Math 345 — Algebraic number theory
Exercise sheet 2: Algebraic integers

https://staff.aub.edu.lb/~nm116/teaching/2018/math345/index.html
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Answers must be submitted by Monday March 12 (provisional), 12:00.

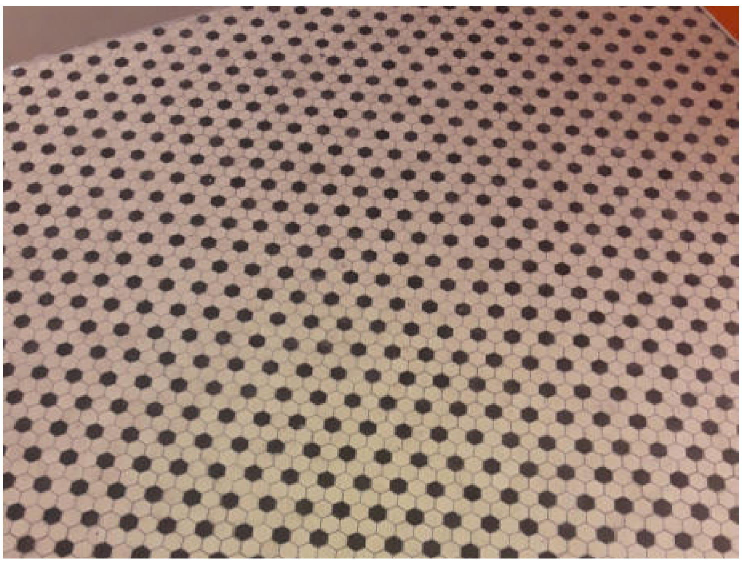
Reminder: $\operatorname{disc}(x^n + bx + c) = (-1)^{n(n-1)/2}\big((1-n)^{n-1}b^n + n^n c^{n-1}\big)$.

Exercise 2.1: To be or not to be integral (10 pts)

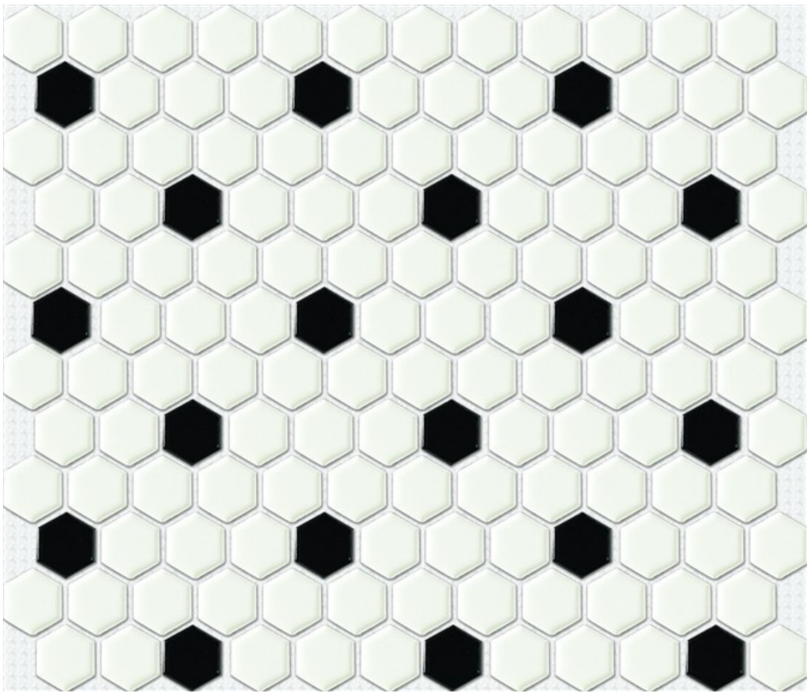
Is $\dfrac{3+2\sqrt{6}}{\sqrt{6}-2}$ an algebraic integer ?

Exercise 2.2: Floor tilings (10 pts)

1. (5 pts) In the picture below, the centre of the hexagonal floor tiles (both black and white ones) form a lattice, and the centre of the black tiles form a sublattice. Compute the index of this sublattice by writing down a change-of-basis (= transition) matrix. What is the proportion of black tiles?



2. (5 pts) Same questions for this other tiling pattern.



Exercise 2.3: A cubic field (30 pts)

Let $f(x) = x^3 + x - 1$.

1. The aim of this question is to prove that $f(x)$ is irreducible over $\mathbb{Q}$.
   (a) (2 points) Prove that if $f(x)$ were reducible, then it would have a rational root.
   (b) (5 points) Prove that this root would in fact be an integer by using the notion of algebraic integer.
   (c) (5 points) Prove that this root could only be ± 1, and conclude that $f(x)$ is irreducible over $\mathbb{Q}$.
2. (3 points) Compute the discriminant of $f(x)$.
3. (10 points) Let $K = \mathbb{Q}(\alpha)$, where α is a root of $f(x)$. Compute the ring of integers of K.
4. (5 points) Determine the signature of K by using discriminants.

Exercise 2.4: A quartic field (50 pts)

Let $f(x) = x^4 - 2x + 4$, which you may assume without proof is irreducible over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha)$, where α satisfies $f(\alpha) = 0$.

1. (5 points) Compute and factor the discriminant of $\mathbb{Z}[\alpha]$.

   Hint: $2^{10} - 3^3 = 997$ *is prime.*

2. (12 points) At this point, what are the possibilities for $\operatorname{disc} K$, and the corresponding values of the index of $\mathbb{Z}[\alpha]$?

3. (5 points) Let $\beta = \frac{\alpha^3}{2} \in K$, and consider the lattice $\mathcal{O} \subset K$ with $\mathbb{Z}$-basis

   $$1, \alpha, \alpha^2, \beta.$$

   Prove that $\mathcal{O}$ is stable under multiplication by β.

   Hint: what is $\beta \cdot \alpha$ *?.*

4. (7 points) Deduce that β is an algebraic integer.

5. (2 points) Which of the possibilities listed in question 2. remain?

6. (7 points) Prove that $\mathcal{O}$ is an order in K.

   Hint: Prove that $\mathcal{O}$ *is also stable under multiplication by* α*.*

7. (12 points) It turns out that $\mathbb{Z}_K = \mathcal{O}$. Give the discriminant of K in factored form.