array}$$

Conversely, given a K-vector space V and $\rho : G \longrightarrow \mathrm{GL}(V)$, we put a $K[G]$-module structure on V by

$$\left(\sum_{g \in G} \lambda_g e_g \right) v = \sum_{g \in G} \lambda_g \rho(g)(v) \quad (\lambda_g \in K, v \in V).$$

G-modules

A $K[G]$-module $=$ a representation of G over K.

Similarly, a sub-$K[G]$-module $=$ a subrepresentation,
and $K[G]$-module morphisms $=$ representation morphisms.

For example, we see immediately that if $f : V \longrightarrow W$ is a morphism of representations of G, then $\operatorname{Ker} f \subseteq V$ and $\operatorname{Im} f \subseteq W$ are subrepresentations.

Simple and indecomposable modules

Let R be a ring, and M an R-module.

Definition (Simple module)

M is simple if $M \neq \{0\}$ and its only submodules are $\{0\}$ and M.

Definition (Indecomposable module)

M is indecomposable if it cannot be expressed non-trivially as

$$M = \bigoplus_{i \in I} M_i.$$

So irreducible representations = simple $K[G]$-modules, and indecomposable = indecomposable.

Semi-simplicity

Semi-simple modules

Definition (Semi-simple module)

Let R be a ring. An R-module M is semi-simple if it can be decomposed as

$$M = \bigoplus_{i \in I} M_i$$

with the M_i simple R-submodules.

Since the M_i are simple, this is a "complete decomposition".

So saying that a representation is semi-simple means that it is "completely decomposable".

Supplements in a module

Definition (Supplement to a submodule)

Let R be a ring, M an R-module, and $N \subseteq M$ a submodule. A *supplement to N is a submodule $N' \subseteq M$ such that* $\overline{M = N \oplus N'}$.

This means that every $m \in M$ can be written as $m = n + n'$, $n \in N$, $n' \in N'$, and that $N \cap N' = \{0\}$.

Remark

In general, a supplement does not exist!

Counter-example

Take $R = \mathbb{Z}$, $M = \mathbb{Z}$, $N = 2\mathbb{Z}$. Then N does not have a supplement: if $\mathbb{Z} = 2\mathbb{Z} \oplus N'$, then given $n' \in N'$, we have $2n' \in N' \cap 2\mathbb{Z}$, absurd.

Semi-simplicity vs. supplements

Theorem

Let R be a ring, and M an R-module. Assume that there is no infinite chain $M \supseteq M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$ of submodules. Then

$$M \text{ semi-simple} \iff \text{Every submodule of } M \text{ has a supplement.}$$

Proof.

- $\Leftarrow$: If M is already simple, OK. Else let $\{0\} \subsetneq N \subsetneq M$ be a submodule. By assumption, it has a supplement N', so $M = N \oplus N'$. Iterate on N and N'. This terminates, else we would have an infinite descending chain.

$\square$

Semi-simplicity vs. supplements

Theorem

Let R be a ring, and M an R-module. Assume that there is no infinite chain $M \supseteq M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$ of submodules. Then

$$M \text{ semi-simple} \iff \text{Every submodule of } M \text{ has a supplement.}$$

Proof.

- $\Rightarrow$: Write $M = \bigoplus_{i \in I} M_i$ with the M_i simple submodules. Then I is finite, else we would have an infinite chain. If $N \subseteq M$ is a submodule, let $J \subseteq I$ be maximal among subsets such that $M' = N \oplus \bigoplus_{i \in J} M_i$ is still a direct sum. Claim: $M' = M$, so that $\bigoplus_{i \in J} M_i$ is a supplement of N. Indeed, else we would have $M_i \not\subseteq M'$ for at least one i, so $M' \cap M_i = \{0\}$ since M_i is simple. But then we could include i in J, contradiction. □

Maschke's theorem

Maschke's theorem

Theorem (Maschke)

Let K be a field, and let G be a finite group of order $n = \#G$. If $n \neq 0 \in K$, then every representation of G over K is semi-simple.

Counter-example

We have seen that the permutation representation induced by $S_3 \circlearrowleft \{1, 2, 3\}$ is not semisimple when $K = \mathbb{Z}/3\mathbb{Z}$.

Remark

In general, the decomposition is not unique! For example, if V has degree $n \geqslant 2$ and is equipped with the trivial action of G, then $V = \mathbb{1} \oplus \cdots \oplus \mathbb{1}$ in many ways.

Reminder: projections

Theorem

Let K be a field, and V a K-vector space.
If $V = V_1 \oplus V_2$, the projection on V_1 parallel to V_2 is

$$\begin{array}{rccc} \pi : & V & \longrightarrow & V \\ & v_1 + v_2 & \longmapsto & v_1. \end{array}$$

It is linear, and satisfies $\operatorname{Im}\pi = V_1$, $\operatorname{Ker}\pi = V_2$, and $\pi^2 = \pi$.

Conversely, if $\pi \in \operatorname{End}_K(V)$ satisfies $\pi^2 = \pi$, then

$$V = \operatorname{Im}\pi \oplus \operatorname{Ker}\pi,$$

and π is the projection on $\operatorname{Im}\pi$ parallel to $\operatorname{Ker}\pi$.

Proof of Maschke's theorem

Let V be a representation of G over K; for simplicity we assume $\dim V < \infty$. Then there cannot be an infinite chain

$$V \supseteq V_0 \supsetneq V_1 \supsetneq V_2 \cdots .$$

Let V_1 be a subrepresentation of G; we want to prove that it has a supplement.

Let $V_2 \subseteq V$ be such that $V = V_1 \oplus V_2$ as K-vector spaces, and let π be the projection on V_1 parallel to V_2.
NB in principle, V_2 need not be a subrepresentation, nor π a morphism of representations.

Define
$$\begin{array}{rccc} \Pi : & V & \longrightarrow & V \\ & v & \longmapsto & \displaystyle\frac{1}{\#G} \sum_{g \in G} g\pi(g^{-1}v). \end{array}$$

Proof of Maschke's theorem

Let V_1 be a subrepresentation of G; we want to prove that it has a supplement.

Let $V_2 \subseteq V$ be such that $V = V_1 \oplus V_2$ as K-vector spaces, and let π be the projection on V_1 parallel to V_2.

Define
$$\begin{array}{rccc} \Pi : & V & \longrightarrow & V \\ & v & \longmapsto & \displaystyle\frac{1}{\#G}\sum_{g\in G} g\pi(g^{-1}v). \end{array}$$

Since V_1 is a subrepresentation, it is stable by G;
therefore $g\pi(\cdots) \in V_1$, so $\operatorname{Im}\Pi \subseteq V_1$.
But also, for all $v_1 \in V_1$, we have $g^{-1}v_1 \in V_1$,
so $\pi(g^{-1}v_1) = g^{-1}v_1$, so $\Pi(v_1) = v_1$.
Hence $\operatorname{Im}\Pi = V_1$, and $\Pi^2 = \Pi$ is a projection so

$$V = \operatorname{Im}\Pi \oplus \operatorname{Ker}\Pi = V_1 \oplus \operatorname{Ker}\Pi.$$

Proof of Maschke's theorem

Define
$$\begin{array}{rccc} \Pi : & V & \longrightarrow & V \\ & v & \longmapsto & \dfrac{1}{\#G} \displaystyle\sum_{g \in G} g\pi(g^{-1}v). \end{array}$$

We have
$$V = \operatorname{Im} \Pi \oplus \operatorname{Ker} \Pi = V_1 \oplus \operatorname{Ker} \Pi.$$

Furthermore, Π is a morphism of representations.
Indeed, whenever $h \in G$ and $v \in V$, we have
$h\Pi(v) = h\frac{1}{\#G} \sum_{g \in G} g\pi(g^{-1}v) = \frac{1}{\#G} \sum_{g \in G} hg\pi(g^{-1}v) = \frac{1}{\#G} \sum_{g \in G} hg\pi((hg)^{-1}hv) = \frac{1}{\#G} \sum_{g \in G} g\pi(g^{-1}hv) = \Pi(hv)$,
as $\begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & hg \end{array}$ is a bijection.

Thus $V = V_1 \oplus \operatorname{Ker} \Pi$ with $\operatorname{Ker} \Pi$ a subrepresentation.