Math 345 — Algebraic number theory
Exercise sheet 1: Number fields

https://staff.aub.edu.lb/~nm116/teaching/2018/math345/index.html
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Answers must be submitted by Wednesday February 28, 12:00.

Exercise 1.1: Review of methods (10 pts)

Let $K = \mathbb{Q}(\sqrt{3})$, and let $\alpha = a + b\sqrt{3}$ ($a, b \in \mathbb{Q}$) be an element of K. Compute the trace, norm, and characteristic polynomial of α in terms of a and b

1. (5 pts) by writing down the matrix of the multiplication-by-α map with respect to the $\mathbb{Q}$-basis of K of your choice,
2. (5 pts) by considering complex embeddings.

Remark: This exercise is meant as a warm-up for the next exercises, so as to remind you that some computations can be done in different ways. In the next exercises, remember that depending on the situation, some methods require less efforts than others!

Exercise 1.2: A biquadratic extension (30 pts)

1. (3 pts) Let $K = \mathbb{Q}(\sqrt{2})$. Prove that $i \notin K$.
2. (5 pts) Let $L = \mathbb{Q}(\sqrt{2}, i)$. Compute $[L : \mathbb{Q}]$.
3. (5 pts) What is the signature of L ?
4. (6 pts) Let $\alpha = \sqrt{2} + i \in L$. Compute the characteristic polynomial $\chi_{\mathbb{Q}}^{L}(\alpha)$ of α with respect to the extension $L/\mathbb{Q}$.
5. (5 pts) Is the polynomial $\chi_{\mathbb{Q}}^{L}(\alpha)$ squarefree? What does this tell us about α?
6. (6 pts) Compute the characteristic polynomial $\chi_{K}^{L}(\alpha)$ of α with respect to the extension L/K.

Exercise 1.3: Computations in $\mathbb{Q}(\sqrt[3]{2})$ (40 pts)

In this exercise, you may assume[1] *that the polynomial* $x^3 - 2$ *is irreducible over* $\mathbb{Q}$.

1. (8 pts) Let $K = \mathbb{Q}(\sqrt[3]{2})$, and let $\alpha = \frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1} \in K$. Find $a, b, c \in \mathbb{Q}$ such that $\alpha = a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$.
2. (4 pts) Are these rational numbers a, b, c unique ?
3. (2 pts) What is the degree of K over $\mathbb{Q}$?
4. (7 pts) Prove that $\sqrt{2} \notin K$.

   Hint: Think in terms of degrees.
5. (7 pts) Prove that $K = \mathbb{Q}(\alpha)$.
6. (9 pts) Compute the trace, norm, and characteristic polynomial of α.
7. (3pts) What is the minimal polynomial of α over $\mathbb{Q}$?

Exercise 1.4: Resultant practice (20 pts)

1. (10 pts) Let $\gamma = \sqrt{3} - \sqrt[3]{2}$. Express a non-zero polynomial in $\mathbb{Q}[x]$ vanishing at γ as a resultant, and then express this resultant as a determinant.

   You are not required to compute this determinant explicitly. All this questions asks is to express this polynomial as a resultant, and then as a determinant, but you are not required to compute this polynomial explicitly.
2. (10 pts) Let $K = \mathbb{Q}(\alpha)$ be a number field, let $A(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α, and let $\beta = B(\alpha) \in K$, where $B(x) \in \mathbb{Q}[x]$ is some polynomial. Express the characteristic polynomial $\chi_{\mathbb{Q}}^{K}(\beta)$ of β in terms of a resultant involving A and B.

   Hint: Think in terms of complex embeddings.

[1] We will see an efficient way (Eisenstein's criterion) to prove this in chapter 3.