Fields, rings, and modules
Exercise sheet 5

`https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU22102/index.html`

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Email your answers to aylwarde@tcd.ie by Monday April 19, 4PM.

Exercise 1 *A non-free module over a non-commutative ring (50 pts)*

Let $M_2 = M_2(\mathbb{R})$ be the ring of 2×2 matrices with real entries, and let $V = \mathbb{R}^2$ be the space of column vectors of size 2 with real entries.

1. (20 pts) Prove that the natural multiplication $M_2 \times V \longrightarrow V$ gives V the structure of an M_2-module (so that the elements of V are the "vectors" and the elements of M_2 are the "scalars").

2. (15 pts) Find a generating set for the M_2-module V containing as few elements as possible.

3. (15 pts) Prove that V is **not** a free M_2-module.

   Hint: Consider the dimensions of the underlying $\mathbb{R}$*-vector spaces.*

Exercise 2 *Finitely generated Abelian groups (50 pts)*

1. (20 pts) Let G be the Abelian group with generators g, h and relations $8g + 12h = 6g + 8h = 0$. Perform an SNF computation to determine what G is isomorphic to.

2. (15 pts) Determine $\#G$. Is G cyclic?

3. (15 pts) Find all Abelian groups of order 2020, up to isomorphism.

These were the only mandatory exercises, that you must submit before the deadline. The following exercise is not mandatory; it is not worth any points, and you do not have to submit them. However, you can try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercises.

Exercise 3 *A very algebraic viewpoint on modules*

In this exercise, whenever G is an Abelian group, we write $\operatorname{End}(G)$ for the set of group morphisms from G to itself, and we equip this set with the addition defined by

$$\forall f, g \in \operatorname{End}(G),\ \forall x \in G,\ (f+g)(x) \overset{\text{def}}{=} f(x) + g(x)$$

and with the multiplication defined by

$$\forall f, g \in \operatorname{End}(G),\ \forall x \in G,\ (f \times g)(x) \overset{\text{def}}{=} f(g(x)).$$

In other words, addition in $\operatorname{End}(G)$ means pointwise addition, and multiplication in $\operatorname{End}(G)$ means composition.

1. Let G be an Abelian group. Prove that the addition and the multiplication in $\operatorname{End}(G)$ defined above define a (not necessarily commutative) ring structure on $\operatorname{End}(G)$. What are its 0, and its 1?

2. Let R be a ring, and let M be an R-module. Then M is in particular an Abelian group, so we can define $\operatorname{End}(M)$ and put a ring structure on it as in the previous question. In other words, $\operatorname{End}(M)$ is the ring of Abelian group morphisms (as opposed to module morphisms) form M to itself.

   (a) Fix $\lambda \in R$, and denote by μ_λ (standing for "multiplication by λ") the map

   $$\begin{array}{rccc} \mu_\lambda : & M & \longrightarrow & M \\ & m & \longmapsto & \lambda m. \end{array}$$

   Prove that $\mu_\lambda \in \operatorname{End}(M)$.

   (b) Prove that the map

   $$\begin{array}{rccc} \mu : & R & \longrightarrow & \operatorname{End}(M) \\ & \lambda & \longmapsto & \mu_\lambda \end{array}$$

   is a ring morphism.

3. Conversely, given an Abelian group G and a ring R, prove that assigning a ring morphism $\mu : R \longrightarrow \operatorname{End}(G)$ equips G with an R-module structure.