Rings, fields, and modules
Exercise sheet 3

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU22102/index.html

Version: March 3, 2021

Email your answers to aylwarde@tcd.ie by Monday March 22, 4PM.

Exercise 1 *Associates to irreducibles (20 pts)*

Let R be a commutative ring, which is not necessarily a domain.

1. (6 pts) Let $x, y \in R$. Prove that if $xy \in R^\times$, then $x \in R^\times$ and $y \in R^\times$.
2. (14 pts) Let $p \in R$ be irreducible, and let $x \in R$. Prove that if x is associate to p, then x is also irreducible.

Exercise 2 *Nilpotent elements (80 pts)*

Let R be a commutative ring. We say that an element $x \in R$ is *nilpotent* if there exists an integer $n \geqslant 1$ such that $x^n = 0$. We write $\operatorname{Nil}(R) \subset R$ for the subset of R formed of the nilpotent elements of R.
Example: if $R = \mathbb{Z}/8\mathbb{Z}$, then $x = \bar{2} \in \operatorname{Nil}(R)$, since $x^3 = \bar{8} = \bar{0}$ in R; in fact

$$\operatorname{Nil}(\mathbb{Z}/8\mathbb{Z}) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}.$$

Note that by definition, the 0 of R lies in $\operatorname{Nil}(R)$.

1. (5 pts) Prove that if R is a domain, then $\operatorname{Nil}(R)$ is reduced to $\{0\}$.
2. (5 pts) Prove that if $x \in \operatorname{Nil}(R)$, then $1 - x \in R^\times$.

   Hint: What is $(1-x)(1+x+x^2+\cdots+x^n)$*?*
3. (10 pts) Let $x, y \in R$. Prove that if x and y are both nilpotent, then so is $x + y$.

   Hint: Use the binomial formula to expand $(x+y)^n$ *for some large enough* $n \in \mathbb{N}$.
4. (5 pts) Prove that $\operatorname{Nil}(R)$ is an ideal of R.
5. (15 pts) Prove that the quotient ring $R/\operatorname{Nil}(R)$ has no nonzero nilpotents, i.e. that $\operatorname{Nil}(R/\operatorname{Nil}(R)) = \{\bar{0}\}$ where $\bar{0}$ denotes the 0 of the quotient $R/\operatorname{Nil}(R)$.

6. (15 pts) Prove that $\mathrm{Nil}(R)$ is contained in the intersection of all prime ideals of R.

   Hint: Let $P \subset R$ be a prime ideal. Which property does the quotient ring R/P have? How does this relate to this exercise?

   From now on, we admit without proof the fact that actually,

$$\mathrm{Nil}(R) = \bigcap_{P \text{ prime ideal } \triangleleft R} P$$

   agrees with the intersection of all prime ideals of R.

7. (25 pts) Let $F(x) \in R[x]$. Prove that if $F(x)$ is invertible in $R[x]$ iff. its constant coefficient is invertible in R and its other coefficients are all nilpotent.

   Hint: Use some of the previous questions. What would $R[x]^\times$ be if R were a domain?