MAU34104
Group representations
3 - Character theory

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Schur's lemma

Schur's lemma

Theorem (Schur's lemma)

Let R be a ring, and let M_1, M_2 be simple R-modules. Then any module morphism from M_1 to M_2 is either 0 or an isomorphism.

Proof.

Let $f : M_1 \longrightarrow M_2$ be a morphism. Then $\operatorname{Ker} f \subseteq M_1$ is a submodule, so it is either $\{0\}$ or M_1.
If $\operatorname{Ker} f = M_1$, then $f = 0$.
Else, $\operatorname{Ker} f = \{0\}$, so f is injective, so $\operatorname{Im} f \subseteq M_2$ is a nonzero submodule. Thus $\operatorname{Im} f = M_2$, so f is also surjective. $\square$

Schur's lemma

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Let R be a ring, and let M_1, M_2 be simple R-modules. Then any module morphism from M_1 to M_2 is either 0 or an isomorphism.

Corollary (Non-examinable)

If M is simple, then $\operatorname{End}(M)$ is a division ring.

Schur's lemma

Corollary (Schur's lemma for representations over $\mathbb{C}$)

Let G be a group, and let V, W be irreducible representations of G over $K = \mathbb{C}$. Then

$$\operatorname{Hom}_G(V, W) = \begin{cases} \{0\} & \text{if } V \not\simeq W, \\ \{\lambda \operatorname{Id},\ \lambda \in \mathbb{C}\} & \text{if } V = W. \end{cases}$$

Proof.

Let $f \in \operatorname{End}_G(V)$. Since $\mathbb{C}$ is algrebraically closed, f has at least one eigenvalue λ.
As $\operatorname{Id} \in \operatorname{End}_G(V)$, we also have $f - \lambda \operatorname{Id} \in \operatorname{End}_G(V)$.
Since $f - \lambda \operatorname{Id}$ is not injective, it must be 0. $\square$

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As $\operatorname{Id} \in \operatorname{End}_G(V)$, we also have $f - \lambda \operatorname{Id} \in \operatorname{End}_G(V)$.
Since $f - \lambda \operatorname{Id}$ is not injective, it must be 0. $\square$

From now on, we will only consider representations of finite groups G over $K = \mathbb{C}$ of finite degree. In particular, Maschke applies.

Reminders on dot products

Dot products over $\mathbb{R}$

Definition

Let V be an $\mathbb{R}$-vector space. A dot product over V is a map

$$\begin{array}{ccc} V \times V & \longrightarrow & \mathbb{R} \\ (v, w) & \longmapsto & (v|w) \end{array}$$

which is bilinear: for all $\lambda \in \mathbb{R}$, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$,

$$(v_1 + v_2|w) = (v_1|w) + (v_2|w), \quad (\lambda v|w) = \lambda(v|w),$$
$$(v|w_1 + w_2) = (v|w_1) + (v|w_2), \quad (v|\lambda w) = \lambda(v|w),$$

symmetric: $(v|w) = (w|v)$,

and such that $(v|v) \geqslant 0$, with equality only when $v = 0$.

Example

- On $V = \mathbb{R}^n$, usual dot product $(v|w) = \sum_{k=1}^n v_k w_k$.
- On $V = \mathbb{R}[x]$, we can define the dot product

$$(P|Q) = \int_0^1 P(x)Q(x)dx.$$

Dot products over $\mathbb{C}$

Definition

Let V be an $\mathbb{C}$-vector space. A dot product over V is a map

$$\begin{array}{ccc} V \times V & \longrightarrow & \mathbb{C} \\ (v, w) & \longmapsto & (v|w) \end{array}$$

which is sesquilinear: for $\lambda \in \mathbb{C}$, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$,

$$(v_1 + v_2|w) = (v_1|w) + (v_2|w), \quad (\lambda v|w) = \lambda(v|w),$$
$$(v|w_1 + w_2) = (v|w_1) + (v|w_2), \quad (v|\lambda w) = \overline{\lambda}(v|w),$$

conjugate-symmetric: $(v|w) = \overline{(w|v)}$,

and such that $(v|v) \in \mathbb{R}_{\geqslant 0}$, with equality only when $v = 0$.

Example

- On $V = \mathbb{C}^n$, usual dot product $(v|w) = \sum_{k=1}^n v_k \overline{w_k}$.
- On $V = \mathbb{C}[x]$, we can define the dot product

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Orthonormality

Definition (Orthogonal, orthonormal)

Let V be a vector space with a dot product over $\mathbb{R}$ or $\mathbb{C}$, and let $(v_j)_{j\in J}$ be a family of elements of V.

- *The v_j are orthogonal if $(v_j|v_k) = 0$ for all $j \neq k$.*
- *They are orthonormal if furthemore $(v_j|v_j) = 1$ for all j.*

Proposition

If $(e_j)_{j\in J}$ if a basis of V which is orthonormal, then the coordinates of any vector $v \in V$ may be recovered as $(v|e_j)$.

Proof.

If $v = \sum_{j\in J} \lambda_j e_j$, then

$$(v|e_k) = \left(\sum_{j\in J} \lambda_j e_j \,\middle|\, e_k\right) = \sum_{j\in J} \lambda_j (e_j|e_k) = \lambda_k. \qquad \square$$

The character of a representation

Conjugagy and class functions

Let G be a group.

Definition (Conjugacy)

Two elements $g, g' \in G$ are conjugate if $g' = hgh^{-1}$ for some $h \in G$.
The conjugacy class of $g \in G$ is the set of its conjugates.

Definition (Class function)

A class function is a function $\psi : G \longrightarrow \mathbb{C}$ such that

$$\psi(hgh^{-1}) = \psi(g)$$

for all $g, h \in G$.

Class functions form a $\mathbb{C}$-vector space, whose dimension is the number of conjugacy classes in G, and which we equip with the dot product $(\phi \mid \psi) = \dfrac{1}{\#G} \sum_{g \in G} \phi(g)\overline{\psi(g)}$.

Reminder: traces

Definition (Trace)

The trace of a square matrix is the sum of its diagonal elements.

Example

$$\mathrm{Tr}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = 1 + 4 = 5.$$

Reminder: traces

Definition (Trace)

The trace of a square matrix is the sum of its diagonal elements.

Proposition

Tr *is linear:* $Tr(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$, $\mathrm{Tr}(\lambda A) = \lambda \, \mathrm{Tr}(A)$.
Furthermore $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, *whence* $\mathrm{Tr}(BAB^{-1}) = \mathrm{Tr}(A)$.

Corollary

We can define the trace of a linear map $T : V \longrightarrow V$.

We record:

$$\mathrm{Tr}\, T = \sum_i \text{coeff. of } e_i \text{ in } T(e_i)$$

for any basis $(e_1, \cdots, e_d)$ of V.

The character of a representation

Definition (Character)

The character of a representation $\rho : G \longrightarrow \mathrm{GL}(V)$ *is*

$$\begin{array}{rcl} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \operatorname{Tr} \rho(g). \end{array}$$

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Proposition

The character of a representation is a class function.

Proof.

$$\operatorname{Tr} \rho(hgh^{-1}) = \operatorname{Tr} \left(\rho(h)\rho(g)\rho(h)^{-1}\right) = \operatorname{Tr} \rho(g).$$ □

The character of a representation

Definition (Character)

The *character* *of a representation* $\rho : G \longrightarrow \mathrm{GL}(V)$ *is*

$$\begin{array}{rcl} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \operatorname{Tr} \rho(g). \end{array}$$

Proposition

If two representations are equivalent, then they have the same character.

Proof.

$\operatorname{Tr}\left(P^{-1}\rho(g)P\right) = \operatorname{Tr}\rho(g).$ □

The character of a representation

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The character of a representation $\rho : G \longrightarrow \mathrm{GL}(V)$ *is*

$$\begin{array}{rcl} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \operatorname{Tr} \rho(g). \end{array}$$

Proposition

The character of the trivial representation $\mathbb{1}$ *is the constant function* 1 *from* G *to* $\mathbb{C}$.

We still denote it by $\mathbb{1}$.

The character of a representation

Definition (Character)

The *<u>character</u>* *of a representation* $\rho : G \longrightarrow \mathrm{GL}(V)$ *is*

$$\begin{array}{ccc} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \operatorname{Tr} \rho(g). \end{array}$$

Proposition

If χ *is the character of a representation of a group* G *of degree* n*, then* $\chi(1_G) = n$*.*

Notation: $\deg \chi = \chi(1_G)$.

The character of a representation

Definition (Character)

The character of a representation $\rho : G \longrightarrow \mathrm{GL}(V)$ *is*

$$\begin{array}{rcl} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \operatorname{Tr} \rho(g). \end{array}$$

Proposition

Let G *be a group,* V_1, V_2 *representations of* G*, and* χ_1, χ_2 *their characters. The the character of* $V_1 \oplus V_2$ *is* $\chi_1 + \chi_2$*.*

Proof.

Let $\rho_1 : G \longrightarrow \mathrm{GL}(V_1)$, $\rho_2 : G \longrightarrow \mathrm{GL}(V_2)$. Then the representation $V_1 \oplus V_2$ is defined by

$$g \mapsto \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}.$$

$\square$

Characters on inverses; kernels

Theorem (Reminder)

Let A be a square matrix with coefficients in a field K.
If $P(x) \in K[x]$ satisfies $P(A) = 0$, has all its roots in K, and no repeated root, then A is diagonalisable, and all its eigenvalues are roots of $P(x)$.

Lemma

Let ρ be a representation of a finite group G of order $n = \#G$. Then for all $g \in G$, $\rho(g)$ is diagonalisable, and its eigenvalues are of the form $e^{2k\pi i/n}$, $k \in \mathbb{Z}$.

Proof.

Let $g \in G$. By Lagrange, $g^n = 1_G$, so $\rho(g)^n = \mathrm{Id}$.
Take $P(x) = x^n - 1 = \prod_{k=0}^{n}(x - e^{2k\pi i/n}) \in \mathbb{C}[x]$. $\square$

Characters on inverses; kernels

Lemma

Let ρ be a representation of a finite group G of order $n = \#G$. Then for all $g \in G$, $\rho(g)$ is diagonalisable, and its eigenvalues are of the form $e^{2k\pi i/n}$, $k \in \mathbb{Z}$.

Corollary

Let χ be the character of ρ. Then for all $g \in G$,

$$g \in \operatorname{Ker} \rho \iff \chi(g) = \deg \rho.$$

Characters on inverses; kernels

Corollary

Let χ be the character of a representation ρ of G.
Then for all $g \in G$, $\chi(g^{-1}) = \overline{\chi(g)}$.

Proof.

For a given $g \in G$, pick a basis of V such that

$$\rho(g) = \begin{pmatrix} \ddots & & 0 \\ & \lambda_j & \\ 0 & & \ddots \end{pmatrix}, \quad \lambda_j = e^{2k_j i\pi/n}.$$

Then $\rho(g^{-1}) = \rho(g)^{-1} = \begin{pmatrix} \ddots & & 0 \\ & \lambda_j^{-1} & \\ 0 & & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & & 0 \\ & \overline{\lambda_j} & \\ 0 & & \ddots \end{pmatrix}$. □

The character of Hom

Invariants

Given a representation V of G, we define

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\} = \bigcap_{g \in G} \operatorname{Ker}\big(\rho(g) - \operatorname{Id}\big).$$

This is a subrepresentation of V; in fact, it is the largest one which is a direct sum of copies of $\mathbb{1}$.

Example

Recall that if V and W are representations of G, then so is $\operatorname{Hom}(V, W)$ by

$$(gT)(v) = g\big(T(g^{-1}v)\big).$$

Then $\operatorname{Hom}(V, W)^G = \operatorname{Hom}_G(V, W)$.

The dimension of invariants

Lemma

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ *be a representation.*

Then $\pi = \frac{1}{\#G} \sum_{g \in G} \rho(g) \in \mathrm{End}(V)$ *is a projection onto* V^G*.*

Proof.

For all $v \in V$ and $h \in G$, we have

$$\rho(h)(\pi(v)) = \rho(h)\left(\frac{1}{\#G}\sum_{g\in G}\rho(g)(v)\right) = \frac{1}{\#G}\sum_{g\in G}\rho(h)(\rho(g)(v))$$

$$= \frac{1}{\#G}\sum_{g\in G}\rho(hg)(v) = \frac{1}{\#G}\sum_{g\in G}\rho(g)(v) = \pi(v).$$

We conclude that $\pi^2 = \pi$ is a projection onto $\mathrm{Im}\,\pi = V^G$. $\square$

The dimension of invariants

Lemma

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ *be a representation.*

Then $\pi = \dfrac{1}{\#G} \displaystyle\sum_{g \in G} \rho(g) \in \mathrm{End}(V)$ *is a projection onto* V^G.

Corollary

$$\dim V^G = \frac{1}{\#G} \sum_{g \in G} \chi(g).$$

Proof.

The matrix of π with respect to a suitable basis is $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$,

whence $\dim V^G = n = \mathrm{Tr}\,\pi = \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr}\,\rho(g)$. $\square$

The character of Hom

Let G be a group. Recall that if V_1 and V_2 are representations of G, then so is $\mathrm{Hom}(V_1, V_2)$ by

$$(gT)(v) = g\big(T(g^{-1}v)\big).$$

The character of Hom

Lemma

The character of $\mathrm{Hom}(V_1, V_2)$ *is* $\overline{\chi_1}\,\chi_2$*, where* χ_1 *is the character of* V_1 *and* χ_2 *that of* V_2*.*

The character of Hom

Proof.

Fix bases $\mathcal{B}_1 = (b_1, \cdots)$ of V_1 and $\mathcal{C} = (c_1, \cdots)$ of V_2.
Then the $T_{i,j}$, defined by $T_{i,j}(b_j) = c_i$, $T_{i,j}(b_k) = 0$ for $k \neq j$, form a basis of $\mathrm{Hom}(V_1, V_2)$.
Pick $g \in G$; we have $gT_{i,j} = \rho_2(g) T_{i,j} \rho_1(g)^{-1}$. Let Q be the matrix of $\rho_1(g)^{-1}$ on $\mathcal{B}$, and R that of $\rho_2(g)$ on $\mathcal{C}$.

$$b_n \overset{\rho_1(g)^{-1}}{\longmapsto} \sum_k Q_{k,n} b_k \overset{T_{i,j}}{\longmapsto} Q_{j,n} c_i \overset{\rho_2(g)}{\longmapsto} Q_{j,n} \sum_m R_{m,i} c_m$$

so $gT_{i,j} = \sum_{m,n} Q_{j,n} R_{m,i} T_{m,n}$.
Thus the trace of g on $\mathrm{Hom}(V_1, V_2)$ is

$$\sum_{i,j} \text{coeff. of } T_{i,j} \text{ in } gT_{i,j} = \sum_{i,j} Q_{j,j} R_{i,i} = \left(\sum_i R_{i,i}\right)\left(\sum_j Q_{j,j}\right)$$

$$= \left(\operatorname{Tr} \rho_2(g)\right)\left(\operatorname{Tr} \rho_1(g^{-1})\right) = \chi_2(g)\overline{\chi_1(g)}.$$

$\square$

The character of Hom

Lemma

The character of $\operatorname{Hom}(V_1, V_2)$ *is* $\overline{\chi_1}\,\chi_2$, *where* χ_1 *is the character of* V_1 *and* χ_2 *that of* V_2.

Corollary

If V *is a representation of character* χ, *then the character of its linear dual* $V^\vee$ *is* $\overline{\chi}$.

Proof.

$$V^\vee = \operatorname{Hom}(V, \mathbb{1}).$$

$\square$

The character of Hom

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The character of $\mathrm{Hom}(V_1, V_2)$ *is* $\overline{\chi_1}\,\chi_2$*, where* χ_1 *is the character of* V_1 *and* χ_2 *that of* V_2*.*

Corollary

Let V_1, V_2 *be a representations of* G *with respective characters* χ_1, χ_2*. Then* $(\chi_1|\chi_2) = \dim \mathrm{Hom}_G(V_1, V_2) \in \mathbb{Z}_{\geq 0}$*.*

Proof.

Let $\chi = \overline{\chi_1}\,\chi_2$ be the character of $\mathrm{Hom}(V_1, V_2)$. Then

$$(\chi_1|\chi_2) = \frac{1}{\#G} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} = \overline{\frac{1}{\#G} \sum_{g \in G} \chi(g)}$$

$$= \overline{\dim \mathrm{Hom}(V_1, V_2)^G} = \dim \mathrm{Hom}_G(V_1, V_2). \qquad \square$$

Characters know everything

First orthogonality of characters

Let G be a finite group, and let $\mathrm{Irr}(G)$ be the set of characters of isomorphism classes of irreducible representations of G.

Theorem

The set $\mathrm{Irr}(G)$ *is orthonormal.*

Proof.

Let $\chi_1, \chi_2 \in \mathrm{Irr}(G)$, and let V_1, V_2 be the corresponding representations. Then by Schur's lemma,

$$(\chi_1|\chi_2) = \dim \mathrm{Hom}_G(V_1, V_2) = \begin{cases} 0 & \text{if } V_1 \not\simeq V_2, \\ 1 & \text{if } V_1 = V_2. \end{cases}$$

$\square$

Consequences (Main slide of the module!)

Theorem

Let V be a representation of G of character χ. Suppose that the decomposition of V into irreducible representations is

$$W_1^{\oplus n_1} \oplus \cdots \oplus W_r^{\oplus n_r}.$$

Then $\chi = \sum_{j=1}^r n_j \chi_j$, where χ_j is the character of W_j; thus

$$n_j = (\chi \mid \chi_j) \quad \text{and} \quad (\chi \mid \chi) = \sum_{j=1}^r n_j^2.$$

Proof.

We have $\chi = \sum_{j=1}^r n_j \chi_j$ by additivity of characters on direct sums; the rest follows from orthonormality. □

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$$n_j = (\chi \,|\, \chi_j) \quad \textit{and} \quad (\chi \,|\, \chi) = \sum_{j=1}^r n_j^2.$$

Corollary

V is irreducible iff. $(\chi \,|\, \chi) = 1$.

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$$n_j = (\chi \,|\, \chi_j) \quad \text{and} \quad (\chi \,|\, \chi) = \sum_{j=1}^r n_j^2.$$

Corollary

For a given V, the integers n_k are unique.

Corollary

Two representations of G are equivalent iff. they have the same character.

Consequences (Main slide of the module!)

Theorem

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Then $\chi = \sum_{j=1}^r n_j \chi_j$, where χ_j is the character of W_j; thus

$$n_j = (\chi \mid \chi_j) \quad \text{and} \quad (\chi \mid \chi) = \sum_{j=1}^r n_j^2.$$

Corollary

$$\# \operatorname{Irr}(G) \leq \# \text{Conj. classes in } G.$$

NB we will prove later that this is actually an <u>equality</u>.

Example: S_3

Take $G = S_3$, order $\#G = 6$.
We already know some representations of G:

Conj. classes	$\{1_G\}$	$\{(12),(13),(23)\}$	$\{(123),(132)\}$
$\mathbb{1}$	1	1	1
ε	1	-1	1
$\triangleleft$	2	0	-1
Perm	3	1	0

We compute $(\mathbb{1}|\mathbb{1}) = (\varepsilon|\varepsilon) = (\triangleleft|\triangleleft) = 1$, so they are irreducible.
However, since $(\text{Perm}|\text{Perm}) = 2$, Perm is <u>not</u> irreducible, but rather the direct sum of two irreducible representations.
Which ones?
We compute $(\text{Perm}|\mathbb{1}) = 1$, $(\text{Perm}|\varepsilon) = 0$, and $(\text{Perm}|\triangleleft) = 1$, so we get that $\text{Perm} \simeq \mathbb{1} \oplus \triangleleft$.

Isotypic components

Morphisms from class functions

Lemma

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a representation, and let $f : G \longrightarrow \mathbb{C}$ be a class function. Then $T_f = \sum_{g \in G} f(g)\rho(g) \in \mathrm{End}_G(V)$.

Proof.

For all $h \in G$, we have $\rho(h) T_f = \rho(h) \displaystyle\sum_{g \in G} f(g)\rho(g)$

$$= \sum_{g \in G} f(g)\rho(h)\rho(g) = \sum_{g \in G} f(g)\rho(hg) = \sum_{g \in G} f(g)\rho(hgh^{-1})\rho(h)$$

but the conjugation $\begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & g' = hgh^{-1} \end{array}$ is bijective, so

$$\underset{\substack{g'=hgh^{-1} \\ g=h^{-1}g'h}}{=} \sum_{g' \in G} f(h^{-1}g'h)\rho(g')\rho(h) = T_f \rho(h). \qquad \square$$

Morphisms from class functions

Lemma

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ *be a representation, and let* $f : G \longrightarrow \mathbb{C}$ *be a class function. Then* $T_f = \sum_{g \in G} f(g)\rho(g) \in \mathrm{End}_G(V)$.

Corollary

If V *is irreducible, then* $T_f = \dfrac{\#G}{\deg V}(\chi \mid \overline{f})\,\mathrm{Id}_V$.

Proof.

By Schur, we have $T_f = \lambda\,\mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$. Take traces. □

Isotypical components

Theorem (Formula for projection on isotypical component)

Let $V = \bigoplus_{\chi \in \operatorname{Irr} G} W_\chi^{\oplus n_\chi}$ *be the complete decomposition of a representation of* G*. Then for each* χ*, the projection on* $W_\chi^{\oplus n_\chi}$ *is given by* $\pi_\chi = \dfrac{\deg \chi}{\# G} \displaystyle\sum_{g \in G} \overline{\chi(g)} e_g \in \mathbb{C}[G]$*.*

Proof.

Let $T = \dfrac{\deg \chi}{\# G} \displaystyle\sum_{g \in G} \overline{\chi(g)} e_g = \dfrac{\deg \chi}{\# G} T_{\overline{\chi}}$.

Then for all $\psi \in \operatorname{Irr}(G)$, T acts on W_ψ by $\lambda \operatorname{Id}$, where

$$\lambda = \frac{\deg \chi}{\# G} \frac{\# G}{\deg \psi} (\psi \,|\, \overline{\overline{\chi}}) = \begin{cases} 1 \text{ if } \psi = \chi, \\ 0 \text{ if } \psi \neq \chi. \end{cases} \qquad \square$$

Isotypical components

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Let $V = \bigoplus_{\chi \in \operatorname{Irr} G} W_\chi^{\oplus n_\chi}$ *be the complete decomposition of a representation of* G*. Then for each* χ*, the projection on* $W_\chi^{\oplus n_\chi}$ *is given by* $\pi_\chi = \dfrac{\deg \chi}{\# G} \sum_{g \in G} \overline{\chi(g)} e_g \in \mathbb{C}[G]$.

Corollary (Isotypical components)

For each χ*, the subset* $W_\chi^{\oplus n_\chi}$ *does not depend of the chosen decomposition of* V*. It is called the* χ*-isotypical component.*

Example: symmetries

Let $G = C_2 = \{1, g\}$ with $g^2 = 1$. A representation ρ of G is determined by $S = \rho(g)$ since $\rho(1) = \mathrm{Id}$. Besides $S^2 = \mathrm{Id}$, so S is diagonalisable with eigenvalues ± 1, so S is a symmetry.

The irreducible representations of G are those of degree 1, namely $\mathbb{1}$ and $\varepsilon : G \xrightarrow{\sim} \{\pm 1\}$.

Given a symmetry $S \in \mathrm{End}(V)$, the isotypical components are

$$V_{\mathbb{1}} = \mathrm{Ker}(S - \mathrm{Id}) = V_+ \quad \text{and} \quad V_\varepsilon = \mathrm{Ker}(S + \mathrm{Id}) = V_-.$$

The decomposition $V = V_+ \oplus V_-$ is canonical;
the decompositions $V_+ = \mathbb{1}^{\oplus \dim V_+}$, $V_- = \varepsilon^{\oplus \dim V_-}$ are not.

The projectors are $\pi_{\mathbb{1}} = \frac{1}{2}(1 + S)$, $\pi_\varepsilon = \frac{1}{2}(1 - S)$.