Galois theory — Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU34101/index.html

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Exercise 1 *The fifth cyclotomic field*

In this exercise, we consider the primitive 5th root $\zeta = e^{2\pi i/5}$, and we set $L = \mathbb{Q}(\zeta)$. We know that L is Galois over $\mathbb{Q}$, so we define $G = \mathrm{Gal}(L/\mathbb{Q})$. We also let

$$c = \frac{\zeta + \zeta^{-1}}{2} = \cos(2\pi/5) = 0.309\cdots,$$

$$C = \mathbb{Q}(c),$$

and finally

$$c' = \frac{\zeta^2 + \zeta^{-2}}{2} = \cos(4\pi/5) = -0.809\cdots.$$

1. Write down explicitly the minimal polynomial of ζ over $\mathbb{Q}$, and express its complex roots in terms of ζ.
2. Deduce that $\zeta + \zeta^2 + \zeta^3 + \zeta^4 = -1$.
3. Prove that G is a cyclic group. What is its order? Find an explicit generator of G.
4. Deduce that $c \notin \mathbb{Q}$.
5. Make the list of all subgroups of G.
6. Draw a diagram showing all the fields E such that $\mathbb{Q} \subset E \subset L$, ordered by inclusion.
7. What are the conjugates of c over $\mathbb{Q}$? Determine explicitly the minimal polynomial of c over $\mathbb{Q}$ (exact computations only, computations with the approximate value of c are forbidden).
8. Deduce that
$$c = \frac{-1 + \sqrt{5}}{4}.$$
9. What are the conjugates of ζ over C (as opposed to over $\mathbb{Q}$)?
10. Deduce that
$$\zeta = \frac{-1 + \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}}{4}.$$

Solution 1

1. The minimal polynomial of ζ over $\mathbb{Q}$ is the 5th cyclotomic polynomial

$$\Phi_5(x) = \frac{x^5-1}{\Phi_1(x)} = \frac{x^5-1}{x-1} = x^4+x^3+x^2+x+1.$$

   Its complex roots are the primitive 5-th roots of 1, namely

$$\mu_5^\times = \{\zeta^k \mid k \in (\mathbb{Z}/5\mathbb{Z})^\times\} = \{\zeta, \zeta^2, \zeta^3, \zeta^4\}.$$

2. By Vieta and the previous question,

$$\zeta+\zeta^2+\zeta^3+\zeta^4 = \sum \text{Roots of } \Phi_5(x) = -\text{Coeff. of } x^3 = -1.$$

   Alternatively,

$$1+\zeta+\zeta^2+\zeta^3+\zeta^4 = \sum \text{Roots of } x^5-1 = -\text{Coeff. of } x^4 = 0.$$

3. We know that G can be identified to $(\mathbb{Z}/5\mathbb{Z})^\times$ by matching $x \in (\mathbb{Z}/5\mathbb{Z})^\times$ to $\sigma_x : z \mapsto z^x \in G$ for all $z \in \mu_5$. In particular, $G \simeq (\mathbb{Z}/5\mathbb{Z})^\times$ is an Abelian group of order $\phi(5) = 4$. To prove that it is cyclic, we can notice that $2 \in (\mathbb{Z}/5\mathbb{Z})^\times$ is a generator since

$$2 \neq 1, \quad 2^2 = 4 \neq 1, \quad 2^3 = 3 \neq 1, \quad 2^4 = 1,$$

   so that $\sigma_2 \in G$ is a generator. We can also argue that since 5 is prime, $\mathbb{Z}/5\mathbb{Z}$ is a finite field, so its multiplicative group is cyclic; but then we still need to find an explicit generator.

4. The element $\sigma_2 \in G$ acts on L by $\zeta \mapsto \zeta^2$, and thus takes $c = \frac{\zeta+\zeta^{-1}}{2}$ to $\frac{\zeta^2+\zeta^{-2}}{2} = c'$. Since $c' \neq c$ (clear from their numerical values), we have $c \notin L^G$; but $L^G = \mathbb{Q}$ since L is Galois over $\mathbb{Q}$.

5. Since $G \simeq \mathbb{Z}/4\mathbb{Z}$ is cyclic of order 4, its only nontrivial subgroup is $2\mathbb{Z}/4\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$, and is generated by $\sigma_2^2 = \sigma_4 = \sigma_{-1}$. Our list is thus

$$\{1\}$$
$$|$$
$$H = \{\sigma_{\pm 1}\}$$
$$|$$
$$G.$$

6. We already know that under the Galois correspondence, $\{\sigma_1\}$ corresponds to L, and G to $\mathbb{Q}$. It remains to identify L^H.

   We know that $[L^H : \mathbb{Q}] = [G : H] = 2$; besides $\sigma_{-1} \in H$ acts by $\zeta \mapsto \zeta^{-1}$ (i.e. is the complex conjugation) and therefore fixes c, so that $c \in L^H$, whence

$$C = \mathbb{Q}(c) \subseteq L^H.$$

Since $c \notin \mathbb{Q}$, we have $C \supsetneq \mathbb{Q}$ and so $[C : \mathbb{Q}] \geqslant 2$, so finally

$$L^H = C.$$

Our diagram is thus

$$\begin{array}{c} L \\ | \\ C \\ | \\ \mathbb{Q}. \end{array}$$

7. The conjugates of c over $\mathbb{Q}$ are the $\sigma(c)$ for $\sigma \in \operatorname{Gal}(L/\mathbb{Q})$ (and also for $\sigma \in \operatorname{Gal}(C/\mathbb{Q})$), i.e.

   - c itself for $\sigma = 1$,
   - $\frac{\zeta^2+\zeta^{-2}}{2} = c'$ for $\sigma = 2$,
   - $\frac{\zeta^{-2}+\zeta^{2}}{2} = c'$ for $\sigma = 3 = -2$,
   - and $\frac{\zeta^{-1}+\zeta}{2} = c$ for $\sigma = 4 = -1$,

   so finally, just c itself and c'. (We can get the same conclusion faster by taking σ in the smaller quotient $\operatorname{Gal}(C/\mathbb{Q}) = G/H = \{\sigma_{\pm 1}, \sigma_{\pm 2}\} \simeq (\mathbb{Z}/5\mathbb{Z})^\times / \pm 1$ of $\operatorname{Gal}(L/\mathbb{Q})$, if we are not afraid to work with this quotient). The minimal polynomial of c over $\mathbb{Q}$ is thus

$$\begin{aligned} \prod_{\beta \text{ conjugate to } c} (x - \beta) &= (x-c)(x-c') \\ &= x^2 - (c+c')x + cc' \\ &= x^2 - \frac{\zeta + \zeta^{-1} + \zeta^2 + \zeta^{-2}}{2} x + \frac{(\zeta+\zeta^{-1})(\zeta^2+\zeta^{-2})}{4} \\ &= x^2 - \frac{\zeta + \zeta^4 + \zeta^2 + \zeta^3}{2} x + \frac{\zeta^3 + \zeta^4 + \zeta + \zeta^2}{4} \\ &= x^2 + \frac{1}{2}x - \frac{1}{4}. \end{aligned}$$

8. By solving $x^2 + \frac{1}{2}x - \frac{1}{4} = 0$, we find that $\Delta = 5/4$, whence

$$c, c' = \frac{-1 \pm \sqrt{5}}{4}.$$

   Since $c > c'$, we deduce that $c = \frac{-1+\sqrt{5}}{4}$ (and also that $c' = \frac{-1-\sqrt{5}}{4}$).

9. The conjugates of ζ over C are the elements of the orbit of ζ under $H = \operatorname{Gal}(L/C)$, that is to say the $\sigma(\zeta)$ for $\sigma \in \operatorname{Gal}(L/C) = H = \{\sigma_{\pm 1}\}$. So they are ζ and ζ^{-1}.

10. Similarly to the previous questions, the minimal polynomial of ζ over C is

$$\prod_{\substack{\beta \text{ conjugate} \\ \text{of } \zeta \text{ over } C}} (x-\beta) = (x-\zeta)(x-\zeta^{-1}) = x^2-(\zeta+\zeta^{-1})x+\zeta\zeta^{-1} = x^2-2cx+1 \in C[x],$$

whose roots are

$$\frac{2c \pm \sqrt{4c^2 - 4}}{2} = c \pm \sqrt{c^2 - 1} = \frac{-1 + \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}}{4}$$

using $c = \frac{-1+\sqrt{5}}{4}$. Since ζ (as opposed to ζ^{-1}) is the root with positive imaginary part (draw a regular pentagon; alternatively $\operatorname{Im} \zeta = \sin(2\pi/5) > 0$ as $2\pi/5 < \pi$), we conclude that

$$\zeta = \frac{-1 + \sqrt{5} + i\sqrt{10 + 2\sqrt{5}}}{4}.$$