Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JS/SS Maths/TP/TJH **Semester 1, 2019**

MAU34101 Galois theory — Mock exam

Some date **Some location** **Some time**

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Instructions to Candidates:

This is a mock exam, and is intended for revision purposes only.
This paper contains **five** questions. You **must** attempt **four** of them: question 1, and **exactly three** of questions 2, 3, 4, and 5.
Should you attempt all questions (not recommended), you will **only** get the marks for question 1 and the best three others.
Non-programmable calculators are permitted for this examination.

You may not start this examination until you are instructed to do so by the Invigilator.

Question 1 *Bookwork*

Let $K \subset L$ be a finite extension, and let $\Omega \supset K$ be algebraically closed. Which inequalities do we always have between $[L : K]$, $\# \operatorname{Aut}_K(L)$, $\# \operatorname{Hom}_K(L, \Omega)$? When are they equalities? State equivalent conditions.

Question 2 *Correspondence in degree 3*

Let K be a field, and $F(x) \in K[x]$ be separable and of degree 3. Denote its 3 roots in its splitting field L by $\alpha,\ \beta,\ \gamma$.

1. What are the possibilities for $\operatorname{Gal}_K(F)$? How can you tell them apart?
2. For each of the cases found in the previous question, sketch the diagram showing all the fields $K \subset E \subset L$ and identifying these fields. In particular, locate $K(\alpha)$, $K(\beta)$, $K(\gamma)$, $K(\alpha, \beta)$, etc.
3. In which of the cases above is the stem field of F isomorphic to its splitting field? (*Warning: there is a catch in this question.*)

Question 3 *Galois group computations*

Determine the Galois group over $\mathbb{Q}$ of the polynomials below, and say if they are solvable by radicals over $\mathbb{Q}$.

1. $x^3 - x^2 - x - 2$,
2. $x^3 - 3x - 1$,
3. $x^3 - 7$,
4. $x^5 + 21x^2 + 35x + 420$,
5. $x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Question 4 *A cosine formula (35 pts)*

Let $c = \cos(2\pi/17)$.

1. Prove that c is algebraic over $\mathbb{Q}$.
2. Determine the conjugates of c over $\mathbb{Q}$, and its degree as an algebraic number over $\mathbb{Q}$.
3. Explain how one could in principle use Galois theory (and a calculator / computer) to find an explicit formula for c.

Question 5 *Extensions of finite field are Galois (35 pts)*

Let $p \in \mathbb{N}$ be prime, $n \in \mathbb{N}$, and $q = p^n$.

1. Give two proofs of the fact that the extension $\mathbb{F}_p \subset \mathbb{F}_q$ is Galois: one by viewing $\mathbb{F}_q$ as a splitting field, and the other by considering the order of $\mathrm{Frob} \in \mathrm{Aut}(\mathbb{F}_q)$.
2. What does the Galois correspondence tell us for $\mathbb{F}_p \subset \mathbb{F}_q$?
3. Generalise to an arbitrary extension of finite fields $\mathbb{F}_q \subset \mathbb{F}_{q'}$.