MAU34104
Group representations
4 - Character tables

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Decomposition of the regular representation

The character of a permutation representation

Let G be a group. If G acts on a set X, we get a permutation representation $\mathbb{C}[X]$.

Proposition

The character of this representation is given by

$$\chi(g) = \# \operatorname{Fix} g$$

where $\operatorname{Fix} g = \{x \in X \mid g \cdot x = x\} \subseteq X$.

Proof.

$\mathbb{C}[X]$ has basis $\{e_x\}_{x \in X}$, so
$\chi(g) = \sum_{x \in X}$ coeff. of e_x in $(ge_x = e_{g \cdot x})$. $\square$

Decomposition of the regular representation

$G \circlearrowright X = G$ by $g \cdot x = gx \rightsquigarrow$ regular representation $\mathbb{C}[G]$.

Remark

This is a faithful ($=$ injective) representation of G, since this action of G on itself is faithful: if $g \cdot x = x$ for all $x \in X$, then $g \cdot 1_G = 1_G$, so $g = 1_G$.

This shows that every finite group admits a faithful representation.

In particular, if G is non-Abelian, then G has a least one irreducible representation of degree ≥ 2; more on this later.

Decomposition of the regular representation

$G \circlearrowright X = G$ by $g \cdot x = gx \rightsquigarrow$ regular representation $\mathbb{C}[G]$.

Theorem

The decomposition of the regular representation is

$$\mathbb{C}[G] \simeq \bigoplus_{\rho \in \operatorname{Irr} G} \rho^{\oplus \deg \rho}.$$

Proof.

Let $n = \#G$. The character of $\mathbb{C}[G]$ is

$$\chi_{\text{reg}}(g) = \# \operatorname{Fix} g = \#\{h \in X = G \mid gh = h\} = \begin{cases} n \text{ if } g = 1_G, \\ 0 \text{ else.} \end{cases}$$

Therefore, if ρ is an irreducible representation of character χ_ρ, then its multiplicity in the decomposition of $\mathbb{C}[G]$ is

$$(\chi_{\text{reg}} \mid \chi_\rho) = \frac{1}{n} \sum_{g \in G} \chi_{\text{reg}}(g) \overline{\chi_\rho(g)} = \frac{1}{n} n \overline{\chi_\rho(1_G)} = \deg \rho. \quad \square$$

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Corollary

$$\# G = \sum_{\rho \in \operatorname{Irr} G} (\deg \rho)^2.$$

Example

For $G = S_3$, we have seen that $\operatorname{Irr} G = \{\mathbb{1}, \varepsilon, \triangleleft\}$, so

$$\mathbb{C}[S_3] \simeq \mathbb{1}^{\oplus \deg \mathbb{1}} \oplus \varepsilon^{\oplus \deg \varepsilon} \oplus \triangleleft^{\oplus \deg \triangleleft} = \mathbb{1} \oplus \varepsilon \oplus \triangleleft \oplus \triangleleft$$

and indeed $\# S_3 = 6 = 1^2 + 1^2 + 2^2$.

Decomposition of the regular representation

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Theorem

The decomposition of the regular representation is

$$\mathbb{C}[G] \simeq \bigoplus_{\rho \in \operatorname{Irr} G} \rho^{\oplus \deg \rho}.$$

Corollary

For all $g \in G$, $$\sum_{\chi \in \operatorname{Irr} G} (\deg \chi)\chi(g) = \begin{cases} \#G \text{ if } g = 1_G, \\ 0 \text{ else.} \end{cases}.$$

We will see later on that this is an avatar of the second orthogonality relations of characters.

Wedderburn on $\mathbb{C}[G]$

Algebras (not examinable)

Definition

Let K be a field. A K-algebra A is a vector space over K which is also a ring, and such that

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \quad \text{for all } \lambda \in K \text{ and } a, b \in A.$$

Example

The polynomial ring $K[x]$ and the matrix ring $\mathcal{M}_n(K)$ are actually K-algebras.

Example

$\mathbb{C}[G]$ is a $\mathbb{C}$-algebra.

Opposite rings

Let R be a ring.

Definition (Opposite ring)

We can define a new ring R^{opp} by keeping the same set, keeping the addition, and defining multiplication by

$$x \times_{R^{\mathrm{opp}}} y = y \times_R x.$$

Example

If R is commutative, then $R^{\mathrm{opp}} = R$.

Example

For any field K and $n \in \mathbb{N}$, we have $\mathcal{M}_n(K)^{\mathrm{opp}} \simeq \mathcal{M}_n(K)$ by transposition.

Opposite rings

Let R be a ring.

Lemma

Let $M = R$ viewed as an R-module. Then $\operatorname{End}_R(M) = R^{\text{opp}}$.

Proof.

The map

$$\begin{array}{rcl} R^{\text{opp}} & \longrightarrow & \operatorname{End}_R(M) \\ r & \longmapsto & (\mu_r : m \mapsto mr) \end{array}$$

is injective because $\mu_r(1) = r$,
surjective because for all $f \in \operatorname{End}_R(M)$, we must have $f(r) = f(r1) = rf(1)$ for all $r \in M = R$ whence $f = \mu_{f(1)}$,
and it is a ring morphism since $\mu_r \circ \mu_s = \mu_{sr}$. $\square$

Wedderburn's theorem (not examinable)

Theorem (Wedderburn)

Let K be a field, and let A be a K-algebra. If A is semi-simple as an A-module, then

$$A \simeq \prod_i \mathcal{M}_{n_i}(D_i)$$

where the D_i are division rings containing K.

This applies in particular to $A = \mathbb{C}[G]$, since the regular representation is semisimple.

Endomorphisms of direct sums

Lemma

Let R be a ring, and M be an R-module. If $M = M_1 \oplus M_2$,

$$\mathrm{End}_R(M) \simeq \begin{pmatrix} \mathrm{End}_R(M_1) & \mathrm{Hom}_R(M_2, M_1) \\ \mathrm{Hom}_R(M_1, M_2) & \mathrm{End}_R(M_2) \end{pmatrix}.$$

Proof.

$$\begin{aligned} f(m_1 + m_2) &= f_1(m_1 + m_2) + f_2(m_1 + m_2) \\ &= f_{1\to 1}(m_1) + f_{2\to 1}(m_2) + f_{1\to 2}(m_1) + f_{2\to 2}(m_2). \end{aligned}$$

This really is a ring isomorphism: if $f, g \in \mathrm{End}_R(M)$ with

$$f = \begin{pmatrix} f_{1\to 1} & f_{2\to 1} \\ f_{1\to 2} & f_{2\to 2} \end{pmatrix}, \quad g = \begin{pmatrix} g_{1\to 1} & g_{2\to 1} \\ g_{1\to 2} & g_{2\to 2} \end{pmatrix},$$

then $(f \circ g)_{1\to 1} = f_{1\to 1} \circ g_{1\to 1} + f_{2\to 1} \circ g_{1\to 2}$, etc. $\square$

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Example

$$\begin{aligned}\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) &\simeq \begin{pmatrix} \operatorname{End}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}) & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \\ \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) & \operatorname{End}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \end{pmatrix} \\ &\simeq \begin{pmatrix} \mathbb{Z}/4\mathbb{Z} & 2\mathbb{Z}/4\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}.\end{aligned}$$

Endomorphisms of direct sums

Lemma

Let R be a ring, and M be an R-module. If $M = M_1 \oplus M_2$,

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Example

If $M = M_1 \oplus M_2$ where $M_1 \not\simeq M_2$ are both <u>simple</u>, then

$$\operatorname{End}_R(M) \simeq \begin{pmatrix} \operatorname{End}_R(M_1) & 0 \\ 0 & \operatorname{End}_R(M_2) \end{pmatrix} \simeq \operatorname{End}_R(M_1) \times \operatorname{End}_R(M_2).$$

Wedderburn on $\mathbb{C}[G]$

Let G be a finite group, and let $\operatorname{Irr} G = \{\rho_1, \cdots, \rho_m\}$, so that

$$\mathbb{C}[G] \simeq \underbrace{\rho_1 \oplus \cdots \oplus \rho_1}_{\deg \rho_1} \oplus \cdots \oplus \underbrace{\rho_m \oplus \cdots \oplus \rho_m}_{\deg \rho_m}.$$

From Schur's lemma and the above, we deduce that

$$\begin{aligned}
\mathbb{C}[G]^{\text{opp}} &\simeq \operatorname{End}_G(\mathbb{C}[G]) \\
&\simeq \begin{pmatrix} \mathcal{M}_{\deg \rho_1}\big(\operatorname{End}_G(\rho_1)\big) & & 0 \\ & \ddots & \\ 0 & & \mathcal{M}_{\deg \rho_m}\big(\operatorname{End}_G(\rho_m)\big) \end{pmatrix} \\
&\simeq \prod_{i=1}^m \mathcal{M}_{\deg \rho_i}\big(\operatorname{End}_G(\rho_i)\big) \simeq \prod_{i=1}^m \mathcal{M}_{\deg \rho_i}(\mathbb{C}).
\end{aligned}$$

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From Schur's lemma and the above, we deduce that

$$\begin{aligned}\mathbb{C}[G]^{\text{opp}} &\simeq \operatorname{End}_G(\mathbb{C}[G]) \\ &\simeq \prod_{i=1}^{m} \mathcal{M}_{\deg \rho_i}\big(\operatorname{End}_G(\rho_i)\big) \simeq \prod_{i=1}^{m} \mathcal{M}_{\deg \rho_i}(\mathbb{C}).\end{aligned}$$

Taking opposites, we conclude that

$$\begin{aligned}\mathbb{C}[G] = \mathbb{C}[G]^{\text{opp}\,\text{opp}} &\simeq \left(\prod_{\rho \in \operatorname{Irr} G} \mathcal{M}_{\deg \rho}(\mathbb{C})\right)^{\text{opp}} \\ &\simeq \prod_{\rho \in \operatorname{Irr} G} \mathcal{M}_{\deg \rho}(\mathbb{C})^{\text{opp}} \simeq \prod_{\rho \in \operatorname{Irr} G} \mathcal{M}_{\deg \rho}(\mathbb{C}).\end{aligned}$$

The number of irreducible representations

Centres

Definition (Centre)

The centre of a ring R is

$$Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}.$$

It is a commutative subring.

Remark

If R is actually an algebra, then $Z(R)$ is a subspace, and therefore a subalgebra.

Theorem (Centre of a matrix algebra = scalar matrices)

For every field K and $n \in \mathbb{N}$, we have

$$Z\big(\mathcal{M}_n(K)\big) = \{\lambda I_n, \ \lambda \in K\} \simeq K.$$

The centre of $\mathbb{C}[G]$

Proposition

Let $x = \sum_{g \in G} \lambda_g e_g \in \mathbb{C}[G]$. *Then*

$$x \in Z(\mathbb{C}[G]) \iff g \mapsto \lambda_g \text{ is a class function.}$$

Proof.

$$\begin{aligned} x \in Z(\mathbb{C}[G]) &\iff xy = yx \text{ for all } y \in \mathbb{C}[G] \\ &\iff xe_h = e_h x \text{ for all } h \in G \\ &\iff \sum_{g \in G} \lambda_g e_{gh} = \sum_{g \in G} \lambda_g e_{hg} \text{ for all } h \in G \\ &\iff g \mapsto \lambda_g \text{ is a class function.} \end{aligned}$$

$\square$

The centre of $\mathbb{C}[G]$

Proposition

Let $x = \sum_{g \in G} \lambda_g e_g \in \mathbb{C}[G]$. Then

$$x \in Z(\mathbb{C}[G]) \iff g \mapsto \lambda_g \text{ is a class function.}$$

Remark

If M is an R-module, then for all $z \in Z(R)$,

$\begin{array}{cccc} f_z : & M & \longrightarrow & M \\ & m & \longmapsto & zm \end{array}$ is an R-module endomorphism,

since for all $\lambda \in R$, $f_z(\lambda m) = z\lambda m = \lambda z m = \lambda f_z(m)$.

This explains why $T_f = \sum_{g \in G} f(g) g$ acts as a representation endomorphism on every representation if f is a class function.

The centre of $\mathbb{C}[G]$

Proposition

Let $x = \sum_{g \in G} \lambda_g e_g \in \mathbb{C}[G]$. *Then*

$$x \in Z(\mathbb{C}[G]) \iff g \mapsto \lambda_g \text{ is a class function.}$$

Corollary

When C *ranges over the conjugacy classes of* G,
the $e_C = \sum_{g \in C} e_g$ *form a* $\mathbb{C}$*-basis of* $Z(\mathbb{C}[G])$.
In particular, $\dim_{\mathbb{C}} Z(\mathbb{C}[G]) = \#$ *conj. classes in* G.

The number of irreducible representations

Theorem (Number of irr reps = number of conj classes)

Let G be a finite group. Then

$$\#\operatorname{Irr}(G) = \#\textit{Conj. classes in } G.$$

Proof.

$$\begin{aligned}
\#\text{Conj. classes in } G &= \dim_{\mathbb{C}} Z(\mathbb{C}[G]) \\
&= \dim_{\mathbb{C}} Z\left(\prod_{\rho \in \operatorname{Irr} G} \mathcal{M}_{\deg \rho}(\mathbb{C})\right) \\
&= \dim_{\mathbb{C}} \prod_{\rho \in \operatorname{Irr} G} Z\big(\mathcal{M}_{\deg \rho}(\mathbb{C})\big) \\
&= \dim_{\mathbb{C}} \prod_{\rho \in \operatorname{Irr} G} \mathbb{C} = \#\operatorname{Irr} G.
\end{aligned}$$

$\square$

The number of irreducible representations

Theorem (Number of irr reps = number of conj classes)

Let G be a finite group. Then

$$\# \operatorname{Irr}(G) = \# \textit{Conj. classes in } G.$$

Corollary

Irreducible characters form an orthonormal basis of the space of class functions.

The number of irreducible representations

Theorem (Number of irr reps = number of conj classes)

Let G be a finite group. Then
$$\#\operatorname{Irr}(G) = \#\textit{Conj. classes in } G.$$

Corollary

G is Abelian $\iff$ All irr. reps. of G have deg. 1.

Proof.

$\#G = \sum_{\rho \in \operatorname{Irr} G} (\dim \rho)^2$. $\square$

Lemma (Decomposition of $\mathcal{M}_n(K)$ as a module over itself)

Let K be a field, $n \in \mathbb{N}$, and let $S = K^n$ viewed as an $\mathcal{M}_n(K)$-module (column vectors). The $\mathcal{M}_n(K)$-module $\mathcal{M}_n(K)$ decomposes as $\underbrace{S \oplus \cdots \oplus S}_{n}$.

Besides, S is a simple $\mathcal{M}_n(K)$-module.

Proof.

Let $A, B \in \mathcal{M}_n(K)$. If the columns of B are $B_1, \cdots, B_n$, then the columns of AB are $A\,B_1, \cdots, A\,B_n$.
Let $0 \neq T \subseteq S$ be a sub-$\mathcal{M}_n(K)$-module, and let $0 \neq t \in T$.
For all $s \in S$, there is $A \in \mathcal{M}_n(K)$ such that $s = At$, so $T \supseteq S$. $\square$

Interpretation of the Wedderburn decomp. of $\mathbb{C}[G]$

Theorem

Up to conjugacy, the Wedderburn isomorphism

$$\mathbb{C}[G] \simeq \prod_{\rho \in \mathrm{Irr}(G)} \mathcal{M}_{\deg \rho}(\mathbb{C})$$

is simply given by $x \mapsto \big(\rho(x)\big)_{\rho \in \mathrm{Irr}(G)}$.

Its inverse takes

$$(0, \cdots, 0, \underset{\rho}{\mathrm{Id}}, 0, \cdots, 0) \in \prod_{\rho \in \mathrm{Irr}(G)} \mathcal{M}_{\deg \rho}(\mathbb{C})$$

to $\pi_\rho = \dfrac{\deg \rho}{\# G} \sum_{g \in G} \overline{\chi_\rho(g)} e_g$ *(Fourier inversion formula).*

Interpretation of the Wedderburn decomp. of $\mathbb{C}[G]$

Proof.

For each $\chi \in \operatorname{Irr}(G)$, let W_χ be the irreducible representation of character χ, and let $e_\chi = \frac{\deg \chi}{\# G} \sum_{g \in G} \overline{\chi(g)} e_g \in Z(\mathbb{C}[G])$.
As $\mathbb{C}[G]$-modules,

$$\prod_{\chi \in \operatorname{Irr}(G)} W_\chi^{\oplus n_\chi} \simeq \mathbb{C}[G] \simeq \prod_{\chi \in \operatorname{Irr}(G)} \mathcal{M}_{\deg \chi}(\mathbb{C}) \simeq \prod_{\chi \in \operatorname{Irr}(G)} S_\chi^{\oplus n_\chi}.$$

Besides, e_χ acts on W_ψ as Id if $\psi = \chi$ and as 0 else, so

$$\begin{array}{ccc} \mathbb{C}[G] & \xrightarrow{\sim} & \displaystyle\prod_{\chi \in \operatorname{Irr}(G)} \mathcal{M}_{\deg \chi}(\mathbb{C}) \\ x & \longmapsto & {}^t(y \mapsto yx)_{| \prod_{\chi \in \operatorname{Irr}(G)} W_\chi^{\oplus n_\chi}} \end{array}$$

takes e_χ to $(0, \cdots, 0, \underset{\chi}{\mathrm{Id}}, 0, \cdots, 0)$. In particular, e_χ acts on S_ψ as Id if $\psi = \chi$ and as 0 else, so $S_\psi \simeq W_\psi$. $\square$

The character table

The character table of a group

Definition (Character table)

The <u>*character table*</u> *of a finite group G is obtained by putting the values of the characters of* $\operatorname{Irr} G$ *on each conjugacy class of G.*
It is a <u>*square*</u> *matrix of size $\#\operatorname{Irr}(G) = \#$Conj. classes in G.*

Example (Character table of S_3)

Rep. of class	Id	(12)	(123)
$\#$ class	1	3	2
$\mathbb{1}$	1	1	1
ε	1	-1	1
$\triangleleft$	2	0	-1

The conjugacy principle

Principle

Let G be a group of "transformations". If $g \in G$ is a "transformation" of "parameters" $x, y, \cdots$, then for all $h \in G$, hgh^{-1} is the "transformation" of "parameters" $h \cdot x, h \cdot y, \cdots$.

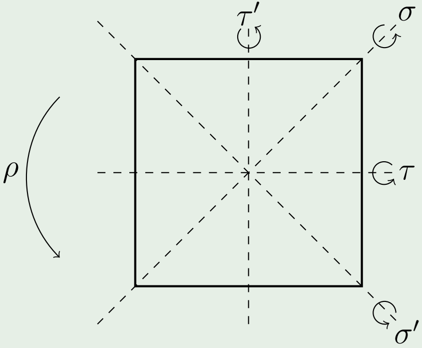
Example

Let $G = \mathrm{SO}_3(\mathbb{R})$. If $g \in G$ is the rotation of axis ℓ and angle θ, then for all $h \in G$, hgh^{-1} is the rotation of axis $h(\ell)$ and angle θ.

The conjugacy principle

Example

The conjugacy classes of $G = D_8$



are $\{\mathrm{Id}\}, \{\rho, \rho^3\}, \{\rho^2\}, \{\sigma, \sigma'\}, \{\tau, \tau'\}$.

The conjugacy principle

Example

Let $G = S_n$ for some $n \in \mathbb{N}$. If $\sigma = (i_1 \cdots i_k)(j_1 \cdots j_l) \cdots \in S_n$, then for all $\tau \in S_n$,

$$\tau\sigma\tau^{-1} = \big(\tau(i_1) \cdots \tau(i_k)\big)\big(\tau(j_1) \cdots \tau(j_l)\big) \cdots .$$

$\rightsquigarrow$ $\sigma, \sigma' \in S_n$ conjugate iff. same shape of cycle decomposition.

$\rightsquigarrow$ Conjugacy classes of $S_n \longleftrightarrow$ partitions $n = n_1 + \cdots + n_r$.

For example, the conjugacy classes of S_5 are:

$$\begin{array}{ccl}
\{\mathrm{Id}\} & \longleftrightarrow & 5 = 1+1+1+1+1, \\
\{(**)\} & \longleftrightarrow & 5 = 2+1+1+1, \\
\{(***)\} & \longleftrightarrow & 5 = 3+1+1, \\
\{(****)\} & \longleftrightarrow & 5 = 4+1, \\
\{(*****)\} & \longleftrightarrow & 5 = 5, \\
\{(**)(**)\} & \longleftrightarrow & 5 = 2+2+1, \\
\{(***)(**)\} & \longleftrightarrow & 5 = 3+2.
\end{array}$$

Example: S_4

Conjugacy classes of $S_4 \longleftrightarrow$ possible cycle decompositions
$\rightsquigarrow$ 5 classes, represented by Id, (12), (123), (1234), (12)(34).

Example: S_4

Rep. of class	Id	(12)	(123)	(1234)	(12)(34)

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Example: S_4

Rep. of class	Id	(12)	(123)	(1234)	(12)(34)
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Conjugacy classes of $S_4 \longleftrightarrow$ possible cycle decompositions
$\rightsquigarrow$ 5 classes, represented by Id, (12), (123), (1234), (12)(34).

$\rightsquigarrow$ 5 irreducible representations, of degree $n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5$.
$n_1 = 1$ because of $\mathbb{1}$.
$n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = \# S_4 = 24$, so $3 \leq n_5 \leq 4$.
If $n_5 = 4$, then $n_2^2 + n_3^2 + n_4^2 = 24 - 1^2 - 4^2 = 7$, impossible.
So $n_5 = 3$, $n_2^2 + n_3^2 + n_4^2 = 24 - 1^2 - 3^2 = 14$, whence
$3 \leq n_4 \leq 3$, so $n_2^2 + n+_3^2 = 5$.

Example: S_4

Rep. of class	Id	(12)	(123)	(1234)	(12)(34)
# class	1	6	8	6	3
$\mathbb{1}$	1				
	1				
	2				
	3				
	3				

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Example: S_4

Rep. of class	Id	(12)	(123)	(1234)	(12)(34)
# class	1	6	8	6	3
$\mathbb{1}$	1	1	1	1	1
ε	1	-1	1	-1	1
	2				
	3				
	3				
Perm	4	2	1	0	0

Throw in some representations we can think of: $\mathbb{1}$, ε, the permutation representation Perm induced by $S_4 \circlearrowright \{1, 2, 3, 4\}$.

Example: S_4

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Throw in some representations we can think of: $\mathbb{1}$, ε, the permutation representation Perm induced by $S_4 \circlearrowleft \{1, 2, 3, 4\}$.
We find $(\text{Perm} \mid \text{Perm}) = 2$ and $(\text{Perm} \mid \mathbb{1}) = 1$,
so $\text{Perm} \simeq \mathbb{1} \oplus \chi$ with χ irreducible of degree 3.

Example: S_4

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	2				
χ	3	1	0	-1	-1
	3				

The character of $\operatorname{Hom}(\chi, \varepsilon)$ is $\chi\overline{\varepsilon} = \chi\varepsilon$. Since $(\chi\varepsilon \,|\, \chi\varepsilon) = (\chi \,|\, \chi) = 1$, it is irreducible.

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χ	3	1	0	-1	-1
$\chi\varepsilon$	3	-1	0	1	-1
χ^2	9	1	0	1	1

The character of $\mathrm{Hom}(\chi, \chi)$ is $\chi\overline{\chi} = \chi^2$. We find $(\chi^2 \,|\, \chi^2) = 4$, $(\chi^2 \,|\, \mathbb{1}) = 1$, $(\chi^2 \,|\, \chi) = 1$, $(\chi^2 \,|\, \chi\varepsilon) = 1$,
$\rightsquigarrow \chi^2 = \mathbb{1} + \chi + \chi\varepsilon + \psi$ where ψ is the remaining character.

Alternatively, the regular representation $\mathbb{C}[S_4]$ has character R given by $R(\mathrm{Id}) = 24$, $R(g) = 0$ if $g \neq \mathrm{Id}$, and decomposes as $R = \mathbb{1} + \varepsilon + 2\psi + 3\chi + 3\chi\varepsilon$.

Example: S_4

Rep. of class	Id	(12)	(123)	(1234)	(12)(34)
# class	1	6	8	6	3
$\mathbb{1}$	1	1	1	1	1
ε	1	-1	1	-1	1
ψ	2	0	-1	0	2
χ	3	1	0	-1	-1
$\chi\varepsilon$	3	-1	0	1	-1

The character of $\mathrm{Hom}(\chi, \chi)$ is $\chi\overline{\chi} = \chi^2$. We find $(\chi^2 \mid \chi^2) = 4$, $(\chi^2 \mid \mathbb{1}) = 1$, $(\chi^2 \mid \chi) = 1$, $(\chi^2 \mid \chi\varepsilon) = 1$,
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Alternatively, the regular representation $\mathbb{C}[S_4]$ has character R given by $R(\mathrm{Id}) = 24$, $R(g) = 0$ if $g \neq \mathrm{Id}$, and decomposes as $R = \mathbb{1} + \varepsilon + 2\psi + 3\chi + 3\chi\varepsilon$.

Normal subgroups on the character table

Theorem (Normal subgroups on character table)

For each $\chi \in \operatorname{Irr} G$, let

$$N_\chi = \{g \in G \mid \chi(g) = \deg \chi\}$$

Then the N_χ are normal subgroups of G. Conversely, every normal subgroup of G is an intersection of some of the N_χ.

Proof.

Recall that if ρ has character χ, then $g \in \operatorname{Ker} \rho \Leftrightarrow \chi(g) = \deg \chi$. Thus $N_\chi = \operatorname{Ker} \rho \triangleleft G$.
Conversely, let $N \triangleleft G$. The regular representation $\mathbb{C}[G/N]$ is a faithful representation of G/N; we may view it as a representation R of G via $G \twoheadrightarrow G/N$ which thus satisfies $\operatorname{Ker} R = N$. Therefore, if $R \simeq \bigoplus \rho^{\oplus n_\rho}$ with each ρ irreducible and $n_\rho \geq 1$, then $N = \bigcap \operatorname{Ker} \rho = \bigcap N_{\chi_\rho}$. $\square$

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Theorem (Derived subgroup on character table)

The derived subgroup of G is $D(G) = \bigcap_{\deg \chi = 1} N_\chi$.

Proof.

As $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$ is Abelian,

$$\operatorname{Irr}\big(G/D(G)\big) = \{\chi \in \operatorname{Irr} G \mid \deg \chi = 1\}. \qquad \square$$

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Theorem (Derived subgroup on character table)

The derived subgroup of G is $D(G) = \bigcap_{\deg \chi = 1} N_\chi$.

Corollary

We can see the normal subgroups and the derived subgroup of G on its character table. In particular, we can see whether G is simple and whether it is perfect.

Example: S_4

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# class	1	6	8	6	3
$\mathbb{1}$	1	1	1	1	1
ε	1	-1	1	-1	1
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χ	3	1	0	-1	-1
$\chi\varepsilon$	3	-1	0	1	-1

We find the normal subgroups $N_\varepsilon = A_4$ and $N_\psi = V_4$. Since $N_\varepsilon \supset N_\psi$, these are all the non-trivial normal subgroups of S_4.
Anyway, S_4 is not a simple group.
The derived subgroup is

$$D(S_4) = \bigcap_{\deg \rho = 1} \operatorname{Ker} \rho = N_\varepsilon = A_4.$$

In particular, S_4 is not perfect.

The second orthogonality relations & extra results

Second orthogonality of characters

Theorem (Second orthogonality of characters)

Let $g, h \in G$, and let $C_h \subset G$ be the conjugacy class of h. Then
$$\sum_{\chi \in \operatorname{Irr} G} \chi(g)\overline{\chi(h)} = \begin{cases} \#G/\#C_h & \text{if } g \in C_h, \\ 0 & \text{else.} \end{cases}$$

Proof.

$f_h : G \longrightarrow \mathbb{C}$ defined by $f(g) = 1$ if $g \in C_h$, $f(g) = 0$ else, is a class function. As $\operatorname{Irr} G$ is an orthonormal basis,
$$f_h = \sum_{\chi \in \operatorname{Irr} G} (f_h \,|\, \chi)\, \chi,$$
$$(f_h \,|\, \chi) = \frac{1}{\#G} \sum_{g \in G} f_h(g)\overline{\chi(g)} = \frac{\#C_h}{\#G}\overline{\chi(h)}$$
$$\rightsquigarrow f_h(g) = \frac{\#C_h}{\#G} \sum_{\chi \in \operatorname{Irr} G} \overline{\chi(h)}\chi(g).$$
$\square$

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Let $g, h \in G$, and let $C_h \subset G$ be the conjugacy class of h. Then
$$\sum_{\chi \in \operatorname{Irr} G} \chi(g)\overline{\chi(h)} = \begin{cases} \#G/\#C_h \text{ if } g \in C_h, \\ 0 \text{ else.} \end{cases}$$

Remark

$$\#G/\#C_h = \#\{g \in G \mid gh = hg\}.$$

Corollary

$$\sum_{\chi \in \operatorname{Irr} G} (\deg \chi)\chi(g) = \begin{cases} \#G \text{ if } g = 1_G, \\ 0 \text{ else.} \end{cases} .$$

Second orthogonality of characters

Theorem (Second orthogonality of characters)

Let $g, h \in G$, and let $C_h \subset G$ be the conjugacy class of h. Then
$$\sum_{\chi \in \operatorname{Irr} G} \chi(g)\overline{\chi(h)} = \begin{cases} \#G/\#C_h & \text{if } g \in C_h, \\ 0 & \text{else.} \end{cases}$$

Example (Finding a character when all the others are known)

We can easily determine the missing character ψ of S_4:

Rep. of class	Id	(12)	(123)	(1234)	(12)(34)
# class	1	6	8	6	3
$\mathbb{1}$	1	1	1	1	1
ε	1	-1	1	-1	1
ψ					
χ	3	1	0	-1	-1
$\chi\varepsilon$	3	-1	0	1	-1

Irreducible degrees divide $\#G$

Theorem (Irreducible degrees divide $\#G$)

Let G be a finite group. Then $\deg \chi \mid \#G$ for all $\chi \in \mathrm{Irr}(G)$.

Proof.

Admitted. □

Example

The degrees of the irreducible representations of $G = S_4$ are $1, 1, 2, 3, 3$, which all divide $\#G = 24$.

Burnside's theorem (Non examinable)

Theorem (Burnside)

Let G be a finite group. If $\#G$ has at most 2 distinct prime factors, then G is solvable.

Proof.

Admitted, but the proof relies on representation theory! □

Corollary

Let G be a finite group. If G is simple and non-Abelian, then $\#G$ has at least 3 distinct prime factors.

Example

The smallest non-Abelian simple group is A_5, whose order is

$$60 = 2^2 \cdot 3 \cdot 5.$$

The next one has order $168 = 2^3 \cdot 3 \cdot 7$.