MAU34104
Group representations
1 - Introduction

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Group actions

Erlangen

Philosophy: Groups do not exist in the void, they are meant to move points of "spaces".

Example

- The linear group $\mathrm{GL}_n(\mathbb{R})$ ($n \in \mathbb{N}$) moves the points of the space $\mathbb{R}^n$.
- The symmetric group S_n ($n \in \mathbb{N}$) permutes the elements of $\{1, 2, \cdots, n\}$.
- The cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$ ($n \in \mathbb{N}$) can be thought of a group of rotations in the plane which stabilise a regular n-gon.
- Etc.

We get a much clearer understanding of groups this way!

Group actions

Definition (Group action)

Let G be a group, and X be a set. An <u>*action*</u> $G \circlearrowright X$ *of G on X is a map* $\begin{array}{ccc} G \times X & \longrightarrow & X \\ (g, x) & \longmapsto & g \cdot x \end{array}$ *such that* $(gh) \cdot x = g \cdot (h \cdot x)$ *and* $1_G \cdot x = x$ *for all* $g, h \in G$ *and* $x \in X$.

Example

A Rubik's cube is <u>not</u> a group, but a set X of configurations on which a group of motions G acts.

Remark

The axioms imply that for each $g \in G$, the map $g : \begin{array}{ccc} X & \longrightarrow & X \\ x & \longmapsto & g \cdot x \end{array}$ is a bijection, with inverse g^{-1}. Thus an action of G on X can also be defined as a morphism of G to the group of bijections $X \longrightarrow X$.

Group representations

Definition (Linear representation, degree)

Let K for a field, and $n \in \mathbb{N}$. A (linear) representation of G over K of degree n is a group morphism

$$\rho : G \longrightarrow \mathrm{GL}_n(K).$$

So a representation of a group G is an action of G on a set which is a vector space, and such that every $g \in G$ acts by linear transformations.

$\rightsquigarrow$ represent the elements of G by matrices, hence the name.

First examples of representations

Example

- $\mathrm{GL}_n(\mathbb{R}) \circlearrowright \mathbb{R}^n$ is actually a representation of $\mathrm{GL}_n(\mathbb{R})$.
- $\begin{array}{ccc} \mathbb{C}^\times & \longrightarrow & \mathrm{GL}_2(\mathbb{R}) \\ a+bi & \longmapsto & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \end{array}$ is a representation of $\mathbb{C}^\times$ of degree 2 over $\mathbb{R}$.
- $\begin{array}{ccc} K & \longrightarrow & \mathrm{GL}_2(K) \\ x & \longmapsto & \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \end{array}$ is a representation of $(K,+)$ of degree 2 over K.
- The trivial map $G \longrightarrow \mathrm{GL}_n(K)$ is a representation.

Alternative definition

Definition

Alternatively, a representation of G over K is a group morphism $\rho : G \longrightarrow \mathrm{GL}(V)$, where V is a K-vector space.

By choosing a basis of V, we get an isomorphism

$$\mathrm{GL}(V) \simeq \mathrm{GL}_{\dim V}(K),$$

so we recover the previous definition.

Remark

Instead of $\rho(g)(v)$, we often write $\rho_g(v)$, or even gv if ρ is clear from the context.

Permutation representations

Given a set X and a field K, we define the vector space

$$K[X] = \left\{ \sum_{x \in X}^{\text{finite}} \lambda_x e_x \;\middle|\; \lambda_x \in K \right\}$$

with basis $\{e_x, \ x \in X\}$ indexed by X.

If a group G acts on X, then we get a representation

$$\rho : G \longrightarrow \mathrm{GL}(K[X])$$

defined by $ge_x = e_{g \cdot x}$ for all $g \in G$, $x \in X$.
It is called the permutation representation attached to $G \circlearrowleft X$.
Its degree is $\#X$.

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Example

Let $G = S_3 \circlearrowleft X = \{1, 2, 3\}$.
Then for $\sigma = (123), \tau = (12) \in G$, we have

$$\rho(\sigma) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Definition (Regular representation)

In the special case where $X = G$ acts on itself by $g \cdot x = gx$, the corresponding representation $\rho : G \longrightarrow \mathrm{GL}(K[G])$ is called the regular representation of G over K.

Morphisms of representations

Equivalent representations

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation of degree n over K.
By picking a basis $\mathcal{B}_1$ of V, we get $\rho_1 : G \longrightarrow \mathrm{GL}_n(K)$.
If we pick another basis $\mathcal{B}_2$ of V, we get $\rho_2 : G \longrightarrow \mathrm{GL}_n(K)$, which is related to ρ_1 by

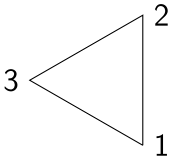
$$\rho_2(g) = P^{-1}\rho_1(g)P$$

for all $g \in G$, where P is the transition matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$.

Definition (Equivalent representations)

We say that ρ_1 and ρ_2 are equivalent.

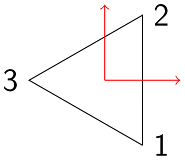
Example of equivalent representations



By labelling the vertices of an equilateral triangle by $1, 2, 3$, we get a representation $\rho : S_3 \longrightarrow \mathrm{GL}(\mathbb{R}^2)$. For example, for $\sigma = (123) \in S_3$ and $\tau = (12) \in S_3$, we have

$$\rho(\sigma) = \text{rotation}, \quad \rho(\tau) = \text{symmetry}.$$

Example of equivalent representations



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$$\rho(\sigma) = \text{rotation}, \quad \rho(\tau) = \text{symmetry}.$$

With respect to the standard basis of $\mathbb{R}^2$, we get

$$\rho_1 : S_3 \longrightarrow \mathrm{GL}_2(\mathbb{R}), \quad \rho_1(\sigma) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad \rho_1(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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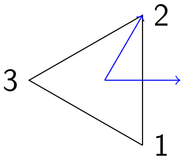

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But with respect to a better basis, we get the equivalent

$$\rho_2 : S_3 \longrightarrow \mathrm{GL}_2(\mathbb{R}), \quad \rho_2(\sigma) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2(\tau) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Morphisms of representations

Definition (Morphism of representations)

Let G be a group, K a field, and

$$\rho_1 : G \longrightarrow \mathrm{GL}(V_1), \quad \rho_2 : G \longrightarrow \mathrm{GL}(V_2)$$

be two representations of G over K.

A morphism from ρ_1 to ρ_2 is a linear map $T : V_1 \longrightarrow V_2$ such that

$$T\big(\rho_1(g)(v)\big) = \rho_2(g)\big(T(v)\big)$$

for all $g \in G$ and $v \in V_1$.

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$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "T"'] & V_1 \arrow[d, "T"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

for all $g \in G$.

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for all $g \in G$.

An isomorphism of representations is a bijective morphism.

Isomorphic representations are also called equivalent.

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The set of morphisms is a vector space denoted by $\mathrm{Hom}_G(V_1, V_2)$, or by $\mathrm{End}_G(V_1)$ if $V_1 = V_2$.

New from old

The trivial representation

Definition (Trivial representation)

Let G be a group, and K be a field. We may view $V = K$ as a vector space over itself.
The <u>trivial representation</u> $\mathbb{1}$ is the trivial morphism

$$\mathbb{1} : G \longrightarrow \mathrm{GL}(V).$$

Direct sum

Let V, W be representations of a group G over K. Then their direct sum $V \oplus W \simeq V \times W$ is also a representation of G, by the rule

$$g(v, w) = (gv, gw) \quad (g \in G, v \in V, w \in W).$$

Linear transformations

Let V, W be representations of a group G over K. Then the space $\mathrm{Hom}(V, W)$ of linear maps from V to W becomes a representation of G by the rule

$$(gT)(v) = g(T(g^{-1}v)) \quad (g \in G, T \in \mathrm{Hom}(V, W), v \in V).$$

Special case: if $V = W$, then we write $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

Special case: if $W = \mathbb{1}$, then $\mathrm{Hom}(V, W) = V^\vee$ is the linear dual of V, with G action

$$(g\ell)(v) = \ell(g^{-1}v) \quad (g \in G, \ell \in V^\vee, v \in V).$$

Irreducibility & indecomposability

Subrepresentations

Let V be a representation of G.

Definition (Subrepresentation)

A subrepresentation of V is a subspace $W \subseteq V$ which is stable by G.

Example

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$$

is a subrepresentation, which is isomorphic to a direct sum of copies of $\mathbb{1}$.

Remark

$$\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W).$$

Irreducible representations

Definition (Irreducible)

V is irreducible if its only subrepresentations are $\{0\}$ and V.

Counter-example

Consider again $G = S_3 \circlearrowright X = \{1, 2, 3\}$, and let $V = K[X]$ the corresponding permutation representation. The vector $v = e_1 + e_2 + e_3$ is invariant by G, so it spans a subrepresentation $W \subsetneq V$ of degree 1. Therefore V is not irreducible.

Note that actually, $W = V^G \simeq \mathbb{1}$.

Indecomposability

Definition

A representation V is indecomposable if is it not isomorphic to a direct sum $V_1 \oplus V_2$ of nontrivial representations $V_1, V_2 \neq \{0\}$.

So irreducible $\implies$ indecomposable.

Counter-example

The converse does not hold!

For instance, take $G = (K, +)$ and $\rho : \begin{array}{ccc} G & \longrightarrow & \mathrm{GL}_2(K) \\ x & \longmapsto & \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \end{array}.$

Then ρ is reducible, since the subspace of K^2 spanned by the first vector is a subrepresentation $\simeq \mathbb{1}$.
But ρ is indecomposable, because this subrepresentation does not have a supplement.

Effect on matrices

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation.

- ρ is reducible iff. there exists a basis of V such that

$$\forall g \in G, \ \rho(g) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

- ρ is decomposable iff. there exists a basis of V such that

$$\forall g \in G, \ \rho(g) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Effect on matrices

Let $\rho : G \longrightarrow \mathrm{GL}_n(K)$ be a representation.

- ρ is reducible iff. there exists $P \in \mathrm{GL}_n(K)$ such that

$$\forall g \in G,\ P^{-1}\rho(g)P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

- ρ is decomposable iff. there exists a basis of V such that

$$\forall g \in G,\ P^{-1}\rho(g)P = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

An example of (in)decomposition

Example

Consider once more the permutation representation $V = K[X]$ attached to $G = S_3 \circlearrowright X = \{1, 2, 3\}$.

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Consider $W' = \left\{ \sum_{x \in X} \lambda_x e_x \in V \;\middle|\; \sum_{x \in X} \lambda_x = 0 \right\}$.
It is also a subrepresentation.

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If $3 \neq 0$ in K, then $V = W \oplus W'$, so V is decomposable.

But if $K = \mathbb{Z}/3\mathbb{Z}$, then $W \subset W'$, and in fact V is indecomposable.