MAU34104
Group representations
5 - Induced representations

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Module web page

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Restriction and induction

Setup

Let G be a finite group, and let $H \leq G$ be a subgroup.

Definition (Restriction)

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation of character χ. We may restrict it into $\rho_{|H} : H \longrightarrow \mathrm{GL}(V)$. We denote this restriction by $\mathrm{Res}_H^G \rho$, and its character by $\mathrm{Res}_H^G \chi$.

Remark

$\mathrm{Res}_H^G \rho$ may be reducible even if ρ was irreducible!

Setup

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We want some sort of reverse construction:
Given $\rho : H \longrightarrow \mathrm{GL}(W)$, we would like to define the induced representation $\mathrm{Ind}_H^G \rho$, which would be a representation of G.

Remark (Induction $\neq$ inflation)

Do not confuse this with inflation, which consists in viewing a representation of a quotient G/N as a representation of G!

$$G \longrightarrow G/N \longrightarrow \mathrm{GL}(V)$$

Definition(s) of the induced representation

Right transversals

Let $H \leq G$.

Definition (Right transversal)

A right transversal of H in G is a subset $T \subseteq G$ such that

$$G = \coprod_{t \in T} Ht.$$

In other words, each $g \in G$ can be uniquely decomposed as $g = h_g t_g$ with $h_g \in H$ and $t_g \in T$.

We may and do make the assumption that $1_G \in T$.
Then the decomposition of every $h \in H$ is $h = h 1_G$.
Actually, for all $g \in G$, $t_g = 1_G$ iff. $g \in H$.

Explicit but dirty construction of Ind

Let W be a representation of H.
Idea: we will define $\operatorname{Ind}_H^G W$ as a product of copies of W indexed by T, so that if $g = ht$ with $h \in H$, $t \in T$, then for all $w \in W$, $g \cdot w$ is $h_g \cdot w$ in the copy of W indexed by t_g.

Definition (Induced representation, first version)

$$\operatorname{Ind}_H^G W = \prod_{t \in T} W_t = \bigoplus_{t \in T} W_t$$

where $W_t = W$ for all t.

Theorem

The formula

$$g \cdot w_t \underset{tg=ht'}{=} h \cdot w_{t'} \quad (h \in H; t, t' \in T)$$

makes $\operatorname{Ind}_H^G W$ a representation of G.

Explicit but dirty construction of Ind

Definition (Induced representation, first version)

$$\operatorname{Ind}_H^G W = \prod_{t \in T} W_t = \bigoplus_{t \in T} W_t$$

where $W_t = W$ *for all* t.

Theorem

The formula

$$g \cdot w_t \underset{tg = ht'}{=} h \cdot w_{t'} \quad (h \in H; t, t' \in T)$$

makes $\operatorname{Ind}_H^G W$ *a representation of* G.

Proof.

Doable, but we will give a better one later. $\square$

Clean but abstract construction of Ind

Let W be a representation of H.

Definition (Induced representation, second version)

$\operatorname{Ind}_H^G W = \{f : G \to W \mid f(hx) = h \cdot f(x) \text{ for all } h \in H, x \in G\}.$

Theorem

The formula

$$(g \cdot f)(x) = f(xg) \quad (g, x \in G)$$

makes $\operatorname{Ind}_H^G W$ *a representation of* G.

Clean but abstract construction of Ind

Definition (Induced representation, second version)

$\mathrm{Ind}_H^G W = \{f : G \to W \mid f(hx) = h \cdot f(x) \text{ for all } h \in H, x \in G\}$, *with* $(g \cdot f)(x) = f(xg) \quad (g, x \in G)$.

Proof.

Note that $\mathrm{Ind}_H^G W$ is a $\mathbb{C}$-vector space.
Let $f \in Ind_H^G W$. Then for all $g, x \in G$ and $h \in H$,
$(g \cdot f)(hx) = f(hxg) = h \cdot f(xg) = h \cdot \big((g \cdot f)(x)\big)$,
so $g \cdot f \in \mathrm{Ind}_H^G W$.
Besides, this is a G-action: for all $f \in \mathrm{Ind}_H^G W$,
clearly $1_G \cdot f = f$, and also $g_1 \cdot (g_2 \cdot f) = (g_1 g_2) \cdot f$ for all $g_1, g_2 \in G$, as $\big(g_1 \cdot (g_2 \cdot f)\big)(x) = (g_2 \cdot f)(xg_1) = f(xg_1g_2)$
$= \big((g_1g_2) \cdot f\big)(x)$ for all $x \in G$.
And this G-action is linear: for $g \in G$, $f_1, f_2 \in \mathrm{Ind}_H^G W$, $\lambda \in \mathbb{C}$,
$g \cdot (f_1 + f_2) = g \cdot f_1 + g \cdot f_2$ and $g \cdot (\lambda f_1) = \lambda (g \cdot f_1)$. □

Equivalence of these constructions

Theorem ($\operatorname{Ind}_H^G W$ well-defined up to isomorphism)

These two constructions of $\operatorname{Ind}_H^G W$ yield equivalent representations.

Proof.

$$\begin{array}{ccc} \{f : G \to W \mid f(hx) = h \cdot f(x)\} & \longleftrightarrow & \prod_{t \in T} W_t \\ f & \longmapsto & \big(f(t)\big)_{t \in T} \\ (f_{\vec{w}} : x \mapsto h_x w_{t_x}) & \longleftarrow & \vec{w} = (w_t)_{t \in T} \end{array}$$

are well-defined, $\mathbb{C}$-linear, and inverses of each other.
$\rightsquigarrow$ we may define a G-action on $\prod_{t \in T} W_t$ by $f_{g \cdot \vec{w}} = g \cdot f_{\vec{w}}$:

$$\begin{aligned} (g \cdot \vec{w})_t &= f_{g \cdot \vec{w}}(t) = (g \cdot f_{\vec{w}})(t) = f_{\vec{w}}(tg) \\ &\underset{tg = ht'}{=} f_{\vec{w}}(ht') = h \cdot f_{\vec{w}}(t') = h \cdot w_{t'}. \end{aligned}$$

$\square$

Equivalence of these constructions

Theorem ($\mathrm{Ind}_H^G W$ well-defined up to isomorphism)

These two constructions of $\mathrm{Ind}_H^G W$ yield equivalent representations.

Remark

The degree of the induced representation is

$$\deg \mathrm{Ind}_H^G W = (\# T) \deg W = [G : H] \deg W,$$

and its restriction $\mathrm{Res}_H^G \mathrm{Ind}_H^G W$ contains a copy of the representation W.

Example: permutation representations

Permutation representations as Ind $\mathbb{1}$

Let G be a group acting transitively on a set $X \neq \emptyset$, meaning that for all $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$.

Fix $x_0 \in X$, let $H = \{g \in G \mid g \cdot x_0 = x_0\}$.
For each $x \in X$, pick $g_x \in G$ such that $g_x \cdot x_0 = x$;
the other choices are the $g_x h$, $h \in H$.
We may and do assume that $g_{x_0} = 1_G$. Then

$$\begin{array}{ccc} G/H & \longleftrightarrow & X \\ gH & \longmapsto & g \cdot x_0 \\ g_x H & \longleftarrow\!\shortmid & x, \end{array}$$

so $G = \coprod_{x \in X} g_x H$: the g_x form a left transversal.

Permutation representations as Ind $\mathbb{1}$

Let G be a group acting transitively on a set $X \neq \emptyset$, meaning that for all $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$.

Fix $x_0 \in X$, let $H = \{g \in G \mid g \cdot x_0 = x_0\}$, and for each $x \in X$, pick $t_x = g_x^{-1} \in G$ such that $t_x \cdot x = x_0$; the other choices are the ht_x, $h \in H$. We may and do assume that $t_{x_0} = 1_G$. Then

$$\begin{array}{ccc} H \backslash G & \longleftrightarrow & X \\ Hg & \longmapsto & g^{-1} \cdot x_0 \\ Ht_x & \longleftarrow & x, \end{array}$$

so $G = \coprod_{x \in X} Ht_x$: the t_x form a right transversal.

Permutation representations as Ind $\mathbb{1}$

From the trivial representation $\mathbb{1}_H$ of H, we get

$$\operatorname{Ind}_H^G \mathbb{1}_H = \bigoplus_{x \in X} \mathbb{C}_x = \prod_{x \in X} \mathbb{C}_x$$

which is a representation of G by the rule

$$g \cdot z_x \underset{t_x g = h t_y}{=} h \cdot z_y = z_y.$$

But $t_x g = h t_y$ means that $g = t_x^{-1} h t_y = g_x h t_y$ takes y to x, so that $y = g^{-1} \cdot x$.

In conclusion, g acts on $\operatorname{Ind}_H^G \mathbb{1}_H = \bigoplus_{x \in X} \mathbb{C}_x$ by permuting the coordinates by g^{-1}, so actually

$$\operatorname{Ind}_H^G \mathbb{1}_H = \mathbb{C}[X]$$

is the <u>permutation representation</u> attached to $G \circlearrowright X$.

Induced characters

Extension by 0

Let W be a representation of H of character $\chi : H \longrightarrow \mathbb{C}$.
What is the character $\operatorname{Ind}_H^G \chi : G \longrightarrow \mathbb{C}$ of $\operatorname{Ind}_H^G W$?

Definition (Extension by 0)

The extension of χ by 0 is

$$\begin{array}{rccl} \chi^0 : & G & \longrightarrow & \mathbb{C} \\ & g & \longmapsto & \begin{cases} \chi(g), & \text{if } g \in H \\ 0, & \text{else.} \end{cases} \end{array}$$

Warning

In general, this is not the character of $\operatorname{Ind}_H^G W$.

Extension by 0

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Lemma

For all $g \in G$ and $h \in H$, $\chi^0(hgh^{-1}) = \chi^0(g)$.

Proof.

If $g \notin H$, then $g' = hgh^{-1} \notin H$ either, else $g = h^{-1}g'h \in H$; so $0 = 0$, OK.
Else, if $g \in H$, then $\chi^0(hgh^{-1}) = \chi(hgh^{-1}) = \chi(g) = \chi^0(g)$ as χ is a class function on H. $\square$

Extension by 0

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Lemma

For all $g \in G$ and $h \in H$, $\chi^0(hgh^{-1}) = \chi^0(g)$.

Warning

In general, this does not hold for $h \in G$.

First formula for induced characters

Lemma (First formula for induced characters)

For all $g \in G$,

$$\operatorname{Ind}_H^G \chi(g) = \frac{1}{\# H} \sum_{x \in G} \chi^0(xgx^{-1}).$$

First formula for induced characters: proof

Let $(e_i)_{i \in I}$ be a basis of W. Then $(e_{i,t})_{\substack{i \in I \\ t \in T}}$ is a basis of $\operatorname{Ind}_H^G W = \bigoplus_{t \in T} W_t$, so for all $g \in G$,

$$\operatorname{Ind}_H^G \chi(g) = \sum_{t \in T} \sum_{i \in I} \text{coef. of } e_{i,t} \text{ in } g \cdot e_{i,t}.$$

But $g \cdot e_{i,t} = h \cdot e_{i,t'} \in W_{t'}$ where $tg = ht'$, $h \in H$, $t' \in T$, so this coef. is 0 unless $t = t'$, that is to say $tgt^{-1} \in H$, and in this case $g \cdot e_{i,t} = h \cdot e_{i,t}$ where $h = tgt^{-1} \in H$, so
$\displaystyle\sum_{i \in I} \text{coef. of } e_{i,t} \text{ in } ge_{i,t} = \sum_{i \in I} \text{coef. of } e_{i,t} \text{ in } h \cdot e_{i,t} = \operatorname{Tr}(h|W_t) = \chi(h).$

$$\rightsquigarrow \operatorname{Ind}_H^G \chi(g) = \sum_{t \in T} \chi^0(tgt^{-1}) = \frac{1}{\#H} \sum_{t \in T} \sum_{h \in H} \chi^0(htgt^{-1}h^{-1})$$

$$\underset{x=ht}{=} \frac{1}{\#H} \sum_{x \in G} \chi^0(xgx^{-1}) \text{ as } G = \coprod_{t \in T} Ht. \qquad \square$$

Reminders on group actions

Let G be a finite group acting on a finite set X. For each $x \in X$, we let

$$G \cdot x = \{g \cdot x, g \in G\} \subseteq X, \quad G_x = \{g \in G \,|g \cdot x = x\} \leq G.$$

As $\begin{array}{ccc} G & \longrightarrow & G \cdot x \\ g & \longmapsto & g \cdot x \end{array}$ is $\#G_x$-to-1, we have

$$\#G = \#G_x \times \#(G \cdot x).$$

Example

Let G act on $X = G$ by $g \cdot x = gxg^{-1}$. Then $G \cdot x$ is the conjugacy class of X, and $G_x = C_G(x) = \{g \in G \mid gx = xg\}$ is called the <u>centraliser</u> of x in G.

Second formula(e) for induced characters

Let G be a group, $g \in G$, and $K \leq G$. Define
$cc_K(g) = \{kgk^{-1}, k \in K\} \subseteq G, \quad C_K(g) = \{k \in K \mid kg = gk\} \leq K.$

Let $H \leq G$. We have $cc_G(g) = \coprod_i cc_H(g_i)$ for some $g_i \in G$.

If $g_i \in H$, then $cc_H(g_i) \subseteq H$; if $g_i \notin H$, then $cc_H(g_i) \cap H = \emptyset$.

$$\rightsquigarrow cc_G(g) \cap H = \coprod_j cc_H(h_j) \text{ for some } h_j \in H.$$

Theorem (Second formula for induced characters)

Let χ be a character on $H \leq G$. For all $g \in G$,

$$\operatorname{Ind}_H^G \chi(g) = \#C_G(g) \sum_j \frac{\chi(h_j)}{\#C_H(h_j)} = \frac{[G:H]}{\#cc_G(g)} \sum_j \#cc_H(h_j)\chi(h_j),$$

where $cc_G(g) \cap H = \coprod_j cc_H(h_j)$.

Second formula(e) for induced characters

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where $cc_G(g) \cap H = \coprod_j cc_H(h_j)$.

Proof.

$$\operatorname{Ind}_H^G \chi(g) = \frac{1}{\#H} \sum_{x \in G} \chi^0(xgx^{-1}) = \frac{\#C_G(g)}{\#H} \sum_{g' \in cc_G(g)} \chi^0(g')$$

$$= \frac{\#C_G(g)}{\#H} \sum_j \sum_{y \in cc_H(h_j)} \chi(y) = \#C_G(g) \sum_j \frac{\#cc_H(h_j)}{\#H} \chi(h_j). \quad \square$$

Second formula(e) for induced characters

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where $cc_G(g) \cap H = \coprod_j cc_H(h_j)$.

Example

Let $G = S_3$, $\sigma = (123), \sigma' = (132), \tau = (12), \tau' = (23), \tau'' = (13)$, $H = \{\mathrm{Id}, \tau\} \leq G$, and $\chi \in \mathrm{Irr}(H)$: $\chi(\mathrm{Id}) = 1, \chi(\tau) = -1$.

- $cc_G(\tau') \cap H = \{\tau, \tau', \tau''\} \cap H = \{\tau\} = cc_H(\tau)$
  $\rightsquigarrow \operatorname{Ind}_H^G \chi(\tau') = \dfrac{[G:H]}{\#cc_G(\tau')} \#cc_H(\tau)\chi(\tau) = -1.$
- $cc_G(\sigma) \cap H = \{\sigma, \sigma'\} \cap H = \emptyset$
  $\rightsquigarrow \operatorname{Ind}_H^G \chi(\sigma) = 0$ as $\sum_\emptyset = 0$.

Frobenius reciprocity

Frobenius reciprocity

Theorem (Frobenius reciprocity)

Let ϕ be a character on H, and ψ be a character on G. Then

$$(\operatorname{Ind}_H^G \phi \mid \psi)_G = (\phi \mid \operatorname{Res}_H^G \psi)_H.$$

Frobenius reciprocity: proof

$$
\begin{aligned}
(\operatorname{Ind}_H^G \phi \mid \psi)_G &= \frac{1}{\#G} \sum_{g \in G} \left(\operatorname{Ind}_H^G \phi(g) \right) \overline{\psi(g)} \\
&= \frac{1}{\#G} \sum_{g \in G} \left(\frac{1}{\#H} \sum_{x \in G} \phi^0(xgx^{-1}) \right) \overline{\psi(g)} \\
&= \frac{1}{\#G\#H} \sum_{x \in G} \sum_{g \in G} \phi^0(xgx^{-1}) \overline{\psi(g)} \\
&\underset{\substack{y = xgx^{-1} \\ g = x^{-1}yx}}{=} \frac{1}{\#G\#H} \sum_{x \in G} \sum_{y \in G} \phi^0(y) \overline{\psi(x^{-1}yx)} \\
&= \frac{1}{\#G\#H} \sum_{y \in G} \sum_{x \in G} \phi^0(y) \overline{\psi(y)} = \frac{1}{\#H} \sum_{y \in G} \phi^0(y) \overline{\psi(y)} \\
&= \frac{1}{\#H} \sum_{y \in H} \phi(y) \overline{\psi(y)} = (\phi \mid \operatorname{Res}_H^G \psi)_H. \qquad \square
\end{aligned}
$$

Frobenius reciprocity: example

Take $G = S_3$,
$H = \{\mathrm{Id}, \tau\} \leq G$
where $\tau = (12)$.

Recall the character table of H:

	Id	(12)	(123)
	1	3	2
$\mathbb{1}_G$	1	1	1
ε	1	-1	1
$\lhd$	2	0	-1.

For any character χ of H, we have

$$(\mathrm{Ind}_H^G \chi \mid \mathbb{1}_G)_G = (\chi \mid \mathrm{Res}_H^G \mathbb{1}_G)_H = \frac{\chi(\mathrm{Id}) + \chi(\tau)}{2},$$

$$(\mathrm{Ind}_H^G \chi \mid \varepsilon)_G = (\chi \mid \mathrm{Res}_H^G \varepsilon)_H = \frac{\chi(\mathrm{Id}) - \chi(\tau)}{2},$$

$$(\mathrm{Ind}_H^G \chi \mid \lhd)_G = (\chi \mid \mathrm{Res}_H^G \lhd)_H = \frac{2\chi(\mathrm{Id}) + 0\chi(\tau)}{2} = \deg \chi.$$

$H \simeq \mathbb{Z}/2\mathbb{Z} \rightsquigarrow \mathrm{Irr}(H) = \{\mathbb{1}_H, \psi\}$, $\psi(\mathrm{Id}) = 1$, $\psi(\sigma) = -1$.

We find that $\mathrm{Ind}_H^G \mathbb{1}_H = \mathbb{1}_G + \lhd$ and that $\mathrm{Ind}_H^G \psi = \varepsilon + \lhd$.

All reps from induced reps

Corollary (Get all reps of G from ind reps from any H)

Let G be a finite group, and let $\chi \in \operatorname{Irr}(G)$.
For all $H \leq G$, there exists $\psi \in \operatorname{Irr}(H)$ such that $\operatorname{Ind}_H^G \psi$ contains χ as an irreducible component.

Proof.

Let ψ be an irreducible component of $\operatorname{Res}_H^G \chi$. Then

$$(\operatorname{Ind}_H^G \psi \mid \chi)_G = (\psi \mid \operatorname{Res}_H^G \chi)_H \geq 1. \qquad \square$$

All reps from induced reps

Example

Take $G = S_4 \geq H = S_3$. Recall the character tables:

S_3	Id	(12)	(123)
# class	1	3	2
$\mathbb{1}_{S_3}$	1	1	1
ε_{S_3}	1	-1	1
$\triangleleft$	2	0	-1

S_4	Id	(12)	(123)	(1234)	(12)(34)
# class	1	6	8	6	3
$\mathbb{1}_{S_4}$	1	1	1	1	1
ε_{S_4}	1	-1	1	-1	1
ψ	2	0	-1	0	2
χ	3	1	0	-1	-1
$\chi\varepsilon_{S_4}$	3	-1	0	1	-1

- $\operatorname{Ind}_{S_3}^{S_4} \mathbb{1}_{S_3} = \mathbb{1}_{S_4} + \chi$,
- $\operatorname{Ind}_{S_3}^{S_4} \varepsilon_{S_3} = \varepsilon_{S_4} + \chi\varepsilon_{S_4}$,
- $\operatorname{Ind}_{S_3}^{S_4} \triangleleft = \psi + \chi + \chi\varepsilon_{S_4}$.