Math 345 — Algebraic number theory
Review sheet 1

https://staff.aub.edu.lb/~nm116/teaching/2018/math345/index.html
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Exercise 1: Quadratic fields with little ramification

1. Let $p \in \mathbb{N}$ be an odd prime. Find all quadratic number fields (real and imaginary) that are ramified only at p.
2. Same question for $p = 2$.

Exercise 2: A quartic field and some big numbers

Let $f(x) = x^4 + 3x^3 - 18x^2 - 24x + 129$, which is an irreducible polynomial over $\mathbb{Q}$ (why ?), and let $K = \mathbb{Q}(\alpha)$, where α is a root of $f(x)$.

1. If I told you that $\operatorname{disc} f = 930\,069$, why would not that be very useful to you ? Which information can you get from that nonetheless ?
2. I now tell you that the roots of f are approximately $-4.1 \pm 0.1i$ and $2.6 \pm 1.0i$. What is the signature of K ? Can you compute the trace of α from these approximate values ? Why is the result obvious ?
3. If I now tell you that $\operatorname{disc} f$ factors as $3^3 \cdot 7^2 \cdot 19 \cdot 37$, what can you say about the ring of integers of K and the primes that ramify in K ?
4. In principle (don't actually do it), how could you test whether $\beta = \frac{\alpha^3 - 2\alpha^2 - \alpha + 2}{7}$ is an algebraic integer ?
5. If I now tell you that the characteristic polynomial of β is $\chi(\beta) = x^4 + 28x^3 + 207x^2 + 154x + 247$, whose discriminant is $\operatorname{disc} \chi(\beta) = 25\,364\,993\,616$, which conclusions can you draw from that ?
6. Given that $\operatorname{disc} \chi(\beta)$ factors as $2^4 \cdot 3^3 \cdot 17^4 \cdot 19 \cdot 37$, what is the index of the order $\mathbb{Z}[\beta]$? What consequence does this have on the expression of a $\mathbb{Z}$-basis of $\mathbb{Z}_K$ in terms of β ?
7. Let $\gamma = \frac{\beta^2 - 3\beta - 3}{34}$, and let $\delta = \frac{\beta^3 - 12\beta - 9}{34}$, whose respective characteristic polynomials are $\chi(\gamma) = x^4 - 13x^3 + 42x^2 + 8x + 1$ and $\chi(\delta) = x^4 + 139x^3 + 5163x^2 + 973$. Prove that $\{1, \beta, \gamma, \delta\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_K$.
8. Compute explicitly the decomposition of 2, 3, and 7 in K.