Galois theory — Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU34101/index.html

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Exercise 1 *The fifth cyclotomic field*

In this exercise, we consider the primitive 5th root $\zeta = e^{2\pi i/5}$, and we set $L = \mathbb{Q}(\zeta)$. We know that L is Galois over $\mathbb{Q}$, so we define $G = \operatorname{Gal}(L/\mathbb{Q})$. We also let

$$c = \frac{\zeta + \zeta^{-1}}{2} = \cos(2\pi/5) = 0.309\cdots,$$

$$C = \mathbb{Q}(c),$$

and finally

$$c' = \frac{\zeta^2 + \zeta^{-2}}{2} = \cos(4\pi/5) = -0.809\cdots.$$

1. Write down explicitly the minimal polynomial of ζ over $\mathbb{Q}$, and express its complex roots in terms of ζ.
2. Deduce that $\zeta + \zeta^2 + \zeta^3 + \zeta^4 = -1$.
3. Prove that G is a cyclic group. What is its order? Find an explicit generator of G.
4. Deduce that $c \notin \mathbb{Q}$.
5. Make the list of all subgroups of G.
6. Draw a diagram showing all the fields E such that $\mathbb{Q} \subset E \subset L$, ordered by inclusion.
7. What are the conjugates of c over $\mathbb{Q}$? Determine explicitly the minimal polynomial of c over $\mathbb{Q}$ (exact computations only, computations with the approximate value of c are forbidden).
8. Deduce that
$$c = \frac{-1+\sqrt{5}}{4}.$$
9. What are the conjugates of ζ over C (as opposed to over $\mathbb{Q}$)?
10. Deduce that
$$\zeta = \frac{-1+\sqrt{5}+i\sqrt{10+2\sqrt{5}}}{4}.$$