Galois theory — Exercise sheet 4

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU34101/index.html

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Email your answers to mascotn@tcd.ie by Monday 22nd November, 4PM.

Exercise 1 *A polynomial with Galois group A_4 (100 pts)*

Let $F(x) = x^4 - 2x^3 + 2x^2 + 2 \in \mathbb{Q}[x]$. We denote the roots of $F(x)$ in $\mathbb{C}$ by α_1, α_2, α_3, and α_4.

In this exercise, you may use without proof the following facts:

- *The discriminant of f is $\Delta_f = 3136 = 2^6 \cdot 7^2$.*
- *The transitive subgroups of the symmetric group S_4 are*
  - *S_4 itself,*
  - *the alternating group A_4,*
  - *the dihedral group D_8 of symmetries of the square acting on the vertices of the square,*
  - *the Klein group $V_4 = \{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$,*
  - *and the cyclic group $\mathbb{Z}/4\mathbb{Z}$.*

1. *(15 pts)* Prove that $F(x)$ is separable and irreducible over $\mathbb{Q}$.
2. *(20 pts)* Prove that $F(x)$ factors mod 3 as a linear factor times an irreducible factor of degree 3.
3. *(25 pts)* Prove that the Galois group of $F(x)$ is A_4.
4. *(20 pts)* Prove that $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathbb{Q}(\alpha_1, \alpha_2)$.
5. *(20 pts)* Determine the degrees of the irreducible factors of $F(x)$ over $\mathbb{Q}(\alpha_1)$.

This was the only mandatory exercise, that you must submit before the deadline. The following exercise is not mandatory; it are not worth any points, and you do not have to submit it. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about it. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *More Galois groups over* $\mathbb{Q}$

Prove that the following polynomials have no repeated root in $\mathbb{C}$, and determine their Galois group over $\mathbb{Q}$. *Warning: Some polynomials may be reducible!*

1. $F_1(x) = x^3 - 4x + 6$,
2. $F_2(x) = x^3 - 7x + 6$,
3. $F_3(x) = x^3 - 21x - 28$,
4. $F_4(x) = x^3 - x^2 + x - 1$,
5. $F_5(x) = x^5 - 6x + 3$, *using without proof the fact that this polynomial has exactly* 3 *real roots.*