Rings, fields, and modules
Exercise sheet 2

https://www.maths.tcd.ie/~mascotn/teaching/2021/MAU22102/index.html

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Email your answers to aylwarde@tcd.ie by Friday March 5, 4PM.

Exercise 1 *The characteristic of a ring (100 pts)*

Let R be a commutative ring. We note that that there exists one and only one ring morphism from $\mathbb{Z}$ to R: indeed, if f is such a morphism, then $f(0_\mathbb{Z}) = 0_R$, $f(1_\mathbb{Z}) = f(1_R)$, and so $f(2_\mathbb{Z}) = f(1_\mathbb{Z} + 1_\mathbb{Z}) = f(1_\mathbb{Z}) + f(1_\mathbb{Z}) = 1_R + 1_R$, and more generally $f(n) = \underbrace{1_R + \cdots + 1_R}_{n}$ for all $n \geq 0$; and finally $f(-n) = -f(n)$, so $f(n) = \underbrace{-1_R - \cdots - 1_R}_{|n|}$ for all $n < 0$, so f is completely determined; and we check easily that this f is indeed a ring morphism (you are *not* required to do this in this exercise). We denote this unique morphism from $\mathbb{Z}$ to R by f_R.

1. (10 pts) Prove that there exists a unique $n \in \mathbb{Z}$, $n \geq 0$, such that $\operatorname{Ker} f_R = n\mathbb{Z}$. You may use without proof the fact that the ring $\mathbb{Z}$ is principal.

This n is called the characteristic *of the ring R; we denote it by* $\operatorname{char} R$.

2. (15+5 pts) Let R be a ring of characteristic c. Prove that R contains a subring isomorphic to $\mathbb{Z}/c\mathbb{Z}$. What does this mean when $c = 0$?
3. Determine the characteristic of the following rings:
   (a) (5 pts) $\mathbb{C}$,
   (b) (5 pts) $\mathbb{Z}/n\mathbb{Z}$, in terms of $n \in \mathbb{Z}$,
   (c) (5 pts) $R[x]$, in terms of $\operatorname{char} R$,
   (d) (10 pts) $R \times S$, where S is another commutative ring, in terms of $\operatorname{char} R$ and $\operatorname{char} S$.
4. (10pts) Prove that if R is a domain, then $\operatorname{char} R$ is either 0 or a prime number.
5. (10pts) Let R and S be commutative rings. Prove that if there exists a morphism from R to S, then $\operatorname{char} R$ is a multiple of $\operatorname{char} S$.
6. (5 pts) Find all ring morphisms from $\mathbb{Z}/2021\mathbb{Z}$ to $\mathbb{C}$.
7. (15+5pts) Prove that there are no ring morphisms from $\mathbb{Q}$ to $\mathbb{Z}$; comment.

This was the only mandatory exercise, that you must submit before the deadline. The following exercise is not mandatory; it is not worth any points, and you do not have to submit it. However, I highly recommend that you try to solve it for practice, and you are welcome to email me if you have questions about it. The solution will be made available with the solution to the mandatory exercise.

Exercise 2 *Ideals in a quotient*

In this exercise, whenever $f : X \longrightarrow Y$ is a map between two sets, for each subset $S \subseteq X$ we define $f(S) = \{f(s), s \in S\} \subseteq Y$; and for each subset $T \subseteq Y$, we define $f^{-1}(T) = \{x \in X \mid f(x) \in T\}$.

1. Let R be a commutative nonzero ring. Prove that R is a field iff. the only ideals of R are $\{0\}$ and R.

2. Let $f : R \longrightarrow S$ be a morphism between commutative rings.

   (a) Prove that if f is surjective, then for every ideal I of R, $f(I)$ is an ideal of S.

   (b) Give a counter-example showing that this statement is no longer true if f is not surjective.

3. Let again $f : R \longrightarrow S$ be a morphism between commutative rings.

   (a) Prove that if J is an ideal of S, then $f^{-1}(J)$ is an ideal of R.

   (b) Which statement proved in class do we recover by taking $J = \{0\}$?

4. Let now R be a commutative ring, I an ideal of R, and $f : R \longrightarrow R/I$ the canonical projection.

   (a) Prove that the maps

$$\begin{array}{rccc} \Phi : & \{\text{Ideals of } R \text{ containing } I\} & \longrightarrow & \{\text{Ideals of } R/I\} \\ & J & \longmapsto & f(J) \end{array}$$

   and

$$\begin{array}{rccc} \Psi : & \{\text{Ideals of } R/I\} & \longrightarrow & \{\text{Ideals of } R \text{ containing } I\} \\ & J & \longmapsto & f^{-1}(J) \end{array}$$

   are well-defined, i.e. that their images land where their definitions claim they land.

   (b) Prove that Φ and Ψ are inclusion-preserving bijections which are inverses of each other.

5. Let R be a commutative ring, and let I be an ideal of R. We say that I is *maximal* if $I \neq R$ and if there are no ideals J of R such that $I \subsetneq J \subsetneq R$ (This definition occurs in the lecture on prime and maximal ideals, but you need not study this lecture to solve this question). By using the previous question(s), prove that R/I is a field iff. I is maximal.

   NB This statement also occurs in the lecture on prime and maximal ideals, but the proof that we give in this exercise is a different one.