Math 345 — Algebraic number theory
Exercise sheet 0: Reminders

https://staff.aub.edu.lb/~nm116/teaching/2018/math345/index.html
Nicolas Mascot (nm116@aub.edu.lb)

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For practise only; does not count for the final grade.

Exercise 0.1

Give an example

1. Of a ring which is not commutative,
2. Of a commutative ring which is not a domain,
3. Of a pair (R, S) with R a commutative domain and $S \subset R$ a part of R which is an additive subgroup but not an ideal,
4. Of a pair (R, I) with R a commutative ring which is not a domain and $I \subset R$ an ideal of R such that the quotient R/I is a domain,
5. Of a pair (R, I) with R a commutative domain and $I \subset R$ an ideal of R such that the quotient R/I is not a domain,
6. Of a domain which is not Euclidean,
7. Of a pair (R, F) where F is a field, R is a ring but not a field, and $R \subset F$,
8. Of a pair (R, F) where F is a field, R is a ring but not a field, and $F \subset R$,
9. Of a polynomial which is irreducible over $\mathbb{Q}$, and factors partially (i.e. is not irreducible but does not factor completely) over $\mathbb{R}$.

Exercise 0.2

Let $A = x^3 - 2$ and $B = x^2 - x$. Find $U, V \in \mathbb{Q}[x]$ such that $AU + BV = 1$.

Exercise 0.3

Let G be the group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, and let

$$H = \{(x,y) \in G \mid x \equiv y \bmod 2\}.$$

Prove that H is a subgroup of G. What is its index ? What is the quotient G/H isomorphic to ?

Exercise 0.4

Let $R = \mathbb{Z}[x]$. Does there exist an ideal I of R (and if yes, give an example) such that the quotient R/I is isomorphic to

1. $\mathbb{Z}$,
2. $(\mathbb{Z}/2\mathbb{Z})[x]$,
3. $\mathbb{Z}/6\mathbb{Z}$,
4. a field,
5. The ring $M_2(\mathbb{Z}/4\mathbb{Z})$ of 2×2 matrices with coefficients in $\mathbb{Z}/4\mathbb{Z}$,
6. The ring $\mathbb{Z}[i] = \{a + bi,\ a, b \in \mathbb{Z}\}$, where $i^2 = -1$?

Exercise 0.5

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

1. Find numbers λ and μ such that $A^2 + \lambda A + \mu I_2 = 0_2$ (where I_2 is the identity matrix of size 2×2, and 0_2 is the 0 matrix of size 2×2).
2. Does there exist a matrix B such that $AB = I_2$?
3. If yes, does B have integer coefficients ?

Exercise 0.6

1. Let A be a square matrix with complex coefficients, and suppose that the characteristic polynomial of A has real coefficients. Does this imply that the coefficients in A are in fact real ?
2. Let A be a square matrix with rational coefficients, and suppose that the coefficients of the characteristic polynomial of A are integers. Does this imply that the coefficients in A are in fact integers ?

Exercise 0.7

Check that $\mathbb{C}$ is a vector space over $\mathbb{R}$. What is its dimension ? Write down the matrix of the map "multiplication by i" with respect to a basis of your choice, and compute its trace, determinant, and characteristic polynomial. What does Cayley-Hamilton tell you in this case ?