Introduction to number theory
Exercise sheet 6

https://www.maths.tcd.ie/~mascotn/teaching/2020/MAU22301/index.html

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Answers are due for Monday December 14th, 2PM.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Pell-Fermat (100pts)*

1. (25pts) Determine the coefficients a_n of the continued fraction expansion of $\sqrt{14}$.
2. (25pts) Use the previous question to find a fundamental unit in $\mathbb{Z}[\sqrt{14}]$. What is its norm?
3. (20pts) Find integers $x, y \in \mathbb{N}$ such that $x^2 - 14y^2 = 1$ and $y > 100$. How many such pairs (x, y) are there?
4. (30pts) Find integers $x, y \in \mathbb{N}$ (in particular, $y \neq 0$) such that $x^2 - 41y^2 = 1$.

Solution 1

1. Since $\sqrt{14}$ is a quadratic irrational, we know that the a_n are ultimately periodic. We compute

n	x_n	a_n
0	$\sqrt{14}$	3
1	$\frac{1}{\sqrt{14}-3} = \frac{3+\sqrt{14}}{(\sqrt{14}-3)(\sqrt{14}+3)} = \frac{3+\sqrt{14}}{5}$	1
2	$\frac{1}{\frac{3+\sqrt{14}}{5}-1} = \frac{5}{3+\sqrt{14}-5} = \frac{5(2+\sqrt{14})}{(-2+\sqrt{14})(2+\sqrt{14})} = \frac{5(2+\sqrt{14})}{10} = \frac{2+\sqrt{14}}{2}$	2
3	$\frac{1}{\frac{2+\sqrt{14}}{2}-2} = \frac{2}{\sqrt{14}-2} = \frac{2(2+\sqrt{14})}{(\sqrt{14}-2)(\sqrt{14}+2)} = \frac{2(2+\sqrt{14})}{10} = \frac{2+\sqrt{14}}{5}$	1
4	$\frac{1}{\frac{2+\sqrt{14}}{5}-1} = \frac{5}{\sqrt{14}-3} = \frac{5(3+\sqrt{14})}{(\sqrt{14}-3)(\sqrt{14}+3)} = \frac{5(3+\sqrt{14})}{5} = 3+\sqrt{14}$	6
5	$\frac{1}{3+\sqrt{14}-6} = \frac{1}{\sqrt{14}-3} = x_1$	$\cdots$

   As $x_5 = x_1$, we enter a cycle of length 4; whence $\sqrt{14} = [3, \overline{1, 2, 1, 6}]$.

2. From the a_n found in the previous question, we compute p_n and q_n, until

$p_n^2 - 14q_n^2 = \pm 1$. We find

n	a_n	p_n	q_n	$p_n^2 - 14q_n^2$
-2		0	1	
-1		1	0	
0	3	3	1	-5
1	1	4	1	$+2$
2	2	11	3	-5
3	1	15	4	$+1$

so we stop here: the fundamental unit is $\varepsilon = 15 + 4\sqrt{14}$, and its norm is $15^2 - 14 \cdot 4^2 = +1$.

3. Since $N(\varepsilon) = +1$, the solutions to $x^2 - 14y^2 = +1$ correspond (up to signs) to all powers ε^n of ε, so getting $y > 100$ is simply a matter of taking $n \in \mathbb{N}$ large enough.

   Already for $n = 2$, we find that

   $$\varepsilon^2 = (15 + 4\sqrt{14})^2 = 15^2 + 4^2 \cdot 14 + 2 \cdot 15 \cdot 4\sqrt{14} = 449 + 120\sqrt{14}$$

   whence the solution $x = 449$, $y = 120 > 100$.

   We could take higher powers, which will give even larger values of x and y, so there are infinitely many choices.

4. We re-do the same computations, with $\sqrt{41}$ instead of $\sqrt{14}$:

n	x_n	a_n	p_n	q_n	$p_n^2 - 41q_n^2$
-2			0	1	
-1			1	0	
0	$\sqrt{41}$	6	6	1	-5
1	$\frac{1}{\sqrt{41}-6} = \frac{6+\sqrt{41}}{(\sqrt{41}-6)(\sqrt{41}+6)} = \frac{6+\sqrt{41}}{5}$	2	13	2	$+5$
2	$\frac{1}{\frac{6+\sqrt{41}}{5}-2} = \frac{5(4+\sqrt{41})}{(\sqrt{41}-4)(\sqrt{41}+4)} = \frac{4+\sqrt{41}}{5}$	2	32	5	-1

   We stop here since $p_2^2 - 41q_2^2 = -1$, whence the fundamental unit $\varepsilon = 32 + 5\sqrt{41}$. Since $N(\varepsilon) = -1$, solutions to $x^2 - 41y^2 = +1$ correspond to powers of $\varepsilon^2 = 2049 + 320\sqrt{41}$; in particular, the smallest non-trivial solution is $x = 2049$, $y = 320$.

This was the only mandatory exercise, that you must submit before the deadline. The following exercises are not mandatory; they are not worth any points, and you do not have to submit them. However, I highly recommend that you try to solve them for practice, and you are welcome to email me if you have questions about them. The solutions will be made available with the solution to the mandatory exercise.

Exercise 2 *The battle of Hastings*

The battle of Hastings, which took place on October 14, 1066, was a major battle in History.

The following fictional historical text, taken from *Amusement in Mathematics* (H. E. Dundeney, 1917), refers to it:

"The men of Harold stood well together, as their wont was, and formed thirteen squares, with a like number of men in every square thereof. (...) When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle cries 'Ut!', 'Olicrosse!', 'Godemite!'."

Use continued fractions to determine how many soldiers this fictional historical text suggests Harold II had at the battle of Hastings.

Solution 2

We are looking for solutions to $13y^2 + 1 = x^2$ with $x, y \in \mathbb{Z}_{\geqslant 0}$. This translates into the Pell-Fermat equation $x^2 - 13y^2 = 1$.

Clearly, the trivial solution $x = 1$, $y = 0$ does not reflect the situation (I doubt Harold II would have gone to battle alone !), so let us compute the continued fraction expansion of $x = \sqrt{13}$ until we find a non-trivial solution.

$$
\begin{aligned}
x_0 &= \sqrt{13}, & a_0 &= \lfloor x_0 \rfloor = 3, & p_0 &= 3, q_0 = 1, & p_0^2 - 13q_0^2 &= -4 \neq \pm 1. \\
x_1 = \frac{1}{\sqrt{13}-3} &= \frac{3+\sqrt{13}}{4}, & a_1 &= \lfloor x_1 \rfloor = 1, & p_1 &= 4, q_1 = 1, & p_1^2 - 13q_1^2 &= 3 \neq \pm 1. \\
x_2 = \frac{1}{\frac{3+\sqrt{13}}{4}-1} &= \frac{1+\sqrt{13}}{3}, & a_2 &= \lfloor x_2 \rfloor = 1, & p_2 &= 7, q_2 = 2, & p_2^2 - 13q_2^2 &= -3 \neq \pm 1. \\
x_3 = \frac{1}{\frac{1+\sqrt{13}}{3}-1} &= \frac{2+\sqrt{13}}{3}, & a_3 &= \lfloor x_3 \rfloor = 1, & p_3 &= 11, q_3 = 3, & p_3^2 - 13q_3^2 &= 4 \neq \pm 1. \\
x_4 = \frac{1}{\frac{2+\sqrt{13}}{3}-1} &= \frac{1+\sqrt{13}}{4}, & a_4 &= \lfloor x_4 \rfloor = 1, & p_4 &= 18, q_4 = 5, & p_4^2 - 13q_4^2 &= -1.
\end{aligned}
$$

We have found the fundamental unit $\varepsilon = 18 + 5\sqrt{13}$ of norm $N(\varepsilon) = -1$. We deduce that the fundamental solution to our equation corresponds to

$$\varepsilon^2 = 649 + 180\sqrt{13},$$

that is to say $x = 649$, $y = 180$.

Since the other solutions, which correspond to powers of ε^2, are even larger, this suggests a number of soldiers on this side of the battle (including Harold II) was at least $649^2 = 421201$. That's really a lot, which confirms that this text is certainly fictional!

Exercise 3 *Continued fraction vs. series*

Let $x \in (0,1)$ be irrational, and let $[a_0, a_1, \cdots, a_n] = p_n/q_n$ ($n \in \mathbb{Z}_{\geq 0}$) be the convergents of the continued fraction expansion of x. Prove that

$$x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$

Hint: Where could the $(-1)^n$ *come from ?*

Solution 3

We know that $q_n p_{n-1} - p_n q_{n-1} = (-1)^n$ for all n. Therefore, we have

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}$$

for all n. Now, obviously

$$\frac{p_m}{q_m} = \left(\frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}}\right) + \left(\frac{p_{m-1}}{q_{m-1}} - \frac{p_{m-2}}{q_{m-2}}\right) + \cdots + \left(\frac{p_1}{q_1} - \frac{p_0}{q_0}\right)\frac{p_0}{q_0} = \frac{p_0}{q_0} + \sum_{n=1}^{m}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right).$$

On the one hand, $p_0 = a_0 = \lfloor x \rfloor = 0$ since $x \in (0,1)$, so $\frac{p_0}{q_0} = 0$. On the other hand, we know that the sequence $\frac{p_n}{q_n}$ converges to x, so we get

$$x = \lim_{m\to\infty} \frac{p_m}{q_m} = \lim_{m\to\infty} \sum_{n=1}^{m}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) = \sum_{n=1}^{+\infty}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{q_n q_{n-1}} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{q_n q_{n+1}}.$$