Math 345 — Algebraic number theory
Review sheet 2

https://staff.aub.edu.lb/~nm116/teaching/2018/math345/index.html
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Exercise 1: Your turn to grade

Find all that is wrong in the following paragraph.

Let K be an imaginary quadratic field. By Dirichlet's theorem, the rank of $\mathbb{Z}_K^\times$ is zero, so the only units in K are ± 1. But let us consider a prime $p \in \mathbb{N}$ which ramifies in K, say $p\mathbb{Z}_K = \mathfrak{p}^2$. Then if $\mathfrak{p} = (\gamma)$, we have $(p) = \mathfrak{p}^2 = (\gamma)^2 = (\gamma^2)$, so we get that $u = \gamma^2/p$ is a unit in K, which contradicts Dirichlet's theorem.

Exercise 2: A real quadratic field

Determine $\mathrm{Cl}(K)$ and $\mathbb{Z}_K^\times$, where $K = \mathbb{Q}(\sqrt{79})$. Does the equation $x^2 - 79y^2 = 2$ have a solution? If yes, how many? What about $x^2 - 79y^2 = -2$?

Exercise 3: A cubic field

Let $P(x) = x^3 + 6x + 6 \in \mathbb{Z}[x]$, and let $K = \mathbb{Q}(\alpha)$, where α is a root of $P(x)$.
You may find the following table useful:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$P(n)$	-149	-82	-39	-14	-1	6	13	26	51	94	161

1. Compute $[K : \mathbb{Q}]$, $\mathbb{Z}_K$, and $\operatorname{disc} K$.
2. Prove that $\mathrm{Cl}(K)$ is generated by $[\mathfrak{p}_2]$, where $\mathfrak{p}_2$ is the prime of K above 2.
3. Find a non-trivial unit u in K, and prove that it generates the group $\mathbb{Z}_K^\times/\mathbb{Z}_K^{\times 3}$ of units modulo cubes of units.

   Hint: Reduce u modulo a prime of K to prove that it is not a cube.
4. Prove that if $\mathfrak{p}_2$ were principal, there would exist a unit v such that $2v$ is a cube in K.
5. Determine $\mathrm{Cl}(K)$.

Exercise 4: The quartic field returns

Let $f(x) = x^4 + 3x^3 - 18x^2 - 24x + 129$, which is an irreducible polynomial over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha)$, where α is a root of $f(x)$. Last time, we saw that $\operatorname{sign} K = (0, 2)$ and that $\operatorname{disc} K = 2^4 \cdot 3^3 \cdot 17^4 \cdot 19 \cdot 37$, and we defined $\beta = \frac{\alpha^3 - 2\alpha^2 - \alpha + 2}{7}$, $\gamma = \frac{\beta^2 - 3\beta - 3}{34}$, and let $\delta = \frac{\beta^3 - 12\beta - 9}{34}$, whose respective characteristic polynomials turned to be $\chi(\beta) = x^4 + 28x^3 + 207x^2 + 154x + 247$, $\chi(\gamma) = x^4 - 13x^3 + 42x^2 + 8x + 1$, and $\chi(\delta) = x^4 + 139x^3 + 5163x^2 + 973$.

1. Determine the rank of $\mathbb{Z}_K^\times$, and bound the size of W_K.
2. Find a non-trivial (that is to say $u \neq \pm 1$) unit u in K.
3. How can you test whether $u \in W_K$?

Exercise 5

Let $f(x) = x^3 - 4x^2 + 2x - 2$, which is an irreducible polynomial over $\mathbb{Q}$ (why ?), and let $K = \mathbb{Q}(\alpha)$, where α is a root of f.

1. Given that $\operatorname{disc} f = -300$, what can you say about the ring of integers of K and the primes that ramify in K? What if, on the top of that, you notice that $f(x + 3) = x^3 + 5x^2 + 5x - 5$?
2. Prove that $\mathbb{Z}_K$ is a PID.
3. Find a generator for each of the primes above 2, 3 and 5.
4. Use the results of the previous question to discover that $u = 2\alpha^2 - \alpha + 1$ is a unit.
5. We use the unique embedding of K into $\mathbb{R}$ to view K as a subfield of $\mathbb{R}$ from now on. Prove that there exists a unit $\varepsilon \in \mathbb{Z}_K^\times$ such that $\mathbb{Z}_K^\times = \{\pm\varepsilon^n, n \in \mathbb{Z}\}$ and $\varepsilon > 1$.
6. By the technique of exercise 3 from exercise sheet number 5, it can be proved that $\varepsilon \geqslant 4.1$. Given that $u \approx 23.3$, prove that u is a fundamental unit. What is the regulator of K?

   Hint : Reduce u modulo the primes above 3 to prove that u is not a square in $\mathbb{Z}_K$.