# Introduction to number theory
# Exercise sheet 5

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU22301/index.html

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Email your answers to makindeo@tcd.ie by Wednesday December 1st, 2PM.
The use of electronic calculators and computer algebra software is allowed.

**Exercise 1** *A Pell-Fermat equation (100 pts)*

In this Question, we consider the Diophantine equation

$$x^2 - 33y^2 = 1, \quad x, y \in \mathbb{Z} \quad (\star).$$

1. (20 pts) Compute the continued fraction of $x = \sqrt{33}$.

   *This means that you must somehow find a formula for* all *the coefficients $a_n$, $n \in \mathbb{Z}_{\geq 0}$.*

2. (20 pts) Compute the convergents of the continued fraction of $\sqrt{33}$ for $n \leqslant 4$.

3. (10 pts) Find a nontrivial $(y \neq 0)$ solution to the equation $(\star)$.

4. (20 pts) Find a solution to the equation $(\star)$ such that $x \geqslant 1000$.

5. (10 pts) What is the fundamental unit in $\mathbb{Z}[\sqrt{33}]$?

6. (20 pts) Does the Diophantine equation $x^2 - 33y^2 = -1$ have solutions $x, y \in \mathbb{Z}$?

   *Hint: Use the previous question.*

   **This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.**

**Exercise 2** *The battle of Hastings*

The battle of Hastings, which took place on October 14, 1066, was a major battle in History.

The following fictional historical text, taken from *Amusement in Mathematics* (H. E. Dundeney, 1917), refers to it:

*"The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof. (. . . ) When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle cries 'Ut!', 'Olicrosse!', 'Godemite!'."*

Use continued fractions to determine how many soldiers this fictional historical text suggests Harold II had at the battle of Hastings.

**Exercise 3** *Continued fraction vs. series*

Let $x \in (0, 1)$ be irrational, and let $[a_0, a_1, \cdots, a_n] = p_n/q_n$ $(n \in \mathbb{Z}_{\geq 0})$ be the convergents of the continued fraction expansion of $x$. Prove that

$$x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$

*Hint: Where could the $(-1)^n$ come from ?*