Math 345 — Algebraic number theory
Exercise sheet 4: The class group

https://staff.aub.edu.lb/~nm116/teaching/2018/math345/index.html
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Answers must be submitted by Friday April 27, 12:00.

Exercise 4.1: Class group computations (40 pts)

Determine the class group of the following number fields:

1. (20 points) $\mathbb{Q}(\sqrt{-29})$,
2. (20 points) $\mathbb{Q}(\sqrt{-33})$.

Exercise 4.2: Arbitrarily large class numbers (10 points)

Let $d > 0$ be a squarefree integer, and let $K = \mathbb{Q}(\sqrt{-d})$. Suppose that $p \in \mathbb{N}$ is a prime which splits in K, and let $\mathfrak{p}$ be a prime ideal above p.

1. (5 points) Prove that for all integers $i \geq 1$ such that $p^i < |\operatorname{disc} K|/4$, the ideal $\mathfrak{p}^i$ is not principal.

   Hint: consider the cases $d \not\equiv 1 \pmod 4$ *and* $d \equiv 1 \pmod 4$ *separately.*

2. (2 points) What does this tell you about the class number of K?
3. (3 points) Using without proof the fact that there exists infinitely many squarefree positive numbers of the form $8k + 7$ for $k \in \mathbb{N}$, prove that for every $X > 0$ there exists a number field K such that $h_K > X$.

Exercise 4.3: A norm equation (20 points)

Let $n \geqslant 0$ be an integer. The goal of this exercise is to determine the number of solutions to the Diophantine equation

$$x^2 + 10y^2 = 7^n \qquad (x, y \in \mathbb{Z}) \tag{1}$$

in terms of n.

We let $K = \mathbb{Q}(\alpha)$ where $\alpha = \sqrt{-10}$, and note that (1) can be rewritten as

$$N^K_{\mathbb{Q}}(x + \alpha y) = 7^n.$$

You may freely use the fact that $\mathbb{Z}_K^\times = \{\pm 1\}$.

1. (5 points) Determine $\mathbb{Z}_K$ and $\mathrm{Cl}(K)$.
2. (4 points) Determine the decomposition of 7 in $\mathbb{Z}_K$, and the image in $\mathrm{Cl}(K)$ of the primes appearing in this factorisation.
3. (5 points) Let $\mathfrak{a}$ be an of $\mathbb{Z}_K$ of norm 7^n. What does the factorisation into primes of $\mathfrak{a}$ look like? What does this tell you about the image of $\mathfrak{a}$ in $\mathrm{Cl}(K)$?
4. (6 points) Express the number of solutions to (1) in terms of n.

Exercise 4.4: A Mordell-Weil equation (30 points)

The goal of this exercise is to solve the Diophantine equation

$$y^2 = x^3 - 148 \qquad (x, y \in \mathbb{Z}) \tag{2}$$

We let $K = \mathbb{Q}(\alpha)$ where $\alpha = \sqrt{-37}$, and note that (2) can be rewritten as

$$(y + 2\alpha)(y - 2\alpha) = x^3.$$

You may freely use the following facts:

- 37 is prime,
- $\mathbb{Z}_K^\times = \{\pm 1\}$.

1. (1 point) Prove that the equation has no solution such that $37 \mid y$.
2. (8 points) Determine $\mathbb{Z}_K$ and $\mathrm{Cl}(K)$, as well as the decomposition of 37 in K.
3. (4 points) Let (x, y) be a hypothetical solution of (2). Prove that there is at most one prime $\mathfrak{p}$ of K that divides both $(y + 2\alpha)$ and $(y - 2\alpha)$.
4. (12 points) Deduce that at least one of $y + 2\alpha$ or $y - 2\alpha$ is a cube or twice a cube in $\mathbb{Z}_K$.

   Hint: Prove that $(y+2\alpha) = \mathfrak{b}^3\mathfrak{p}_2^r$ *and* $(y-2\alpha) = \mathfrak{b}'^3\mathfrak{p}_2^{r'}$ *for some ideals* $\mathfrak{b}, \mathfrak{b}' \subset \mathbb{Z}_K$ *and integers* $r, r' \geqslant 0$. *How small can you make* r *and* r'*?*
5. (4 points) Find all the solutions of (2).