Galois theory — Exercise sheet 2

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU34101/index.html

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Email your answers to mascotn@tcd.ie by Friday October 22nd, 4PM.

Exercise 1 *A cyclic biquadratic extension (100 pts)*

Let $\alpha = \sqrt{13}$, $K = \mathbb{Q}(\alpha)$, $\beta = i\sqrt{65 + 18\sqrt{13}}$ (where $i^2 = -1$), $\beta' = i\sqrt{65 - 18\sqrt{13}}$ (note that $65 > 18\sqrt{13}$), and $L = \mathbb{Q}(\beta)$.

1. *(6 pts)* Prove that the minimal polynomial of β over $\mathbb{Q}$ is
$$M(x) = (x^2 + 65)^2 - 18^2 \cdot 13 = x^4 + 130x^2 + 13.$$
2. *(10 pts)* What are the Galois conjugates of β over $\mathbb{Q}$?
3. *(14 pts)* Prove that L is a Galois extension of $\mathbb{Q}$.
Hint: Check that $\beta\beta' = -\alpha$.
4. *(6 pts)* Explain why there exists an element $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that $\sigma(\beta) = \beta'$.
5. *(10 pts)* Let $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ be such that $\sigma(\beta) = \beta'$ as above. Explain why $\sigma(\alpha)$ makes sense, and determine $\sigma(\alpha)$.
6. *(10 pts)* Let again $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ be such that $\sigma(\beta) = \beta'$ as above. Determine the action of σ on the conjugates of β.
Hint: Again, $\beta\beta' = -\alpha$.
7. *(20 pts)* Deduce that $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathbb{Z}/4\mathbb{Z}$.
8. *(12 pts)* Sketch a diagram showing all the fields $\mathbb{Q} \subseteq E \subseteq L$, ordered by inclusion.
9. *(12 pts)* Does $i\sqrt{13} \in L$?

Solution 1

1. First of all, we have $\beta^2 = -(65 + 18\sqrt{13})$ so $(\beta^2 + 65)^2 = (18\sqrt{13})^2$, so β is indeed a root of $M(x)$. Besides, the expanded form of $M(x)$ reveals that it is Eisenstein at 13, so it is irreducible over $\mathbb{Q}$; since it is also monic, it is the minimal polynomial of β over $\mathbb{Q}$.

2. The Galois conjugates of β over $\mathbb{Q}$ are by definition the roots of its minimal polynomial over $\mathbb{Q}$, namely $M(x)$. Since it is of degree 4, there are at most 4 of them (in fact exactly 4, because we are in characteristic 0 so this irreducible polynomial must be separable). But one checks as above that $\pm\beta$ and $\pm\beta'$ are roots of $M(x)$; since these 4 numbers are distinct, they are the Galois conjugates of β.

3. We find indeed that

$$\beta\beta' = -\sqrt{(65+18\sqrt{13})(65-18\sqrt{13})} = \sqrt{65^2 - 18^2 \cdot 13} = -\sqrt{13}.$$

Besides, $L \ni \beta^2 = -(65+18\sqrt{13})$, so $\sqrt{13} \in L$ since $65, 18 \in L \supset \mathbb{Q}$. Therefore $\beta' = -\sqrt{13}/\beta \in L$. It follows that the conjugates of β lie in L, so $L = \mathbb{Q}(\beta)$ is the splitting field of $M(x)$ over $\mathbb{Q}$, and is therefore a normal extension of $\mathbb{Q}$. It must also be separable, since the characteristic of $\mathbb{Q}$ is 0.

4. Since $L/\mathbb{Q}$ is Galois, given any conjugate γ of β, there exists at least one $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that $\sigma(\beta) = \gamma$.

5. σ is a function from L to L, so $\sigma(\alpha)$ makes sense since we have shown that $\alpha \in L$. More specifically, we have that

$$\alpha = -\frac{\beta^2 + 65}{18},$$

so

$$\sigma(\alpha) = \sigma\left(-\frac{\beta^2+65}{18}\right) = -\frac{\sigma(\beta^2)+65}{18} = -\frac{\beta'^2+65}{18} = -\alpha$$

since $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ is a field automorphism which fixes the rationals.

6. These conjugates are $\pm\beta$ and $\pm\beta'$, and we already know that $\sigma(\beta) = \beta'$, which immediately implies that $\sigma(-\beta) = -\beta'$. Besides, since $\beta' = -\alpha/\beta$, we have

$$\sigma(\beta') = -\sigma(\alpha)/\sigma(\beta) = \alpha/\beta' = -\beta,$$

which immediately implies that $\sigma(-\beta') = \beta$.

7. We know that $\#\,\mathrm{Gal}(L/\mathbb{Q}) = [L:\mathbb{Q}] = [\mathbb{Q}(\beta):\mathbb{Q}] = \deg M(x) = 4$. Lagrange therefore implies that the order of σ is 1 or 2 or 4. But the above question shows that neither σ nor σ^2 is the identity, so σ has order 4. As result, $\mathrm{Gal}(L/\mathbb{Q})$, which is a group of order 4 which contains an element of order 4, must be cyclic (and generated by this element σ; more specifically, we see that σ acts on the conjugates of β by the 4-cycle $\beta \mapsto \beta' \mapsto -\beta \mapsto -\beta' \mapsto \beta$).

8. Since $\mathrm{Gal}(L/\mathbb{Q}) = \langle\sigma\rangle$ is cyclic, its only nontrivial subgroup is $H = \langle\sigma^2\rangle$, which has cardinal 2 and therefore index 2. The Galois correspondence thus shows that

$$\mathbb{Q} \subset L^H \subset L$$

is the complete list of intermediate fields, where both inclusions are of degree 2. On the other hand, we know that $\alpha \in L$, so $K = \mathbb{Q}(\alpha)$ is a subfield of L. The minimal polynomial of α over $\mathbb{Q}$ is $x^2 - 13$ (Eisenstein at 13), so $[K:\mathbb{Q}] = 2$; therefore $K = L^H$. Final answer:

$$\mathbb{Q} \subset K \subset L.$$

9. If $i\sqrt{13} \in L$, then $E = \mathbb{Q}(i\sqrt{13})$ is a subfield of L, of degree 2 over $\mathbb{Q}$ (same argument: the minimal polynomial of $i\sqrt{13}$ is x^2+13), so by the above question we must have $E = K$. But this is absurd, for instance because $K \subset \mathbb{R}$ whereas $E \not\subset \mathbb{R}$. So $i\sqrt{13} \notin L$.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *Yes or no?*

Let $f(x) = x^3 + x + 1 \in \mathbb{Q}[x]$ (you may assume without proof that f is irreducible over $\mathbb{Q}$), and let $L = \mathbb{Q}[x]/(f)$.

1. Is L a separable extension of $\mathbb{Q}$? Explain.
2. Is L a normal extension of $\mathbb{Q}$? Explain.

   Hint: What does the fact that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing tell you about the complex roots of f?
3. Is L a Galois extension of $\mathbb{Q}$? Explain.

Solution 2

1. Yes, since all fields of characteristic 0 are perfect.
2. Since $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing, f has exactly one real root α (intermediate value theorem) and thus one complex-conjugate pair of roots β, $\bar{\beta}$. The images of L by its $[L : \mathbb{Q}] = 3$ $\mathbb{Q}$-embeddings into $\mathbb{C}$ are $\mathbb{Q}(\alpha) \subset \mathbb{R}$, $\mathbb{Q}(\beta) \not\subset \mathbb{R}$, and $\mathbb{Q}(\overline{\beta}) \not\subset \mathbb{R}$. Since some are $\subset \mathbb{R}$ but others are not, they do not all agree, so L is not normal over $\mathbb{Q}$.
3. No, since it is not normal over $\mathbb{Q}$.

Exercise 3 *Square roots: warm-up*

This exercise is not Galois theory per se*, but is meant as a warm-up for the next exercise. The results it establishes may also be used profitably on future exercises.*

Recall that each positive integer can be factored uniquely *into a product of primes, and that each rational number can be written* uniquely *as n/d with $n \in \mathbb{Z}$, $d \in \mathbb{Z}_{\geqslant 1}$, and* $\gcd(n, d) = 1$.

1. Let $r = n/d \in \mathbb{Q}^\times$ be a nonzero rational number, where $n \in \mathbb{Z}$, $d \in \mathbb{Z}_{\geqslant 1}$, and $\gcd(n, d) = 1$. Prove that r is a square in $\mathbb{Q}$ iff. n and d are squares in $\mathbb{N}$.
2. Let $a, b \in \mathbb{Q}^\times$. Prove that $\mathbb{Q}(\sqrt{a}) = \mathbb{Q}(\sqrt{b})$ iff. a/b is a square in $\mathbb{Q}$.

Solution 3

1. Clearly, if $n = m^2$ and $d = e^2$ are squares in $\mathbb{N}$, then $r = (m/e)^2$ is a square in $\mathbb{Q}$. Conversely, suppose r is a square in $\mathbb{Q}$, say $r = s^2$ with $s = m/e \in \mathbb{Q}$ where $\gcd(m, e) = 1$. Then $\gcd(m^2, e^2) = 1$ (any nontrivial common factor of m^2 and e^2 would have a prime factor, which would show up in the factorisation of m^2 and thus of m, and also in that of e^2 and thus of e, absurd), so $r = m^2/e^2$ is of the desired form.

2. We distinguish 3 cases.

   - If a and b are both squares in $\mathbb{Q}$, then so is a/b; moreover $\mathbb{Q}(\sqrt{a}) = \mathbb{Q}(\sqrt{b}) = \mathbb{Q}$, so the equivalence is satisfied.
   - If one of a and b is a square in $\mathbb{Q}$, say b is but a is not, then neither is a/b (else $a = a/b \times b$ would be), and $\mathbb{Q}(\sqrt{b}) = \mathbb{Q} \not\ni \sqrt{a}$ so $\mathbb{Q}(\sqrt{a}) \supsetneq \mathbb{Q}(\sqrt{b})$, so the equivalence is again satisfied.
   - It remains to check the equivalence when neither a nor b are squares in $\mathbb{Q}$.
     - Suppose a/b is a square in $\mathbb{Q}$, say $a/b = r^2$ where $r \in \mathbb{Q}^\times$. Then $\sqrt{b} = r\sqrt{a} \in \mathbb{Q}(\sqrt{a})$ so $\mathbb{Q}(\sqrt{b}) \subseteq \mathbb{Q}(\sqrt{a})$, and $\sqrt{a} = \frac{1}{r}\sqrt{b} \in \mathbb{Q}(\sqrt{b})$ so $\mathbb{Q}(\sqrt{a}) \subseteq \mathbb{Q}(\sqrt{b})$, whence $\mathbb{Q}(\sqrt{a}) = \mathbb{Q}(\sqrt{b})$.
     - Conversely, suppose $\mathbb{Q}(\sqrt{a}) = \mathbb{Q}(\sqrt{b})$. Then $\sqrt{b} \in \mathbb{Q}(\sqrt{a}) = \{x + y\sqrt{a} \mid x, y \in \mathbb{Q}\}$, so $\sqrt{b} = u + v\sqrt{a}$ for some $u, v \in \mathbb{Q}$. Squaring yields $b = (u^2 + av^2) + 2uv\sqrt{a}$; since a is not a square in $\mathbb{Q}$, $\mathbb{Q}(\sqrt{a})$ admits $1, \sqrt{a}$ as a $\mathbb{Q}$-basis, whence $u^2 + av^2 = b$ and $2uv = 0$ by identifying coefficients on this basis. If $v = 0$, then $u^2 = b$, contrary to our assumption that b is not a square. Therefore $u = 0$, so $av^2 = b$, whence $a/b = (1/v)^2$ is a square in $\mathbb{Q}$.

Remark: One shows similarly that given a field K and a, b in K, we have $K(\sqrt{a}) = K(\sqrt{b})$ iff. a/b is a square in K. However, the latter condition is more delicate to assess, since we do not have an analogue of the first question for a general field K.

Exercise 4 *Square roots*

You may want to use the results established in the previous exercise to solve this exercise.

Let $L = \mathbb{Q}(\sqrt{10}, \sqrt{42})$.

1. Prove that L is a Galois extension of $\mathbb{Q}$.
2. Prove that $[L : \mathbb{Q}] = 4$.
3. Describe all the elements of $\mathrm{Gal}(L/\mathbb{Q})$. What is $\mathrm{Gal}(L/\mathbb{Q})$ isomorphic to?
4. Sketch the diagram showing all intermediate extensions $\mathbb{Q} \subseteq E \subseteq L$, ordered by inclusion. Explain clearly which field corresponds to which subgroup.
5. Does $\sqrt{15} \in L$? Use the previous question to answer.

Solution 4

1. L is the splitting field over $\mathbb{Q}$ of $(x^2-10)(x^2-42) \in \mathbb{Q}[x]$ which is separable (not multiple root), so it is Galois over $\mathbb{Q}$.

2. Since 10 is not a square, $\mathbb{Q}(\sqrt{10}) \neq \mathbb{Q}$, so $[\mathbb{Q}(\sqrt{10}) : \mathbb{Q}] = 2$. In order to conclude that $[L : \mathbb{Q}] = 4$, we need to prove that $[L : \mathbb{Q}(\sqrt{10})]$ is 2 and not 1, i.e. that $\sqrt{42} \notin \mathbb{Q}(\sqrt{10})$. This follows from the previous exercise, since $\frac{42}{10} = \frac{21}{5}$ is not a square in $\mathbb{Q}$ as 21 is not a square in $\mathbb{N}$.

3. We already know that $\#\operatorname{Gal}(L/\mathbb{Q}) = [L : \mathbb{Q}] = 4$ since L is Galois over $\mathbb{Q}$. Besides, an element $\sigma \in \operatorname{Gal}(L/\mathbb{Q})$ must take $\sqrt{10} \in L$ to a root of $x^2 - 10 \in \mathbb{Q}[x]$, i.e. to $\pm\sqrt{10}$; and similarly $\sigma(\sqrt{42}) = \pm\sqrt{42}$. Since σ is completely determined by what it does to $\sqrt{10}$ and to $\sqrt{42}$, this leaves us with only 4 possibilities for σ. But since $\#\operatorname{Gal}(L/\mathbb{Q}) = 4$, all these possibilities must occur. Therefore, $\operatorname{Gal}(L/\mathbb{Q})$ is made up of

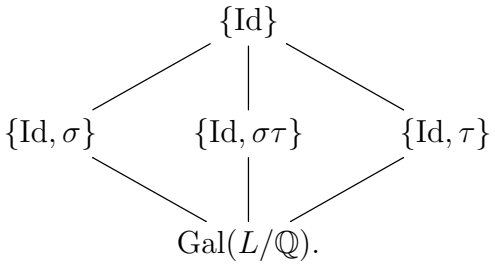
   - Id,
   - $\sigma : \sqrt{10} \mapsto -\sqrt{10},\ \sqrt{42} \mapsto \sqrt{42}$,
   - $\tau : \sqrt{10} \mapsto \sqrt{10},\ \sqrt{42} \mapsto -\sqrt{42}$,
   - $\sigma\tau : \sqrt{10} \mapsto -\sqrt{10},\ \sqrt{42} \mapsto -\sqrt{42}$.

   We see that $\sigma\tau = \tau\sigma$, and that $\sigma^2 = \tau^2 = (\sigma\tau)^2 = \mathrm{Id}$. Therefore

   $$\begin{array}{ccc} (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \operatorname{Gal}(L/\mathbb{Q}) \\ (a,b) & \longmapsto & \sigma^a\tau^b \end{array}$$

   is a group isomorphism.

4. We know from class that since $\operatorname{Gal}(L/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, its subgroup diagram is

   Let us now find the corresponding fields.

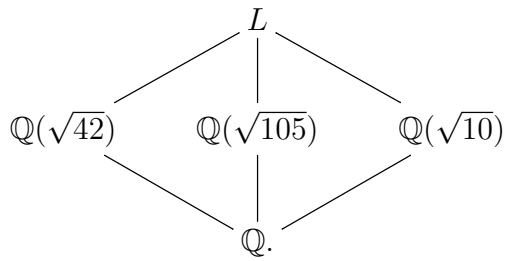
   - Clearly, $L^{\{\mathrm{Id}\}} = L$.
   - We also have $L^{\operatorname{Gal}(L/\mathbb{Q})} = \mathbb{Q}$ since L is Galois over $\mathbb{Q}$.
   - We know that $L^{\{\mathrm{Id},\sigma\}}$ is an extension of $\mathbb{Q}$ of degree $[\operatorname{Gal}(L/\mathbb{Q}) : \{\mathrm{Id}, \sigma\}] = 2$. It is the subfield of L formed of the elements fixed by σ, so it contains $\sqrt{42}$ and thus $\mathbb{Q}(\sqrt{42})$. Since the latter is already an extension of $\mathbb{Q}$ of degree 2, it must agree with $L^{\{\mathrm{Id},\sigma\}}$.
   - Similarly, $L^{\{\mathrm{Id},\tau\}}$ is an extension of degree 2 of $\mathbb{Q}$, which contains $\sqrt{10}$ as it is fixed by τ, so $L^{\{\mathrm{Id},\tau\}} = \mathbb{Q}(\sqrt{10})$.

- Finally, $L^{\{\mathrm{Id},\sigma\tau\}}$ is an extension of degree 2 of $\mathbb{Q}$, but it contains neither $\sqrt{10}$ nor $\sqrt{42}$ since they are not fixed by $\sigma\tau$. However, $\sqrt{10}\sqrt{42} = \sqrt{420}$ is fixed by $\sigma\tau$ since

$$\sigma\tau(\sqrt{10}\sqrt{42}) = (-\sqrt{10})(-\sqrt{42}),$$

so $L^{\{\mathrm{Id},\sigma\tau\}} = \mathbb{Q}(\sqrt{420}) = \mathbb{Q}(\sqrt{105})$.

The field diagram is thus



5. No. Indeed, if $\sqrt{15} \in L$, then $\mathbb{Q}(\sqrt{15})$ is an intermediate field, but that contradicts the previous question: in view of the previous exercise, $\mathbb{Q}(\sqrt{15})$ is neither of $\mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt{42})$, $\mathbb{Q}(\sqrt{105})$ as neither $\frac{15}{10} = \frac{3}{2}$, $\frac{15}{42} = \frac{5}{14}$, $\frac{15}{105} = \frac{1}{7}$ are squares in $\mathbb{Q}$.

Exercise 5 *Bioche vs. Galois*

The goal of this exercise is to give a Galois-theoretic interpretation of Bioche's rules (cf. `https://en.wikipedia.org/wiki/Bioche%27s_rules` *), which are rules suggesting appropriate substitutions to turn integrals involving trigonometric functions into integrals of rational fractions.Knowledge of Bioche's rules is not required to solve this exercise.*

In this exercise, we use the shorthands s for the sine function and c for the cosine function, and we denote by $\mathbb{C}(s,c)$ the set of expressions such as

$$\frac{2sc^3 - i}{c - 7s + 3} = \frac{2\sin x \cos^3 x - i}{\cos x - 7\sin x + 3}$$

which are rational fractions in $s = \sin x$ and $c = \cos x$ with complex coefficients. Observe that $\mathbb{C}(s,c)$ is a field with respect to point-wise addition and multiplication.

We write $\mathbb{C}(c)$ for the subfield of $\mathbb{C}(s,c)$ consisting of rational fractions which can be expressed in terms of c only, and similarly $\mathbb{C}(s)$ for rational fractions in s only. For example, $\frac{c^3-2c^2+2i}{ic-1} \in \mathbb{C}(c)$, but $s \notin \mathbb{C}(c)$ since all the elements of $\mathbb{C}(c)$ are even functions whereas s is not; observe however that $s^2 \in \mathbb{C}(c)$ since $s^2 = 1 - c^2$.

We also define $K = \mathbb{C}(s) \cap \mathbb{C}(c) \subset \mathbb{C}(s,c)$, so that for instance the function $c_2 = \cos(2x)$ lies in K since $c_2 = 2c^2 - 1 = 1 - 2s^2$.

Finally, we define

$$\begin{array}{rcl} \mu : \mathbb{C}(s,c) & \to & \mathbb{C}(s,c) \\ f(x) & \mapsto & f(-x), \end{array} \qquad \begin{array}{rcl} \tau : \mathbb{C}(s,c) & \to & \mathbb{C}(s,c) \\ f(x) & \mapsto & f(x+\pi), \end{array} \qquad \begin{array}{rcl} \sigma : \mathbb{C}(s,c) & \to & \mathbb{C}(s,c) \\ f(x) & \mapsto & f(\pi - x); \end{array}$$

observe that these are field automorphisms of $\mathbb{C}(s,c)$ which are involutive and commute with each other, so they generate the subgroup

$$G = \{\mathrm{Id},\, \mu = \sigma\tau,\, \tau = \mu\sigma,\, \sigma = \mu\tau\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

of $\mathrm{Aut}\big(\mathbb{C}(s,c)\big)$.

1. Prove that the four inclusions $K \subset \mathbb{C}(s) \subset \mathbb{C}(s,c)$ and $K \subset \mathbb{C}(s) \subset \mathbb{C}(s,c)$ are all strict.

2. Prove that $[\mathbb{C}(s) : K] = [\mathbb{C}(s,c) : \mathbb{C}(s)] = [\mathbb{C}(c), K] = [\mathbb{C}(s,c) : \mathbb{C}(c)] = 2$.

3. Prove that $K = \mathbb{C}(c_2)$, where $\mathbb{C}(c_2)$ is the field of rational fractions expressible in terms of c_2 only.

4. Prove that the extension $\mathbb{C}(s,c)/K$ is Galois, and describe its Galois group.

5. Let $f \in \mathbb{C}(s,c)$. Prove that if f is invariant by any two of μ, τ, σ, then it is also invariant by the third one, and that in this case $f \in \mathbb{C}(c_2)$.

6. Determine the minimal polynomials over K of the elements $t = \tan x = s/c$ and $s_2 = \sin(2x) = 2sc$ of $\mathbb{C}(s,c)$.

7. Draw a diagram showing all the subgroups of $\mathrm{Gal}(\mathbb{C}(s,c)/K)$.

8. Draw a diagram showing all the intermediate fields E between K and $\mathbb{C}(s,c)$. Where are the fields $\mathbb{C}(t)$, $\mathbb{C}(s_2,c_2)$, and $\mathbb{C}(s_2)$ on this diagram?

Make sure find an explanation for all the surprising conclusions you may be led to!

Solution 5

1. Every element of $\mathbb{C}(c)$ is even since c is; therefore $s \notin \mathbb{C}(c)$, so $\mathbb{C}(s,c) = \mathbb{C}(c)(s) \supsetneq \mathbb{C}(s)$. The same argument shows that $s \notin K \subset \mathbb{C}(c)$, so $\mathbb{C}(s) \supsetneq K$. Besides, s is invariant by σ whereas c is not, so $c \notin \mathbb{C}(s)$ so $\mathbb{C}(s,c) \supsetneq \mathbb{C}(s)$, and similarly $c \notin K$ so $\mathbb{C}(c) \supsetneq K$.

2. Since $s^2+c^2 = 1$, s is a root of the polynomial $x^2-(1-c^2) \in \mathbb{C}(c)[x]$; therefore, s is algebraic over $\mathbb{C}(c)$ over degree at most 2; since $\mathbb{C}(s,c) = \mathbb{C}(c)(s)$, this shows that $[\mathbb{C}(s,c) : \mathbb{C}(c)] \leq 2$. Since this degree cannot be 1 by the previous question, it must be 2. Similarly, $[\mathbb{C}(s,c) : \mathbb{C}(s)] = 2$.

   The identity $c_2 = 2c^2 - 1$ proves that c is a root of $2x^2 - 1 - c_2 \in K[x]$, so c is algebraic of degree at most 2 over K. We have $\mathbb{C}(c) \subseteq K(c)$ since $\mathbb{C} \subset K$, and $K(c) \subseteq \mathbb{C}(c)$ since $K \subset \mathbb{C}(c)$, so $\mathbb{C}(c) = K(c)$ is an extension of K of degree at most 2, hence exactly 2 by the previous question. Similarly, $\mathbb{C}(s) = K(s)$ is an extension of K of degree at most 2, and hence 2, since s is a root of $2x^2 + c_2 - 1 \in K[x]$.

3. We know that $\mathbb{C}(c_2) \subseteq K \subsetneq \mathbb{C}(c)$; besides, since $\mathbb{C}(c) = \mathbb{C}(c, c_2) = \mathbb{C}(c_2)(c)$ as $c_2 = 2c^2-1 \in \mathbb{C}(c)$, the fact that the polynomial $2x^2-1-c_2$ used in the previous question actually lies in $\mathbb{C}(c_2)[x]$ shows that we have $[\mathbb{C}(c) : \mathbb{C}(c_2)] \leq 2$. The tower law allows us to conclude that $[K : \mathbb{C}(c_2)] \leq 1$.

4. The tower law shows that $[\mathbb{C}(s,c) : \mathbb{C}(c_2)] = [\mathbb{C}(s,c) : \mathbb{C}(c)][\mathbb{C}(c) : K] = 2 \times 2 = 4$, so $\#\operatorname{Aut}_K\big(\mathbb{C}(s,c)\big) \leq 4$, with equality iff. $\mathbb{C}(s,c)$ is Galois over K. But since c_2 is fixed by Id, μ, τ, and σ, these 4 automorphisms induce the identity on $\mathbb{C}(c_2) = K$; therefore $\#\operatorname{Aut}_K\big(\mathbb{C}(s,c)\big) \geq 4$. In conclusion, $\#\operatorname{Aut}_K\big(\mathbb{C}(s,c)\big) = 4 = [\mathbb{C}(s,c) : K]$, so $\mathbb{C}(s,c)$ is Galois over K with Galois group $\mathrm{Gal}(\mathbb{C}(s,c)/K) = \{\mathrm{Id}, \mu, \sigma, \tau\} = G$.

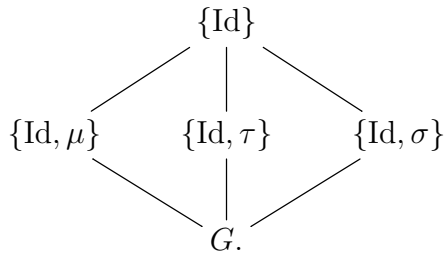
5. Since any two of μ, τ, σ generate G, any element of $\mathbb{C}(s,c)$ fixed by two of those is actually fixed by the whole of $G = \mathrm{Gal}(\mathbb{C}(s,c)/K)$, and therefore lies in $\mathbb{C}(s,c)^{\mathrm{Gal}(\mathbb{C}(s,c)/K)} = K = \mathbb{C}(c_2)$.

6. Since $\mathbb{C}(s,c)$ is Galois over K, the minimal polynomial of any $\alpha \in \mathbb{C}(s,c)$ is the polynomial whose roots are the orbit of α under $\mathrm{Gal}(\mathbb{C}(s,c)) = G$.

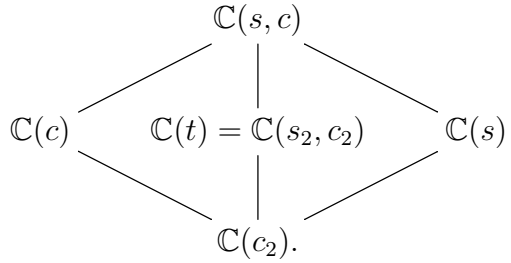
   In the case $\alpha = t$, this orbit is $\{\mathrm{Id}\, t = t, \mu t = -t, \tau t = t, \sigma t = -t\} = \{t, -t\}$, so the minimal polynomial of t over K is $(x-t)(x+t) = x^2 - t^2$. It must lie in $K[x]$, so we necessarily have $t^2 \in K = \mathbb{C}(c_2)$; indeed, we find that $t^2 = \frac{s^2}{c^2} = \frac{1-c_2}{1+c_2} \in \mathbb{C}(c_2)$.

   Similarly, since the orbit of s_2 under G is $\{s_2, -s_2\}$, the minimal polynomial of s_2 over K is $(x-s_2)(x+s_2) = x^2 - s_2^2$, so we must have $s_2^2 \in K = \mathbb{C}(c_2)$; and indeed $s_2^2 = (2sc)^2 = (2s^2)(2c^2) = (1+c_2)(1-c_2) = 1 - c_2 \in \mathbb{C}(c_2)$ — that is simply $s_2^2 + c_2^2 = 1$.

7. Since $\mathrm{Gal}(\mathbb{C}(s,c)/K) = G \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, its subgroup lattice is



8. We apply the Galois correspondence. The subfields corresponding to $\{\mathrm{Id}\}$ and G are of course $\mathbb{C}(s,c)$ and $K = \mathbb{C}(c_2)$, respectively. The subfield corresponding to $\{\mathrm{Id}, \mu\}$ contains c since μ fixes c, and is an extension of K of degree $[G : \{\mathrm{Id}, \mu\}] = 4/2 = 2$, so it is $\mathbb{C}(c)$ by the second question. Similarly, the subfield corresponding to $\{\mathrm{Id}, \sigma\}$ is $\mathbb{C}(s)$. Finally, the subfield corresponding to $\{\mathrm{Id}, \tau\}$ is also an extension of K of degree 2; besides, it contains t since t is invariant by τ. By the previous question, $\mathbb{C}(t)$ is an extension of K of degree at most 2; but $t \notin K$ since t is not fixed by μ, so this extension has degree exactly 2, so it is the subfield corresponding to $\{\mathrm{Id}, \tau\}$. The same thing can be said about $K(s_2)$, so we are led to the curious conclusion that $\mathbb{C}(t) = K(s_2) = \mathbb{C}(s_2, c_2)$; and indeed $t = \frac{s}{c} = \frac{2sc}{2c^2} = \frac{s_2}{1+c_2} \in \mathbb{C}(s_2, c_2)$ whereas $s_2 = 2sc = 2tc^2 = \frac{2tc^2}{s^2+c^2} = \frac{2t}{t^2+1} \in \mathbb{C}(t)$.



   As for $\mathbb{C}(s_2)$, it does not appear on this diagram, simply because $\mathbb{C}(c_2) \not\subset \mathbb{C}(s_2)$! (so yes, that was a trap.) Indeed, every element of $\mathbb{C}(s_2)$ is invariant by $x \mapsto \pi/2 - x$ since s_2 is; but c_2 is not.