Math 345 — Algebraic number theory
Exercise sheet 3: Ideals and factorisation

https://staff.aub.edu.lb/~nm116/teaching/2018/math345/index.html
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Answers must be submitted by Wednesday April 4 (provisional), 12:00.

Exercise 3.1: Ideals of fixed norm (40 pts)

1. (20 points) How many ideals of norm 900 are there in the ring of integers of $\mathbb{Q}(\sqrt{7})$?

   Hint: Compute the decomposition in $\mathbb{Q}(\sqrt{7})$ *of the primes* $p \in \mathbb{N}$ *that divide 900.*

2. (20 points) How many ideals of norm 80 are there in the ring of integers of $\mathbb{Q}(\zeta)$, where ζ is a primitive 60th root of unity?

Exercise 3.2: Non-isomorphic fields that look alike (60 pts)

The goal of this exercise is to prove that the number fields $\mathbb{Q}(\sqrt[3]{6})$ and $\mathbb{Q}(\sqrt[3]{12})$ have the same degree and discriminant, but are not isomorphic.

To ease notation, we let $\alpha = \sqrt[3]{6}$, $\beta = \sqrt[3]{12}$, $K = \mathbb{Q}(\alpha)$ and $L = \mathbb{Q}(\beta)$.

1. (3 points) Prove that $[K : \mathbb{Q}] = 3$.

2. (8 points) Prove that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$ and compute $\operatorname{disc} K$.

3. (10 points) Prove that $[L : \mathbb{Q}] = 3$ and that $\operatorname{disc} L$ is of the form $-2^a 3^5$ for some integer $a \geqslant 0$. What are the possible values of a ?

4. (4 points) Prove that $L \simeq \mathbb{Q}(\sqrt[3]{18})$.

   Hint: Take a look at $\gamma = \beta^2/2$.

5. (7 points) Deduce that $\operatorname{disc} L = \operatorname{disc} K$.

6. (2 points) Which primes $p \in \mathbb{N}$ ramify in K? What about L?

7. (8 points) Compute explicitly the decomposition of 7 in K and in L.

8. (3 points) Deduce that K and L are not isomorphic.

9. (6 points) Compute explicitly the decomposition of 2 and 3 in K and in L.

10. (9 points) Deduce the factorisation of the ideals $\alpha\mathbb{Z}_K$, $\beta\mathbb{Z}_L$ and $\gamma\mathbb{Z}_L$ into primes.

UNASSESSED QUESTIONS

The next questions are not worth any points. I still recommend you to try to solve them, for practice. Correction will be available online, just as for the marked questions.

Exercise 3.3: A cubic field (0 pts)

Let $f(x) = x^3 - 5x + 5$, and let $K = \mathbb{Q}(\alpha)$, where α is a root of $f(x)$.

1. Compute the degree $[K : \mathbb{Q}]$ and the ring of integers $\mathbb{Z}_K$ of K.
2. Which primes $p \in \mathbb{N}$ ramify in K ?
3. For $n \in \mathbb{N}$, $n \leqslant 7$, compute explicitly the decomposition of $n\mathbb{Z}_K$ as a product of prime ideals.
4. Prove that the prime(s) above 5 are principal, and find explicitly a generator for them.
5. List all the ideals $\mathfrak{a}$ of $\mathbb{Z}_K$ such that $N(\mathfrak{a}) \leqslant 7$.
6. Factor the ideals $(\alpha - 2)$ and $(\alpha + 1)$ into primes.
   Hint : $N_{\mathbb{Q}}^{K}(\alpha + n) = -f(-n)$ *for all* $n \in \mathbb{Q}$.
7. Use the previous question to find a non-trivial unit in $\mathbb{Z}_K$.